L3 Astro - Section 3 - Stars and Stellar Evolution

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1 Hydrostatic Equilibrium

- Stars are self-gravitating systems.
- Interiors consist of highly ionized plasma and radiation.
  - kinetic and, in general, radiation pressure
  - mean-free-path $\ll$ radius of star: pressure locally isotropic
- For stars in *hydrostatic equilibrium* pressure gradient dP/dr is balanced by gravitational force density.
  - $\boxed{dP/dr = -g\rho}$
  - the equation of hydrostatic equilibrium (EoHE) expresses *conservation of momentum*
  - and, to order or magnitude, for a system with mass M and size R, so $g \sim GM/R^2$, says
  - $P/\rho \sim GM/R$
  - or, since P is essentially the thermal energy density, that the mean thermal energy per particle divided by particle mass is on the order of $\phi \sim GM/R$, the depth of the Newtonian potential well

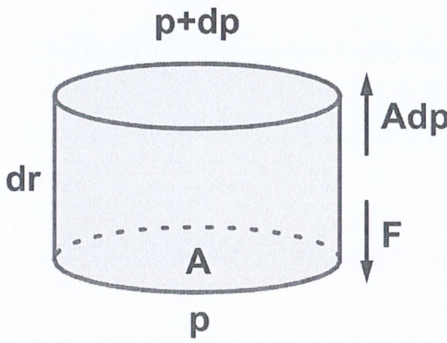


Figure 1: Hydrostatic equilibrium in a 'plane parallel' atmosphere. We consider a cylindrical volume in a stratified atmosphere where there is a pressure $P(z)$ and a gravity field g acting parallel to the z axis. The amount of mass in the cylinder is $dM = \rho A dz$ so the downward gravitational force is $F = gdM = g\rho A dz$. In equilibrium, this must be balanced by an excess upward pressure force on the bottom of the cylinder, as compared to the pressure acting downward on the top. This 'pressure gradient' force is $F = AdP = A(dP/dz)dz$. Hence the condition for hydrostatic equilibrium is $dP/dz = -g\rho$.

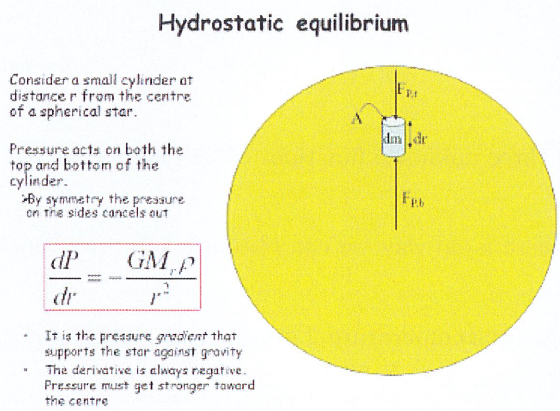


Figure 2: The hydrostatic formula derived for a plane parallel atmosphere applies to a spherical star also. The caption on the left evokes the idea that pressure is something which, 'acting' on a surface, gives a force (per unit area) normal to the surface. But the surface is not a real thing - it's just something drawn on the diagram. Is there a way to understand hydrostatic balance without invoking volumes and surfaces? What is happening *locally* in a self-gravitating ball of pressurised gas? The pressure can be said to be the *flux density of momentum*. How does that make sense? The gravitational force here is really a *force density* as it is distributed. What is the force density associated with the pressure? Is there a flux of momentum associated with the gravitational field?

2 The Main Sequence

2.1 The HR diagram

Stars come with a very wide range of luminosities; The 'Pistol Star' (actually a binary star) is about 10^5 times as luminous as the Sun. And they have a wide range of temperatures. But not all combinations are realised. Most stars lie on or close to a curve in the temperature-luminosity plane known as the *'main sequence'*

- originally seen in the *'Herzsprung-Russell diagram'*.

- in which luminosity was plotted versus 'spectral type'
- it was only later apparent that the latter was essentially measuring the temperature

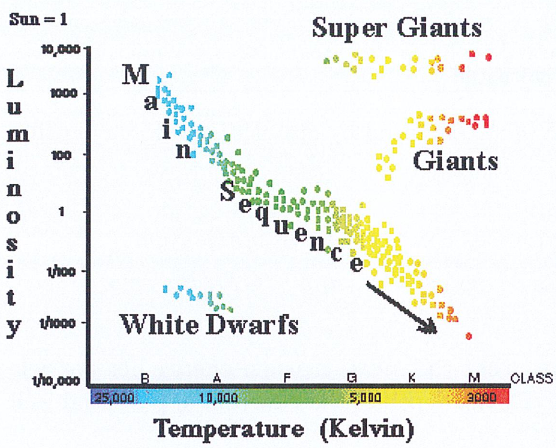


Figure 3: Stars have a wide range of temperatures and a very wide range of luminosities. In the very early 1900's Herzsprung and Russel classified the spectra of stars into 'spectral types' denoted by a letter (O, B, A, F, G, K, M). This spectral classification was actually a (inverse) measure of the temperature of the stars. With distances from parallaxes they were able to determine the absolute magnitude (or intrinsic luminosity). Plotting log-luminosity vs. spectral type they found that the stars did not cover the entire $T - L$ plane. Rather most stars lie on a well defined curve; the *'main sequence'* in which luminosity increases with surface temperature. It was also found that in older stellar systems (like globular clusters) the upper part was depopulated.

So what gives rise to the main sequence? Why would stars – or most stars, at least – be a 1-parameter family?

- Q: does it derive from the EoHE?
- this relates the pressure gradient, the density and the gravity
- but the gravity is determined by the mass – an integral of the density – so really it is a relation between pressure and density
- pressure, for a gas or plasma, is related to the other thermodynamic variables – the density *and temperature* – through the *'equation of state'* (EoS)
  - $\boxed{P = P(\rho, T)}$
  - for kinetic pressure, this relation is $P = nk_{\rm B}T$
  - for radiation pressure, $P = aT^4/3$
- the EoHE and the EoS provides only two constraints on three unknown functions of radius: $\rho(r)$, $P(r)$ and $T(r)$
  - so the system of equations is *under-constrained*; there is no way we can obtain a solution from these alone
- but if we had some other law or relation providing e.g. the temperature $T(r)$ then we could build solutions
  - for example, if we were to assume that the gas/plasma is *isothermal* with $T(r) = T_0$ – perhaps because of thermal conductivity having erased temperature gradients – then one could find a solution for $\rho(r)$ and $P(r)$
  - another possible conjecture might be that the *entropy* of the plasma is independent of radius, so the pressure would be a constant times $\rho^{5/3}$ (for gas pressure)
  - setting the initial density (or pressure) would then determine $\rho(r)$, $P(r)$ and $T(r)$, so these would form a 1-parameter set of solutions
  - neither of these hypotheses, however, are realisitic

