L3 Astro - Section 5 - The Milky Way and the Local Group

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1 Star Clusters and Stellar Populations

There are two quite distinct types of stellar clusters in the Milky Way: *Open clusters* and *globular clusters*.

1.1 Open Clusters

Open clusters are sites of active star formation.

- They lie in the plane of the disk
- They contain lots of young and massive stars ("O,B" stars)
  - massive and hot $\rightarrow$ UV radiation, nebular emission
  - so-called *'population I stars'*
- typically gravitationally unbound
  - can gauge age from dispersion time-scale $t \sim r/\sigma_v$
- 'laboratories' for studying e.g. the 'zero-age main sequence'
  - large numbers of 'co-eval' stars
  - different aged clusters show progression of the 'turn-off' down the main sequence of the HR diagram

Figure 1: Left: the Pleiades: a small, but nearby, open cluster in the MW. Nebulosity of hot gas being heated by UV radiation from the hot stars is visible. Centre: NGC265: an open cluster in the Small Magellanic Cloud. Right: the globular cluster M13. Open clusters contain many hot, young stars ('population I') and populate the disk of the MW. Globular clusters contain old ('population II') stars and populate the halo of the MW.

1.2 Globular Clusters

Globular clusters contain old stars.

- they are distributed in roughly spherical 'halo'
- the stars populate the low-mass end of the main sequence
  - they have lower 'metallicity' than young stars in star forming regions
  - so-called *'population II stars'*
  - so presumably formed earlier than pop-I stars, when the ambient medium was less polluted
- they are gravitationally bound - typically $\sim 10^6$ stars
- they have short dynamical (i.e. orbital) times
- some have 'relaxation times' (see next chapter) shorter than or comparable to the age of the universe

1.3 Stellar Populations and Galactic Evolution

Astronomers refer to all elements except the primary 'primordial' light elements – hydrogen and helium – as 'metals'. The overall picture is that early stars formed from gas which was of low metallicity. The older stars evolve and pollute the IGM with metals. Later stars form from enriched gas.

The terminology is that young stars forming today such as in open clusters are *population I*. Older stars such as those in globulars are called *population II*. This is backwards in terms of evolution: The pop-II stars are old and formed before pop-I (when there was much lower metallicity).

It has long been debated that there may have been a distinct earlier phase of formation of *population III* stars. This debate has been rekindled by the discovery of somewhat unexpected massive black-hole binaries by LIGO.

1.4 Structure of the Milky Way

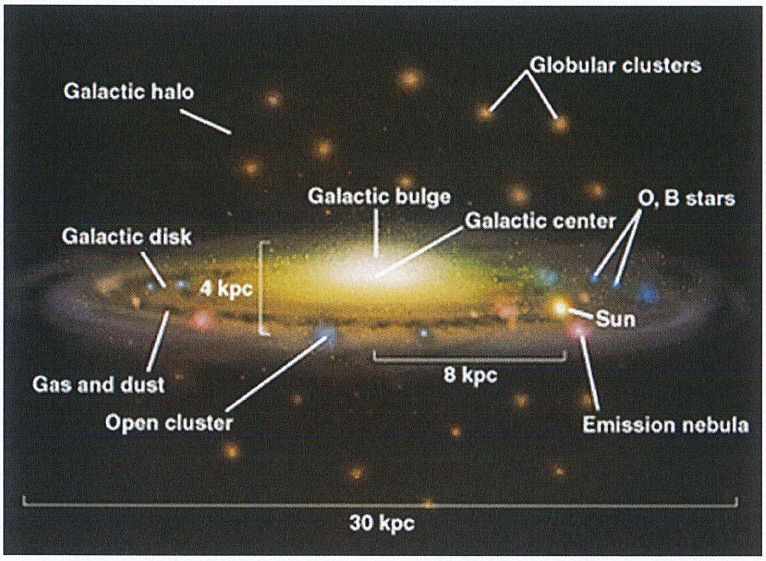

Figure 2: The Milky Way has three main visible components: The disk, the bulge, and the halo. The bulge ($L \sim 5 \times 10^9 L_\odot$) is spheroidal and is composed of old (pop-II) stars and resembles a small elliptical galaxy. It is dynamically 'hot': it is supported by kinetic pressure. The disk ($L \sim 2 \times 10^{10} L_\odot$) is composed of younger stars and is thin. It is supported by rotation. The halo ($L \sim 10^9 L_\odot$) is composed of pop-II stars – both in globular clusters and dispersed – and has a roughly power law profile $n \propto r^{-3}$. Not shown is the main constituent; the dark matter halo ($M \sim 10^{12} M_\odot$), which has a roughly power-law profile $\rho \propto r^{-2}$ extending to at least $\sim$ 200kpc.