2.2 The equations of stellar structure

- A proper explanation of the main sequence and stellar evolutionary processes had to await the development of quantum mechanics and nuclear physics.
- The necessary ingredients were
  - Understanding the *source of energy*
    * nuclear reactions in the centres of the star where density and temperature are greatest.
    * described by some complicated function of density and temperature:
      · $\boxed{\mathcal{E} = \mathcal{E}(\rho, T)}$
      · the rate of energy creation per unit volume per unit time
    * it turns out to be a *very* strongly increasing function of ρ and T
    * it also depends on the chemical composition
    * it is related to the energy flux density $\mathcal{F}$ by the *equation of continuity of energy*
      · $\boxed{d(r^2\mathcal{F})/dr = r^2\mathcal{E}}$
    * so if $\mathcal{E} = 0$ then $\mathcal{F} \propto 1/r^2$, which makes sense, while
    * if $\mathcal{E} \neq 0$ the change $\Delta(4\pi r^2 \mathcal{F})$
      · i.e. the rate at which energy is leaving a shell from the upper surface at $r + \Delta r$ minus the rate at which energy is entering at r
    * is equal to $4\pi r^2 \Delta r \mathcal{E}$
      · which is the rate at which energy is being generated by nuclear reactions within the shell
  - Understanding how the energy gets out
    * involves radiative transfer (including atomic and molecular opacity)
    * the coefficient of *diffusive conductivity* $\kappa(\rho, T)$
    * describes primarily conduction by photon diffusion
    * depends on density, temperature and chemical composition
    * provides a relation between the heat flux density $\mathcal{F}$ and the temperature gradient dT/dr:
      · $\boxed{\mathcal{F} = -\kappa dT/dr}$
      · the *equation of thermal conductivity*
    * though *convectional conduction* also plays a role (see below)
- Together with the EoHE and the EoS his gives a system of 5 equations – the *equations of stellar structure* – for 5 unknown functions $\rho(r)$, $P(r)$, $T(r)$, $\mathcal{E}(r)$ and $\mathcal{F}(r)$
  - these equations embody two conservation laws
    * momentum in the EoHE
    * energy in the equation of continuity of energy
  - along with three 'constituent relations'
    * the equation of state, relating P, ρ and T
    * the conductivity, relating $\mathcal{F}$ and dT/dr
    * the rate of nuclear energy generation $\mathcal{E}$
  - and there is implicitly the assumption that there be no change in the density of either momentum or energy, the star being assumed to have relaxed to a static configuration
    * if one were to create a spherical star for which, initially, the pressure gradient did *not* balance $g\rho$ then there would be a net force density which would drive a change in the momentum density and the system would adjust itself and 'relax' to a situation where the EoHE *is* obeyed (the disturbance of the fluid eventually dissipating)
    * and similarly, if one were to create a star where the temperature gradient dT/dr were such that $d(r^2\mathcal{F})/dr = -d(r^2\kappa dT/dr)/dr$ did not balance the local rate of energy generation $r^2\mathcal{E}$, then the density of thermal energy would have to be changing and the star would again adjust itself appropriately

* note that the *timescales* for these two relaxation processes are different: departures from hydrostatic equilibrium will get erased on the *dynamical time* which is equal to the time it takes for sound waves to cross the star which is very short while the adjustment of the temperature gradient takes place on the thermal conduction timescale.
* for stars on the main-sequence both of these timescales are short compared to the evolutionary timescale, so the assumptions of a static system are valid.

- And, assuming we have the *'cosmic abundance'* (75% hydrogen, 25% helium by weight), these give a 1-parameter family of solutions, where the parameter can be taken to be the mass.

2.3 Photon diffusion and conduction

- Photon diffusion and heat conduction are described in figures 4 and 5.
- The photons in a star are scattered by electrons so they perform a random walk (see figure 4)
  - if the cross section for scattering is σ (the Thomson cross-section for electrons) and the scatterers have space-density n, the mean free path λ is such that $n\sigma\lambda = 1$
  - for a random walk, the net distance travelled scales only as the square root of the number of paths, so it takes radiation a long time to escape from inside a star
  - Q: roughly how long does it take a photon in the Sun to random walk to the surface
- The thermal conductivity is dominated by the transport of energy by photons (see figure 5)
  - if we sit at a fixed height in a star we will see photons that were last scattered a distance $\sim \lambda$ from us
  - the ones coming up came from a region with a slightly higher number density than those going down so there is a slight excess of photons travelling up that is proportional to λ times the temperature gradient
  - and the ones coming up have slightly higher energy – the difference also being proportional to ∇T
  - the upshot is an energy flux density $\mathcal{F} \sim c n_\gamma k_{\rm B} \lambda dT/dr$
  - with the coefficient of dT/dr giving us, to order of magnitude, the thermal conductivity κ

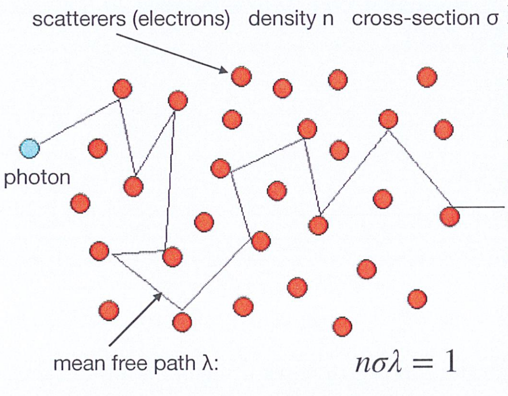


Figure 4: Photon mean free path. Consider photons deep in a star where the plasma is fully ionised and the scattering of photons is dominated by electrons. The photons perform a *random walk*. The *mean free path* can be calculated as follows: Let $P(r)$ be the probability that a photon, starting at $r = 0$ has survived a distance r without scattering. This obeys $dP = -Pn\sigma dr$ (because the fraction of area of a slab of thickness dr covered by disks of area σ is $n\sigma dr$). So $P(r) \propto \exp(-n\sigma r)$. The mean of this distribution is $\lambda \equiv \int dr r P(r) / \int dr P(r) = 1/n\sigma$. The root mean squared distance travelled on a path with N legs is $\langle D^2 \rangle^{1/2} \sim \sqrt{N}\lambda$, whereas the time taken is $N\lambda/c$. So the time taken to diffuse a distance D is $t(D) \sim c^{-1}(D/\lambda)^2 \times \lambda = D^2/c\lambda$.

2.4 Convective conduction

The model above assumes that energy gets out by *radiative conduction*.

For some stars – or for some range of radii in some stars – energy is transported via *convection*. This was first analysed by Schwarzschild. He realised that a density $\rho(r)$ and temperature $T(r)$ profile (with corresponding pressure profile $p(r)$) that satisfies the equations of stellar structure may be unstable to convection.

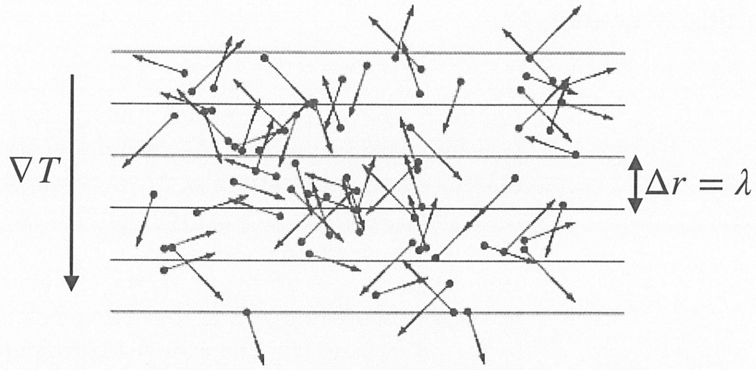


Figure 5: Photon diffusive conductivity. The number of photons per unit time crossing a horizontal area A upward is $\dot{N}_\uparrow \sim c n_\gamma A$, where n_γ is the number density of photons. A similar number will be going down, but not exactly the same. That's because the ones going up (down) were last scattered a distance $\Delta r \sim \lambda$ below (above) the surface where the density of photons was higher (lower). As $n_\gamma = \alpha T^3$ for thermal radiation, there is thus a net flux going up equal to $\Delta \dot{N} = \dot{N}_\uparrow - \dot{N}_\downarrow \sim c n_\gamma A \times (\lambda \nabla T / T)$. There is also a slight difference in the energy carried by these upward and downward travelling photons. The upshot is a *net flux of energy* across the surface is $dE/dt \sim c n_\gamma A k_\mathrm{B} \lambda dT/dr$ so the *energy flux density* is $\mathcal{F} = A^{-1} dE/dt \sim c n_\gamma k_\mathrm{B} \lambda dT/dr$ so the conductivity is $\kappa \sim c n_\gamma k_\mathrm{B} \lambda = c n_\gamma k_\mathrm{B} / n_e \sigma_\mathrm{T}$.

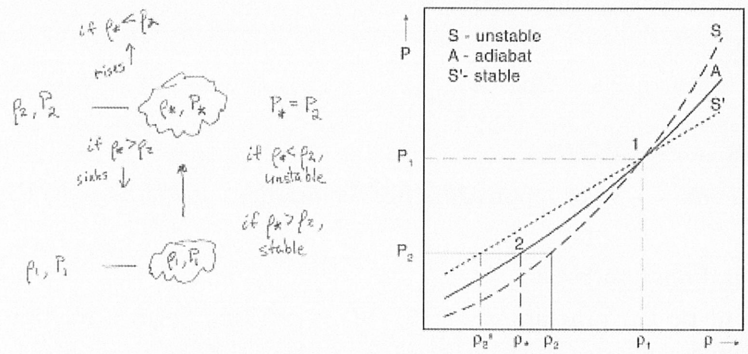


Figure 6: Schwarzschild asked, if we displace an element of fluid upwards in a stratified medium with some gradient of pressure, will it continue to rise (unstable) or will it sink back (stable)? We assume here that the element is sufficiently large, and the time-scale sufficiently short, that heat conduction is negligible, so the element conserves the specific entropy s (entropy per particle or per unit mass). The criterion for stability is that s be increasing with height.

- *Schwarzschild's stability criterion* is derived by considering what happens if an element of fluid is displaced radially.
- the element will, in general, change its volume to maintain pressure equilibrium with its surroundings.
- if it is denser (rarer) than the ambient fluid then the configuration is convectively (un)stable.
- the result is that stability requires the *specific entropy* s to be an increasing function of height.

Convection is included in stellar structure calculations as follows:

- in a *radiative zone* – i.e. a region where the equations of structure give $ds/dr > 0$ and which are therefore convectively stability – the temperature profile is determined from the conduction equation
- in a *convective zone* the temperature profile is determined by requiring $ds/dr = 0$

Schwartzschild's criterion was derived for an *ideal gas*. I.e. conduction into or out of the element of fluid on the relevant time-scale is neglected; the element is assumed to adapt to its changing environment *adiabatically*..

- The same, or similar, physics applies to instability of stratified lava tubes in volcanoes.
- Entropy gradients are also associated with trapping of pollution in the atmosphere when there are so-called "inversion layers".
- Good astronomical sites are those where the telescope is usually above the inversion layer.

2.5 The mass-luminosity scaling law

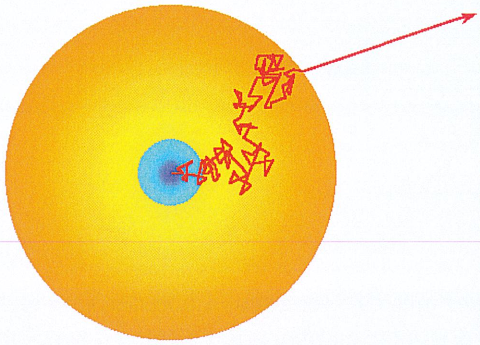


Figure 7: One can understand crudely the luminosity-mass relation for stars as follows: We assume kinetic gas gas, so $k_{\rm B}T \sim GMm_{\rm p}/R$ and therefore $T \propto M/R$. We assume that the opacity is electron scattering, with temperature independent cross section, so the mean free path is $\lambda \propto 1/n \propto R^3/M$ and the escape time is then $t_{\rm esc} \sim R^2/c\lambda \propto M/R$. The energy E in radiation in the star is black-body, so $E \propto T^4R^3 \propto M^4/R$ and the luminosity is $L \sim E/t_{\rm esc} \propto M^3$, the radius R dropping out. Interestingly, we didn't need to assume anything about the source of energy generation. The radius of the star – and hence its central density and temperature – will adjust itself so that the nuclear energy generation rate is equal to the L dictated by the total mass.

The equations of stellar structure can be solved numerically and show the main sequence to be well described by models with 'cosmic abundance' fusing hydrogen to helium.