- The Disk
  - highly flattenened - ordered rotational motiion
  - $L \sim 2 \times 10^{10} L_\odot$
  - population I - plus components of ISM
  - roughly exponential distribution (in radius and height)
    - $n \propto \exp(-r/r_\star) \times \exp(-z/z_\star)$
    - radial scale length $r_\star \sim$ 3kpc (we lie at 8 kpc)
    - vertical 'scale-height' $z_\star \sim$ 0.3kpc
  - thin and thick disk components
    - stars in the thick disk have lower metallicity
    - this supports the idea that stars are born with low velocities in the plane of the disk but get 'heated' over time by gravitational interactions so the disk thickens with age
- The Bulge
  - spheroidal spatial distribution - supported by 'random' motions (kinetic pressure)
  - 'de Vaucouleurs' profile
    - $n \propto \exp(-(r/r_\star)^{1/4})$ or, in more detail,
    - $n \propto \exp(-(ax^2 + by^2 + cz^2)^{1/8})$ – 'spheroidal' isopleths
  - total luminosity $L \sim 5 \times 10^9 L_\odot$

- scale-length ~ 1 kpc. somewhat flattened
- older stars - little star formation
- similar to a modest luminosity elliptical

- The Halo
  - roughly spherical, roughly power law distribution
  - $n \sim r^{-3}$
  - $L \sim 10^9 L_\odot$
  - population II - also includes globular clusters
  - extends to at least $\sim$ 100kpc

2 Weighing the Disk

In the 'local standard of rest' frame we see stars moving with random motions (in directions perpendicular to as well as in the plane).

Astronomers have measured the 'velocity dispersion' for vertical motions: $\sigma \sim$ tens of km/s (greater for older stars) and the density profile (roughly exponential $n(z) = \exp(-z/z_*)$). From this we can determine the density – and density profile – of the disk.

2.1 Uniform 'Slab' model

Consider a uniform density slab.

- Poisson's equation: $\nabla^2 \phi = 4\pi G \rho \quad \Rightarrow \quad \phi(z) = 2\pi G \rho_0 z^2$
- so the gravitational acceleration is $g = -\partial\phi/\partial z = -4\pi G \rho_0 z$
  - so $\ddot{z} = -4\pi G \rho_0 z$: a simple harmonic oscillator: $z = z_0 \cos(\omega t)$ with $\omega = \sqrt{4\pi G \rho_0}$
  - so $\dot{z} = -\omega z_0 \sin(\omega t)$
  - and therefore – to order of magnitude – $\omega \sim \sigma_z / z_0 \sim \sqrt{4\pi G \rho_0}$
  - or $\rho_0 \sim (\sigma_z/z_0)^2/(4\pi G)$.
- this is very rough – we'll next see how to make it precise