A rough understanding of why the dependence of the luminosity on the mass of main sequence stars can be obtained from simple order of magnitude argument as follows:

- Consider a self-gravitating ball of plasma of mass M and radius R
- Ignoring the details of the radial structure
  - the density is $\rho \sim M/R^3$
  - while the gravitational acceleration is $g \sim GM/R^2$
  - hydrostatic equilibrium requires $dP/dr = -\rho g$ or, approximating the gradient as $dP/dr \sim P/R$
    * $P \sim \rho \times R \times GM/R^2 \sim \Phi\rho$
    * where $\Phi \sim GM/R$ is the gravitational potential
  - assuming the pressure is mainly kinetic gas pressure, so $P = nkT \sim \rho kT/m_p$, gives a relation between temperature, mass and radius:
    * $kT \simeq GMm_p/R$
    * note that this is essentially the *virial theorem* $2 \times \mathrm{KE} + \mathrm{PE} = 0$:
      · the left hand side is the mean kinetic energy per particle and the right hand side is, to order of magnitude, the gravitational binding energy per particle
    * and the essential ingredient we shall use here is that temperature scales with mass and radius as
    * $\boxed{T \propto M/R}$
- We assume that energy flux is limited by photons scattering off electrons
  - the *mean free path* $\lambda = 1/(n_e\sigma)$, where σ is Thompson scattering cross-section
  - so after N_c collisions, a photon will have travelled a net distance $l \simeq \sqrt{N_c}\lambda$
  - so to escape ($l = R$) a photon has to scatter $N_c \sim (R/\lambda)^2$ times
  - this takes time $t = N_c \times (\lambda/c)$
  - and the *escape time* is
    * $t_{\rm esc} \sim R^2/\lambda c$
  - which, since $\lambda \propto 1/n_e$, and $n_e \propto M/R^3$, as we are assuming a fully ionised plasma, scales with mass and radius as
    * $\boxed{t_{\rm esc} \propto M/R}$
- The energy in the radiation is $E \sim u(T)R^3 = aT^4R^3$, so, with $T \propto M/R$ from the virial theorem (or hydrostatic equilibrium) this scales with mass and radius as

– $\boxed{E \propto M^4/R}$

- And finally, if this energy escapes in time $t_{\rm esc}$, the luminosity is $L \sim E/t_{\rm esc} \propto M^4/R/(M/R)$
- So in this model the luminosity is only a function of the *mass*

  – $\boxed{L \propto M^3}$

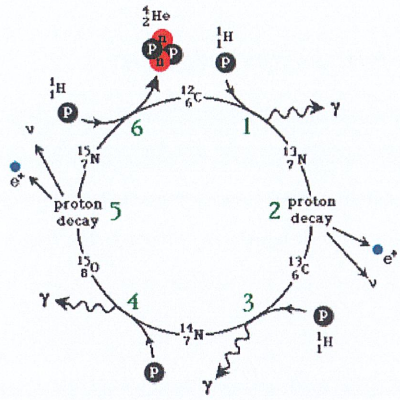
- Despite the very crude modelling, this is actually quite a reasonable approximation. More detailed modelling shows that
  – $L \propto M^\gamma$ with $\gamma \simeq 3.5$ for low-to-intermediate mass stars
  – where, for lower mass stars, we need to include atomic opacity as well as electron scattering
- A key result: *the luminosity scales as a high power of mass*
  – so *lifetime* $t \sim \epsilon M/L$ decreases (strongly) with mass
    * here $\epsilon \sim 1\%$ (roughly the fractional excess mass of a neutron compared to a proton) characterizes the efficiency of nuclear reactions
  – so *massive stars live fast and die young*
  – this explains why, in an old globular cluster, we only see stars with L below what is called the *main sequence turn-off*
  – it also explains why spiral galaxies, with ongoing star-formation and with bright, hot young stars still present, tend to be blue, while elliptical galaxies, in which star-formation has ceased, are *'red and dead'* as it is only the low-mass cooler stars that are dominating the luminosity
- Perhaps surprisingly, we were able to obtain the $L - M$ relation solely by considering hydrostatic equilibrium and thermal conductivity from photon diffusion: the law $L \propto M^3$ *is independent of the details of the source of energy.*
  – so we do not obtain, from this argument, an expression for the radius
  – to determine R – which in turn, with L would determine the surface temperature T, since $L \sim acT^4R^2$ – we would need to model the process of heat generation by nuclear fusion
  – but that is highly complex, and also sensitive to the inner structure of the star, so cannot be simply included
  – however, the strong positive dependence of the heat generation on density and temperature means that we would expect stars, in this model, to be *self-regulating*
    * imagine that the heat generation were larger than that needed to provide the luminosity
    * the star would then heat up and expand to reduce the energy generation rate
    * so it will adjust its size (and temperature) to maintain the luminosity required by hydrostasis and photon conduction

  – lifetime $\propto \frac{1}{M^2}$

3 Stellar evolution

3.1 Sub-solar and main sequence stars

- sub-stellar objects $M \ll M_\odot$ do not get hot enough to ignite hydrogen fusion
  – their heat source is *gravitational contraction*
- main-sequence: these are hot and dense enough to fuse hydrogen to helium
  – the Sun burns via the 'P-P chain'. More massive stars burn via the 'CNO-cycle'



$$^{12}C_6 + {}^1H_1 \longrightarrow {}^{13}N_7 + \gamma$$
$$^{13}N_7 \longrightarrow {}^{13}C_6 + e^+ + \nu$$
$$^{13}C_6 + {}^1H_1 \longrightarrow {}^{14}N_7 + \gamma$$
$$^{14}N_7 + {}^1H_1 \longrightarrow {}^{15}O_8 + \gamma$$
$$^{15}O_8 \longrightarrow {}^{15}N_7 + e^+ + \nu$$
$$^{15}N_7 + {}^1H_1 \longrightarrow {}^{12}C_6 + {}^4He_2$$

Figure 8: Low mass stars like the sun fuse hydrogen via the 'proton-proton chain'. In more massive stars, helium is also generated via the 'CNO-cycle', which is a *catalytic* process involving carbon, nitrogen and oxygen. In this process, a ^{12}C carbon nucleus changes progressively, absorbing protons, some of which get converted to neutrons (emitting positrons and neutrinos – an extra source of heat) until eventually it can decay back to ^{12}C by emitting a helium nucleus.

3.2 Post main sequence evolution

- once sufficient helium 'ash' accumulates the stars switch to burning hydrogen in a shell surrounding the helium core and surrounded, in turn, by a bloated red-giant 'envelope'.
  - the star evolves off the main sequence (to higher L but lower T) (see figure 9)
- this continues until central density/temperature high enough to ignite He burning
  - for high enough mass stars
  - for lower mass stars the He core remains inert
- multiple shells form as higher mass nuclei fuse
  - short evolutionary time-scale
- but fusion is only exothermal up to iron
- what happens then? what is the ultimate fate of a star?

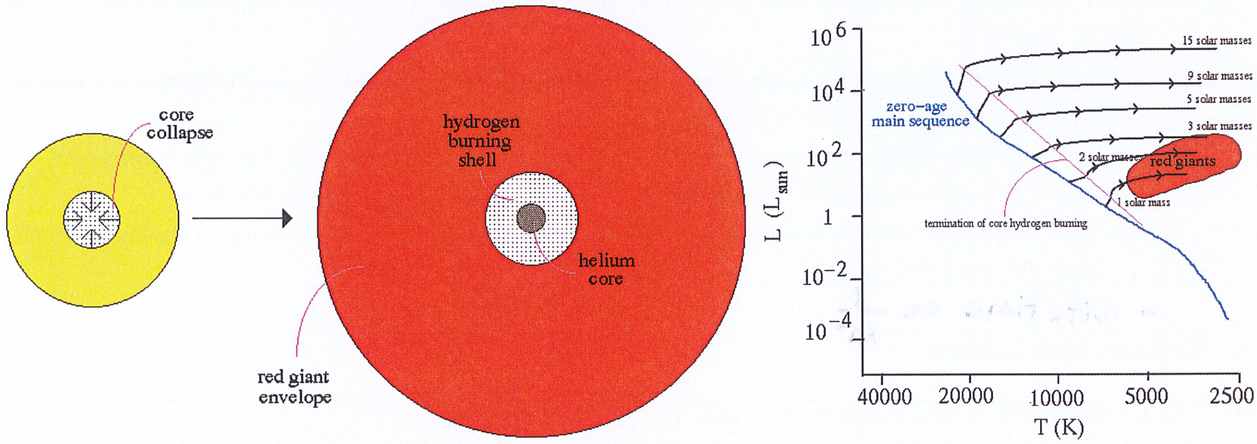


Figure 9: Main-sequence stars burn hydrogen, building up 'ash' of helium in their centres. As the helium builds up, the stars move slightly off the so-called 'zero-age main sequence', but at a certain point the core collapses and the star evolves dramatically off the main sequence, becoming a 'red-giant'. The overall properties of the star are still determined largely by hydrostasis and photon diffusion, so the stars move on tracks at (very roughly) constant luminosity.

4 White dwarfs

4.1 Observed properties

Peculiar properties recognised in the early 20th century. The first example was Sirius-B

- this is in a binary system - with a distance measurable from its parallax so its orbital velocity and orbit radius gives the mass:
  - $M \simeq 1 M_\odot$ $\Leftarrow$ solar mass
- but its low flux density (combined with its distance) $\Rightarrow$ very low luminosity
- and its *high effective temperature* $\Rightarrow$ very compact
- so it is *extremely dense:* $\rho \simeq 10^9$ kg/m^3
  - this implies that light from the surface should be *gravitationally redshifted*
  - a prediction that was observationally confirmed
- Q: how does this density compare with that of the sun (hint: solar and lunar tides are roughly similar).

4.2 Eddington's "paradox"

Stars of this density posed a challenge to the then current theory. Eddington: *"It would seem that the star will be in an awkward predicament when its supply of subatomic energy fails."*

Fowler expressed the conundrum as follows:

- The electrostatic energy per unit volume of dense plasma with ions (of charge Ze) is embedded in a quasi-uniform distribution of negative charge
  - $u_{\rm elec} \sim -Z^2 e^2 n^{4/3}$
  - i.e. essentially e^2/r divided by a volume $r^3 \sim 1/n$
  - or like the electrostatic binding energy of an atom (per atomic volume)
- while the thermal energy ($(3/2)kT$ per particle) has a density
  - $u_{\rm thermal} \sim n k_{\rm B} T$
- in normal stars, the thermal energy dominates and the stars are 'springy'
  - if compressed adiabatically, $-dP/dr$ exceeds $g\rho$, and the star springs back
  - Q: check that this is correct – figure out the change in gravitational binding energy and the change in the thermal energy - which one "wins"?
- But, for a given T, there is a density n above which $u_{\rm elec} + u_{\rm thermal} < 0$
  - so even ignoring gravity, in this regime it requires energy *input* to expand a volume of fluid so it cannot 'spring back'
  - and white dwarfs seem to live in this regime
- As Fowler put it: *"If part of* [the material in a WD] *were removed from the star and the pressure taken off, what could it do?"*

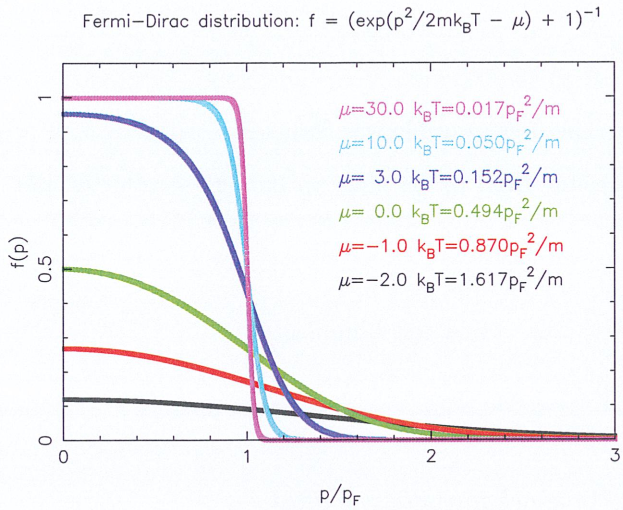


Figure 10: In thermal equilibrium at temperature T, the electron momentum distribution is $f(\mathbf{p}) = 1/(e^{E/k_BT-\mu}+1)$ where $E = |\mathbf{p}|^2/2m$ and μ is the *'chemical potential'*. If μ is large and negative, the mean occupation number f is very small, so the electrons are highly non-degenerate, and the distribution is essentially Maxwellian. But if the electrons lose energy and cool, the chemical potential increases. At low enough temperatures this changes the distribution function in a qualitative manner. As the occupation number cannot exceed unity, one ends up with the electrons filling all the momentum/spin states up to the *'Fermi momentum'* $p_F = h(3n_e/8\pi)^{1/3}$. The plot shows how $f(p)$ changes if the electrons cool at constant volume.

4.3 Electron degeneracy pressure

Fowler (1926) resolved the 'paradox'.