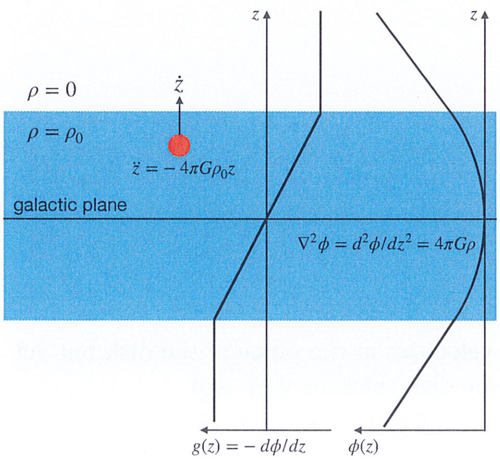

Figure 3: A simple 'slab' model for the disk of the MW. We model the density as being constant within some distance of the mid-plane of the galaxy. That implies that the gravitational potential is a parabola and therefore that the gravity increases linearly with distance from the mid-plane. Stars, in this model, oscillate up and down in the disk, obeying the equation of motion of a simple harmonic oscillator, with frequency ω being equal to the inverse of the *'dynamical time'* $t_{\rm dyn} = 1/\sqrt{4\pi G \rho_0}$. If we measure the mean square vertical velocity $\sigma_z^2 = \langle \dot{z}^2 \rangle$ of stars near the mid-plane and estimate the height distribution of stars in the disk to obtain the mean height z_0 then we have $\rho_0 \sim (\sigma_z/z_0)^2/(4\pi G)$ which we can use to determine the disk density.

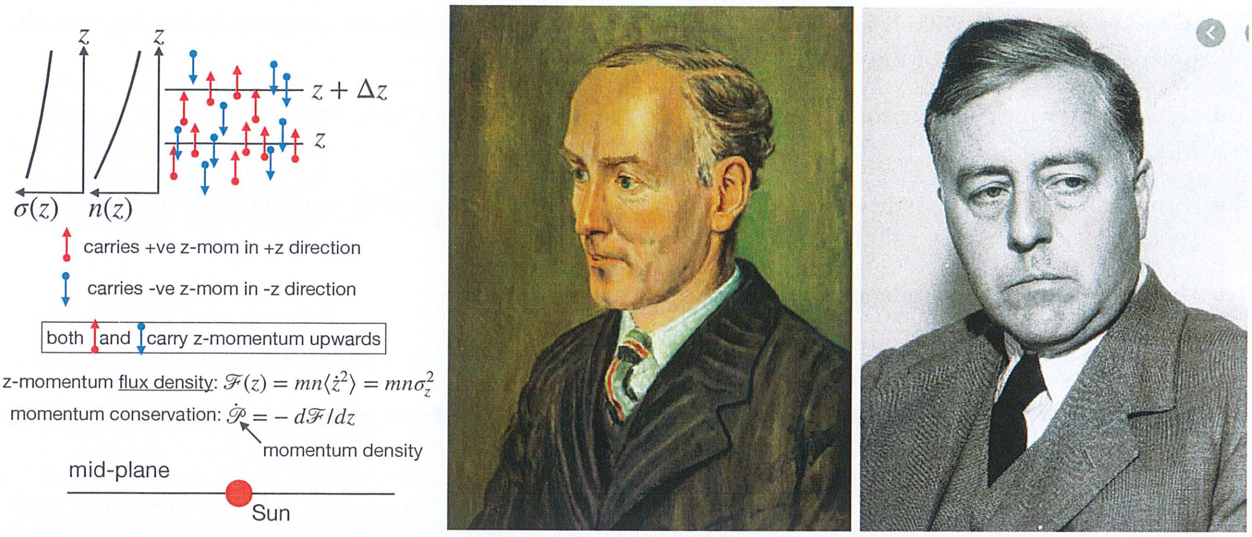

Figure 4: The Jeans equation in 1D. On the right is Sir James Jeans – the famous astrophysicist who shares with Rayleigh the name for the long-wavelength end of the black-body spectrum. In the centre is Jan Oort, who first used Jeans's equation to measure the density of matter in the disk of the galaxy. Left panel shows the problem: lets say we're at the bottom near the plane of the disk and we look up out of the disk and measure distances and vertical component of velocities of stars (they're not assumed to be moving purely up and down as the figure might suggest), and from this we measure the profile of number density and z-component of the velocity dispersion. The key thing to realise here is that both the red (redshifted - as they're moving away from us) and blue stars are transporting +ve z-momentum in the +ve z-direction. The flux density of z-momentum across a plane of constant z is $\mathcal{F} = mn(z)\sigma_z^2(z)$. But both n and σ_z^2 are decreasing with height, so there's more momentum flowing into the slab at the bottom than out of it at the top. As the disk may be assumed to be in a static state, the rate of change of momentum density $\mathcal{P}$ in the disk – (minus) the gradient of the momentum flux density – must be balanced by the rate at which the stars in the slab are losing momentum because of the gravitational acceleration mng, so we can solve for $g = n^{-1}d(n\sigma^2)/dz$. It looks like we made the sweeping – and unreasonable – assumption that all stars have the same mass. But the mass cancels out.

2.2 The 1-dimensional Jean's equation

Let's assume we look up out of the plane and make observations of vertical distances z and line-of-sight velocities v for some population of stars

- And from this we determine $n(z)$ and $\sigma^2(z) \equiv \langle v^2 \rangle$
  - note that this will, in general, depend on type of stars
  - thick disk (older) stars have bigger scale height and velocity dispersion

Consider a slab at height z and thickness Δz.

- and let's assume – for the moment – that all the stars have the same mass
- What is the flux of vertical momentum through the lower surface?
  - stars with velocity v carry vertical momentum $p = mv$
  - and flux (number per unit area per unit time) is vdn where dn is the number density of stars with velocity $v \to v + dv$.
  - so $dp/dtdAdv = mv^2 dn/dv$
  - and taking sum over all velocities and defining $\sigma^2 = \sum dnv^2 / \sum dn$ gives
  - $dp/dtdA = mn\sigma^2$
    * momentum/time/area = force / area = pressure

- if $n\sigma^2$ is decreasing with height – as, in fact, it is – there is more kinetic momentum flowing into the bottom face than out of the top
- the rate at which momentum is accumulating (per unit area per unit time is) $\delta(nm\sigma^2) = m\Delta z \times \partial(n\sigma^2)/\partial z$
- if the disk is in a steady state this must be balanced by the rate at which momentum is being removed (per unit time per unit volume) by gravity: $dp/dV dt = gnm$
- this gives the 1D time independent Jeans's equation
  - $\boxed{\partial(n\sigma^2)/\partial z = ng}$.
- so differentiating the observable $n\sigma^2$ w.r.t. height and dividing by n gives the vertical component of the *gravity* g.
- and differentiating this gives $\partial^2\phi/\partial z^2 = \nabla^2\phi$ which in Poisson's equation gives 4π times the mass density.
- Q:
  - $mn\sigma^2$ is the rate at which stars are transporting z-momentum in the z-direction.
  - It is one component – the zz component – of the *kinetic pressure tensor* $P_{ij} = mn\langle v_i v_j\rangle$
    * which is also called the *'stress tensor'*
    * ij^{th} component is the rate at which stars carry i-momentum in the j^{th} direction
  - its *divergence* $\sum_i \partial/\partial x_i P_{ij}$ gives the rate at which particles are transporting j-momentum out of a volume
  - in a steady state this gets balanced by the momentum being removed by the gravitational field
  - this suggests that one should be able to define a stress tensor T_{ij} for the gravitational field
    * giving the *gravitational field momentum flux density*
    * and whose divergence is $\sum_i \partial/\partial x_i T_{ij} = nmg_j$
  - what would its components be? (Hint: think about magnetic or electric fields)
  - apply this to the Earth. Is there a field momentum flux density where you are standing? Which way is z-momentum flowing? How big is it (in tons per square inch)? How does it compared to e.g. the momentum flux density in the air. Where is it going? (work out how the flux through a spherical surface falls off with height). Does any of this worry you?

Some interesting features of Jeans's equation:

- It is reminiscent of hydrostatic equilibrium $dp/dr = -g\rho$ – i.e. pressure gradient equals gravity times density
  - where p and ρ are the physical pressure and density
- now it looks like we made a strong – and almost certainly erroneous – assumption that all of the stars have the same mass m
  - but, on closer scrutiny, we never actually made use of that assumption
  - the actual mass of the stars in question does not appear in the final result
  - and the result above is valid even if we are dealing with a very heterogeneous sample of stars
- what's going on is that if we have a very large number of stars – as we do – then Jeans's equation applies for each type of star – and then it equally applies in an average sense if we sum both left and right hand sides over different types
  - whether or not we 'weight' the stars by their mass in this sum
- moreover, using different *'tracers'*

  - e.g. old vs. young stars
  - which might not just have different masses, but will have different density and velocity dispersion profiles

- should give the *same* gravitational potential and hence density.
  - and we can average the results from the different tracers to get the best estimate

- the fundamental reason for this is that we are assuming that there is no coupling between the different tracers through collisions
  - if on the other hand collisions *were* important – as in a ionized plasma, for instance – each component would have a Maxwellian distribution characterised by the temperature, and with partial pressure proportional to the number of particles, while the gravity would act preferentially on the more massive particles – the ions rather than the electrons – and the collisions would transfer momentum from the ions to the electrons

- without collisions we cannot assert that the velocity distribution is Maxwellian – and in general it won't be – but it is not necessary.