- Electron de-Broglie waves in a box: density of states (per unit volume) $dn = 8\pi p^2 dp/h^3$
  - factor 2 from 2 'spin-states' per Fourier mode
- Fermi-Dirac: mean occupation number
- $\boxed{f(p) = (\exp(E(\mathbf{p})/k_BT - \mu) + 1)^{-1}}$
  - reminiscent of thermal radiation occupation number $f = (e^{E/k_BT} - 1)^{-1}$
    * but with a '+' sign rather than '-'
    * so whereas for photons (bosons) $f \gg 1$ in the low-energy 'Rayleigh-Jeans' regime f can never exceed unity for fermions
  - 2-parameters: inverse temperature $\beta = 1/k_BT$ and 'chemical potential' μ fixed by the total energy and number of particles (Lagrange multipliers).
    * unlike thermal radiation, we don't assume that the number of particles is free to adjust itself – which is why there is no μ in the black-body formula
  - as $T \to 0$, $f(p)$ becomes a step function equal to 1 (0) below (above) the *Fermi momentum* p_F (see figure 10).
    * momentum for which $n_e\lambda_{dB}^3 \sim 1$
  - parameter μ (dimensionless) measures how 'degenerate' the electrons are
    * but one often sees $\mu' = \mu k_BT$ so μ' has units of energy
  - limiting cases:
    * $\mu \ll 0$
      · highly non-degenerate
      · Maxwell-Boltzmann distribution $f \propto \exp(-E/k_BT)$
      · all modes have small occupation number
    * $\mu \gg 0$:
      · highly degenerate
      · all modes below p_F fully occupied
      · all modes above p_F empty
- mode spacing: $\Delta k = 2\pi/L \quad \Rightarrow \quad \Delta p = \hbar\Delta k = h/L$
- so if all the modes up to p_F are full we have

- $(8/3)\pi(p_{\rm F}/\Delta p)^3 = N \quad \Rightarrow \quad (8/3)\pi p_{\rm F}^3 = Nh^3/L^3 = nh^3$

- so the *Fermi momentum* is
  - $\boxed{p_{\rm F} = h(3n/8\pi)^{1/3}}$
  - thus, with $E = p^2/2m_e$, $u \sim np_{\rm F}^2/2m_e \sim h^2 n^{5/3}/m$
  - pressure increases as $n^{5/3}$, not as n (as would be the case for thermal energy density at fixed T)
    - like for adiabatic compression of a monatomic gas
  - this resolves Eddington's paradox
    - since the electrostatic energy density only goes up like $n^{4/3}$
    - so if the white dwarf is compressed, it springs back

4.4 White dwarf properties

- Hydrostatic equilibrium (or virial theorem) says KE $\sim$ PE:
  - $p_{\rm F}^2/2m_e \sim GMm_p/R$
- so $p_{\rm F}^2/2m_e \sim GM^{2/3}(M/R^3)^{1/3}m_p \sim GM^{2/3}(nm_p)^{1/3}m_p \sim GM^{2/3}m_p^{4/3}p_{\rm F}/h$ or
  - $\boxed{p_{\rm F}/m_e c \sim GM^{2/3}m_p^{4/3}/hc}$
- So hydrostatic equilibrium $\Rightarrow p_{\rm F} \propto M^{2/3}$, while quantum mechanics $\Rightarrow p_{\rm F} \propto n^{1/3} \propto M^{1/3}/R$
  - together these give the *radius-mass relation* for WDs:
    - $\boxed{R \propto M^{-1/3}}$
  - bigger mass $\Rightarrow$ *smaller* WD
- But this assumes electrons are non-relativistic – i.e. $p_{\rm F} \ll m_e c$
  - $p_{\rm F}$ is increasing with M, so this is valid for small enough M,
  - but breaks down at $M \simeq M_\star \sim (hc/G)^{3/2}m_p^{-2}$
- Q: what happens if we have a star of mass $M > M_\star$ and let it cool?
  - as it contracts, the Fermi momentum rises and reaches the point where $p_{\rm F} \sim m_e c$
  - beyond that point the majority of the electrons are relativistic
  - but relativistic electrons – like photons – have an 'adiabatic index' $\gamma = 4/3$
    - as compared to $\gamma = 5/3$ for non-relativistic particles
    - so the star is no longer 'springy' – at best marginally stable
  - while the gravitational force increases because 'pressure gravitates in GR'
  - upshot: degeneracy pressure cannot stabilise the star

4.5 The Chandrasekhar mass

We assumed non-relativistic electrons $v \ll c$ above. But, as mentioned, this breaks down at

$$\boxed{M_\star = (hc/G)^{3/2}m_p^{-2}} \tag{1}$$

which (give or take geometric factors) is the mass of a WD for which the electrons reach relativistic speeds.

- $M_\star$ is formed from the 3 fundamental constants of nature and the proton mass
- interestingly, it is independent of the mass or charge of the electron

- it can also be expressed as $M_\star \sim m_{\rm Pl}^3/m_p^2$ where
  - $m_{\rm Pl} = (hc/G)^{1/2} \simeq 10^{-5}$g is the *Planck mass*
  - the only quantity with units of mass one can form from c, G and h
- Detailed relativistic analysis of hydrostatic equilibrium shows that the radius of the WD falls to *zero* – as illustrated in figure 11 – at $M_{\rm C} = (3/16\pi)(hc/G)^{3/2}(\mu m_p)^{-2} \simeq 1.74 M_\odot/\mu^2$
  - where here μ is the *mean molecular weight per electron*
    - *not* the chemical potential - sorry!
  - so $M_{\rm C} \simeq 1.4 M_\odot$ for $\mu \simeq 2$
  - this is called the *Chandrasekhar mass*
- it is impossible to have a WD above this mass

4.6 Supernovae

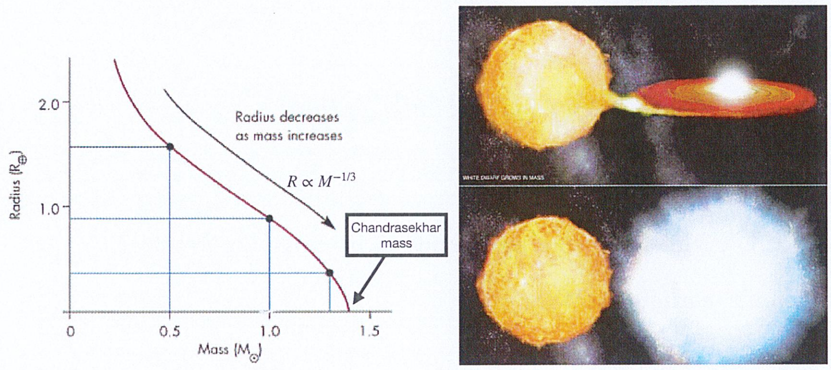


Figure 11: Left panel shows the white dwarf mass-radius relation. The power law behaviour is for masses such that the Fermi momentum is non-relativistic. WDs cannot exist for M beyond the Chandrasekhar mass $M_\star$. Type 1a supernovae are believed to form in binaries where accretion drives the mass above $M_\star$, at which point they explode.

4.6.1 Type 1a supernovae

- Type 1a supernovae are believed to be white dwarfs in a binary system that are accreting matter from the companion.
- When their mass reaches $1.4 M_\odot$ they become unstable and explode as they become neutron stars
- It is this uniformity of their physical nature
  - being determined from fundamental constants
  - with only weak dependence on their composition
- that makes them remarkably uniform in their luminosities
  - thus making them excellent *standard candles* for cosmological studies
- this was used to show that the expansion of the Universe is *speeding up*
  - reported in 1998 by two groups
  - awarded the Nobel prize in 2011
  - interpreted to show that the Universe has become dominated by some mysterious *dark energy*

4.6.2 Type 2 supernovae

- Type 2 – or *core-collapse* supernovae are thought to be the result of late-stage evolution of more massive stars
  - they may become neutron stars
  - or black holes
- depending on the mass of the progenitor

A Statistical mechanics of degenerate electrons

A.1 The density of states

We consider electron to be 'de-Broglie waves' in a box of volume $V = L^3$ just as we did for non-degenerate particles

This gives the density of states:

- the spacing in momentum of allowed states is $\Delta p = h/L$
- so the number of states in a shell of thickness dp is
  - $dN_{\rm states} = 2 \times 4\pi p^2 dp/(\Delta p)^3$
- with the factor 2 from the 2 'spin-states' per mode
- we will follow Bose's terminology and consider shells in momentum space labelled by an index s such that $p = s\Delta p = hs/L$.

A.2 The complexion and the entropy

As Bose did with photons, we let A^s denote the number of modes – 'cells' in Bose's terminology – in the $s^{\rm th}$ shell and p_r^s be the number of modes in that shell with occupation number r.

The *complexion* W is defined to be the number of sets of occupation numbers $\{n_j\}$ where $j = 1 \ldots A^s$ consistent with p_r^s and is given by

$$W = \prod_s A^s! / \prod_r p_r^s! \tag{2}$$

just as for photons – though here we will only allow $r = 0$ or 1.

- the entropy S is defined to be the logarithm of the complexion:
  - invoking Stirling's formula we have
  - $S = \log W = -\sum_s (A^s \log A^s - \sum_r p^s \log p_r^s)$
  - or
  - $S = -\sum_s A^s \sum_r (p_r^s/A^s) \log(p_r^s/A^s)$
  - where the number of cells in the $s^{\rm th}$ shell is $A^s = dN_{\rm states}$
  - or, since we can only have occupation number $r = 0, 1$ (Fermi exclusion principle), and defining $p_1^s/A^s = f_s$
  - $\boxed{S = -\sum_s A^s[(1 - f_s)\log(1 - f_s) + f_s \log f_s]}$

A.3 The thermal, or 'Fermi-Dirac', distribution function

- maximising S subject to the constraints on the number of particles
  - $N_e = \sum_s A^s f_s$
- and of the total energy (non-relativistic electrons)
  - $E_{\rm tot} = \sum_s A^s f_s p^2/2m = (h^2/2mL^2)\sum_s A^s f_s s^2$
- by means of Lagrange multipliers μ and β, the thermal equilibrium distribution must satisfy
  - $\delta(S + \mu N_e - \beta E_{\rm tot})/\delta f_s = A^s[\log[f_s/(1 - f_s)] + \mu - \beta p^2/2m] = 0$
- whose solution, with $\beta = 1/k_{\rm B}T$, is the *Fermi-Dirac distribution function* for which the mean occupation number $f(p) = f_s$ is
  - $\boxed{f(p) = (\exp(p^2/2mk_{\rm B}T - \mu) + 1)^{-1}}$

- the Lagrange multiplier μ is known as the 'chemical potential'. It is actually more common to define this as $\mu' = k_B T\mu$, with units of energy, so the argument of the exponential is $(p^2/2m - \mu')k_B T$. But I prefer not to use this and to keep μ dimensionless – which is simpler, to my mind – and befits the fact that it is the Lagrange multiplier for the number of particles.

- the actual distribution over momentum, defined such that the number of electrons in the momentum volume d^3p is $d^3N_e(p)$ is given by $d^3N_e = A^s f_s$ which, with $A^s = dN_{\text{states}}$, is
  - $d^3N_e(p) = (8\pi L^3/h^3)dpp^2/(\exp(p^2/2mk_B T - \mu) + 1)$

- limiting cases:
  - for μ large and negative this is a *Maxwellian distribution*
  - for μ large and positive, $f(p)$ is the *Heaviside function* which is unity (zero) below (above) a momentum, known as the *Fermi-momentum*, that is determined by the density of particles $n_e = N_e/L^3$ (see figure 10)

A.4 The Fermi momentum

- with $f(p)$ being zero and 1 above and below p_F we must have $N = (8\pi/3)p_F^3 L^3/h^3$ or
  - $\boxed{p_F = (3n_e/8\pi)^{1/3}h}$

- a particle with $p = p_F$ would have a de Broglie wavelength $\lambda = h/p$ on the order of the mean interparticle separation $n^{-1/3}$

A.5 The relation between μ, p_F and T

We have presented above the distribution function, or mean occupation number, for given μ and temperature T. Often we know the density of the electrons – and hence the Fermi momentum – and the temperature and we wish to solve for μ.

- integrating d^3N_e with the substitution $p = \sqrt{mk_B T}y$ gives
  - $N_e = \int d^3N_e = (8\pi/3)(L/h)^3(mk_B T)^{3/2}I(\mu)$
  - where $I(\mu) \equiv 3\int dyy^2/(e^{y^2/2-\mu} + 1)$ is an increasing function of μ (see figure 12)

- implying
  - $mk_B T/p_F^2 = I(\mu)^{-2/3}$

- so we can consider, if we like, μ to be a function of the ratio of the thermal energy to the Fermi energy

- or, equivalently, a function of the ratio of the mean separation $n_e^{-1/3}$ to the de Broglie wavelength of a particle with $p^2/2m = k_B T$

- so μ is a measure of the degree of degeneracy

A.6 The probability distribution for the momentum $P(p)$

- writing $d^3N_e = N_e dP(p)$, so $P(p)$ is a normalised probability, we have for the distribution of electrons per interval of (natural) logarithm of momentum:
  - $dP(p)/d\log p = 3(p/p_F)^3 f(p)$
  - which is shown in figure 13 for a range of values of μ.