- also note that we have not assumed anywhere that the mass be distributed like the stars
  - as is assumed in the *virial theorem*
  - this is a great strength of the Jeans equation

2.3 Implications of Jeans equation for the mass density in the disk

Returning to the time-independent 1D equation, we can use the observed vertical density and velocity dispersion and tracer density to determine the gravity as a function of height above the plane of the disk and, as discussed, the mass density $\rho(z)$.

- Modern estimates of the density in the plane give $\rho(0) \simeq 6 \times 10^{-2} M_\odot/\mathrm{pc}^3$
  - Different tracers have very different scale heights but also have different velocity dispersions, and the mass density estimates from different tracers are – reassuringly – consistent with one another.
  - Integrating the 3D density $\rho(z)$ gives the *surface mass density* Σ. The result, which has to be truncated at some radius as the data give out, is known as the *Oort limit* after Jan Oort, who first did this in the 30's. Modern values are $\Sigma \simeq 75 M_\odot/\mathrm{pc}^2$

- It is then obviously interesting to compare the space mass density with the mass of 'visible stuff' (mostly stars, but also gas, dust etc.).
  - This is a complex, and difficult, accounting exercise.
  - In the 80's there were two somewhat conflicting analyses:
  - John Bahcall at Princeton found that the visible stuff fell short of the density from Jeans equation by about a factor 2.
    * so this indicated considerable dark matter in the disk
  - while Gerry Gilmore at Cambridge found that the dynamical and visible mass estimates were the same

- The currently accepted view is that the amount of dark matter in our vicinity is not more than about 15% of the visible mass density.

3 Dark matter from the rotation curve of the Milky Way

- Observations of HI flux density as a function of angle and frequency – with frequency shifts interpreted as the Doppler effect – indicate that the rotation curve of our galaxy is, beyond a few kpc at least, very flat.
  - recall that local measurements give Oort's constants A and B that together give the local rotation rate Ω_0 and its rate of change with radius.
  - this indicates that the velocity is roughly independent of radius
    * being about 220 km/s.
  - that's only a local measurement, but it can be extended
    * it is particularly simple when looking at distribution of line of sight velocity along directions that probe material interior to our orbit
    * since the maximum velocity – an easy to identify cusp in the data – is comes from the material whose radius vector is perpendicular to the line of sight
  - all this assumes that the disk is circular – which may be somewhat worrying given that many spiral galaxies contain bars in the centre, and spiral arms further out, but
    * a bar would only affect the centre – not the outer parts where the evidence for large amounts of DM resides
    * non-circular motions associated with spiral arms appear to be quite modest, and
    * similar results are found for other spiral galaxies.
  - so the evidence for DM in extended halos of spiral galaxies is insurmountable
- this does assume, however, that Newton's law of gravitation applies

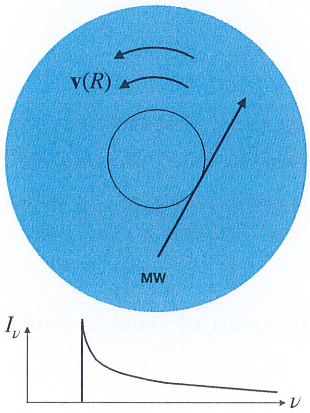

Figure 5: If we observe neutral hydrogen emission in the plane of the disk along lines with modest angle from the galactic centre then we can readily measure the rotation velocity at the 'tangent point' from the brightness as a function of frequency. This is because the line-of-sight velocity only changes quadratically with distance x from the tangent point $v = v_0 + \alpha x^2$. While the amount of gas grows linearly with $x = \sqrt{\Delta v/\alpha}$. So the integrated brightness $I(< \Delta\nu) = \int_{\nu_0}^{\nu_0+\Delta\nu} d\nu I_\nu$ out to $\Delta\nu$ grows like x which is proportional to $\sqrt{\Delta v}$ or equivalently proportional to $\sqrt{\Delta\nu}$. So $I_\nu = dI(< \Delta\nu)/d\Delta\nu \propto 1/\sqrt{\Delta v}$, which is a *cusp*.