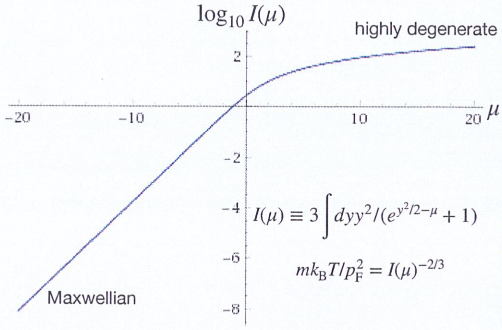


Figure 12: The dimensionless parameter μ appearing in the Fermi-Dirac distribution function $f = (e^{E/k_B T - \mu} + 1)^{-1}$ is the chemical potential. It arises from the constraint on the number of particles and is the Lagrange multiplier for this constraint. In situations where interactions can efficiently create and destroy particles $\mu = 0$. The normalisation of the distribution function gives a relation between the temperature T, the chemical potential μ and the density of particles n (or equivalently the Fermi momentum) $mk_B T/p_F^2 = I(\mu)^{-2/3}$ where the function $I(\mu)$ is plotted on the left. The chemical potential is a measure of the degree of degeneracy.

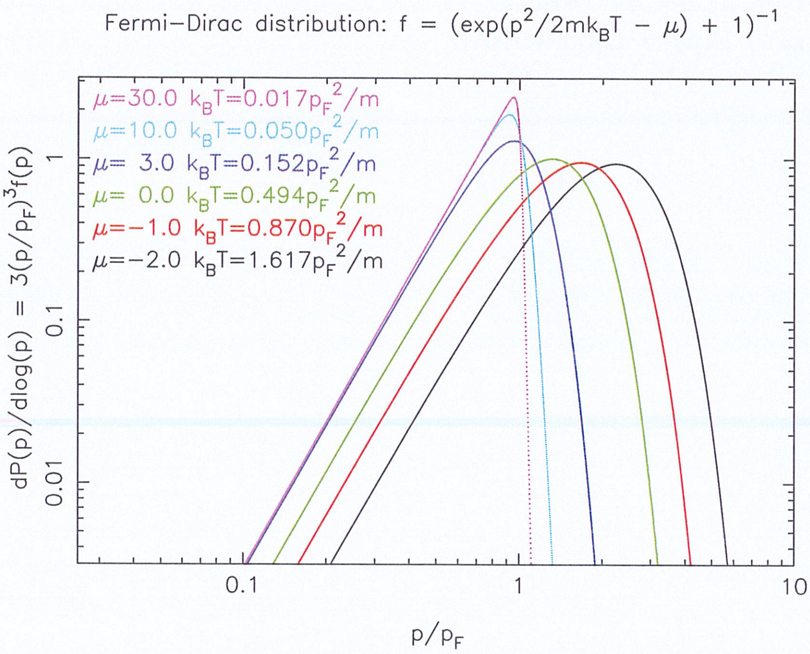


Figure 13: This plot contains the same information as figure 10 but shows how the electrons are distributed over momentum. Here $dP(p)$ is the fraction of particles with momentum in some interval dp, so $P(p)$ is the normalised distribution of momenta. The ordinate is $dP/d\log p$, the fraction of particles per natural log of the momentum, plotted as a function of p in units of the Fermi momentum. For negative μ (rightmost curves) we see Maxwellian distributions, which vary as $dP/d\log p \propto p^3$ for $p \ll \sqrt{k_B T m}$ and are cut-off exponentially at larger p. For more positive μ-values, the cut-off becomes sharper as the electrons become more and more degenerate.

A.7 Fermi-Dirac vs. Bose-Einstein distributions

- the *Bose distribution* for black body radiation is
  - $f(p) = (e^{E(p)/k_B T} - 1)^{-1}$
- the generalisation to a situation where there is a constraint on the number of photons (as in the case where scattering is able to thermalise the momentum distribution but is unable to generate a fully thermal distribution) is the *Bose-Einstein distribution:*
  - $f(p) = (e^{E(p)/k_B T - \mu} - 1)^{-1}$
- which is very similar in form to the *Fermi-Dirac distribution:*
  - $f(p) = (e^{E(p)/k_B T - \mu} + 1)^{-1}$
- whose specialisation to the case where interactions can change the number of particles – and thus drive the chemical potential to zero – is
  - $f(p) = (e^{E(p)/k_B T} + 1)^{-1}$

A.8 Fermi-Dirac and Bose-Einstein from the collisional Boltzmann equation

Quantum field theory gives cross sections for scattering processes and also the 'stimulated emission' and 'Fermi-blocking' factors $1 + f$ (for bosons) and $1 - f$ (for fermions).

The rate of scattering of particles out of momentum states $\mathbf{p}_1$ and $\mathbf{p}_2$ into states $\mathbf{p}'_1$ and $\mathbf{p}'_2$ can be written as an integral over d^3p_2 of a differential cross section times a factor

$$\underbrace{f(\mathbf{p}_1)f(\mathbf{p}_2)(1 \pm f(\mathbf{p}'_1))(1 \pm f(\mathbf{p}'_1))}_{\text{'forward' reactions: } \mathbf{p}_1\mathbf{p}_2 \Rightarrow \mathbf{p}'_1\mathbf{p}'_2} - \underbrace{f(\mathbf{p}'_1)f(\mathbf{p}'_2)(1 \pm f(\mathbf{p}_1))(1 \pm f(\mathbf{p}_1))}_{\text{'inverse' reactions: } \mathbf{p}'_1\mathbf{p}'_2 \Rightarrow \mathbf{p}_1\mathbf{p}_2} \tag{3}$$

with '+' and '-' signs for bosons and fermions respectively.

In equilibrium this must vanish. This implies

$$\log \frac{f(\mathbf{p}_1)}{1 \pm f(\mathbf{p}_1)} + \log \frac{f(\mathbf{p}_2)}{1 \pm f(\mathbf{p}_2)} = \log \frac{f(\mathbf{p}'_1)}{1 \pm f(\mathbf{p}'_1)} + \log \frac{f(\mathbf{p}'_2)}{1 \pm f(\mathbf{p}'_2)}. \tag{4}$$

But energy is conserved in these reactions, so

$$E(\mathbf{p}_1) + E(\mathbf{p}_2) = E(\mathbf{p}'_1) + E(\mathbf{p}'_2). \tag{5}$$

These are compatible if $\log(f(\mathbf{p})/(1 \pm f(\mathbf{p}))) = E(\mathbf{p}) + \text{constant}$ or

$$\frac{f(\mathbf{p})}{1 \pm f(\mathbf{p})} = \exp(\beta E(\mathbf{p}) - \mu). \tag{6}$$

or, solving for $f(\mathbf{p})$,

$$\boxed{f(\mathbf{p}) = (e^{E(\mathbf{p})-\mu} \mp 1)^{-1}.} \tag{7}$$