- This is very different from what would be expected if the mass were distributed like the visible stars.
  - The mass interior to r is nicely convergent for an exponential disk model
    * this would suggest a 'Keplerian' rotation curve with $v \propto 1/\sqrt{r}$
  - the mass of stars in the stellar halo is mildly – i.e. logarithmically – divergent
    * but the total mass in the halo is not large and including this still predicts a falling rotation curve
- While the inner parts of the MW – within a few kpc of the centre – are dominated by the mass in stars, the conclusion is inescapable that – *if Newtonian gravity is an accurate model* – at large radius the total mass distribution has a profile $\rho(r) \sim r^{-2}$.
  - This is much flatter than any of the visible components.
  - and, when integrated, gives a *divergent* total mass

- so the flat rotation curve halo has to be truncated at some radius

- The MW halo extends at least to $r \sim 100$kpc.
- *The mass of the MW is dominated by a massive extended halo composed of dark matter.* (DM)
  - Similar results are found from rotation curves of other spiral galaxies
  - Another way to estimate the total mass of the Milky Way is to apply the so-called 'timing argument' to the local group (the Milky Way plus the Andromeda galaxy) which is just 'turning around' (see below).
    - this allows us to cast the net wider and probe larger radii
  - and one of the particularly strong early indications of DM on these scales came from the high relative velocities of pairs of galaxies
    - in particular a sample of 'binary galaxy' systems with redshifts measured by Ed Turner
    - with pioneering analysis by Bernard Jones and also by Amos Yahil and Jerry Ostriker
  - Gravitational lensing also confirms that, in an average sense, galaxies are surrounded by DM haloes with $\rho_{\rm DM} \sim r^{-2}$.
  - Various other dynamical estimates – such as in clusters of galaxies – suggest that there is even more DM at larger radius.

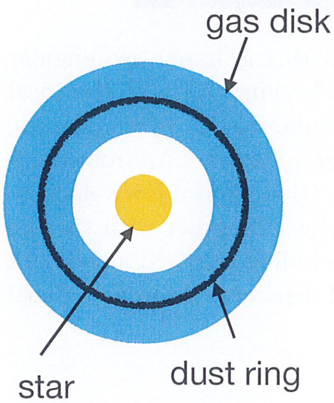

Figure 6: Q: Here's a nice problem involving disks. People studying exoplanetary systems often see stars with broad gaseous disks (observed in molecular lines with ALMA for example). They also sometimes see IR emission – interpreted as emission from dust – that is on a very thin ring within the disk. Can you explain this? Hint: the gas disk has pressure. How does that affect its rotation velocity? Will there be a radius at which the gas is moving at Keplerian speed? What does the velocity difference (disk vs. Keplerian) do near that radius? How might that affect dust grains? As a corollary, the random motions of stars in the MW disk are a kinetic pressure. How does this affect the stellar rotation curve. Is this an important effect for the Milky Way?

4 Mass of Milky Way from the Local Group 'timing argument'

The Milky Way (MW) and its partner Andromeda (M31) dominate the luminosity of the Local Group (LG). MW and M31 have a current separation of about 800 kpc and are approaching each other with a relative speed of about 100 km/s, while proper motions indicate that the transverse motions are much smaller. This motivates a simple model of the LG as two equal point masses $M_{\rm MW} = M_{\rm M31} = M$ on radial orbits that started at the same point and reached a maximum separation $d_{\rm max} \simeq 1.0$Mpc (slightly greater than the current separation) after a time $t_{\rm max} \simeq 10^{10}$ yr (slightly less than the current age of the universe) and are now falling back together for the first time.

Let's use this to estimate the mass M of MW and M31.

- We start with the equation of motion for the half-separation $r = d(t)/2$ (or, equivalently the distance of either galaxy from the centre of mass of the LG) in terms of r and M:
  - $\ddot{r} = -GM/4r^2$
- Next, we show that a parametric solution is given by
  - $r(\eta) = A(1 - \cos\eta)$
  - and

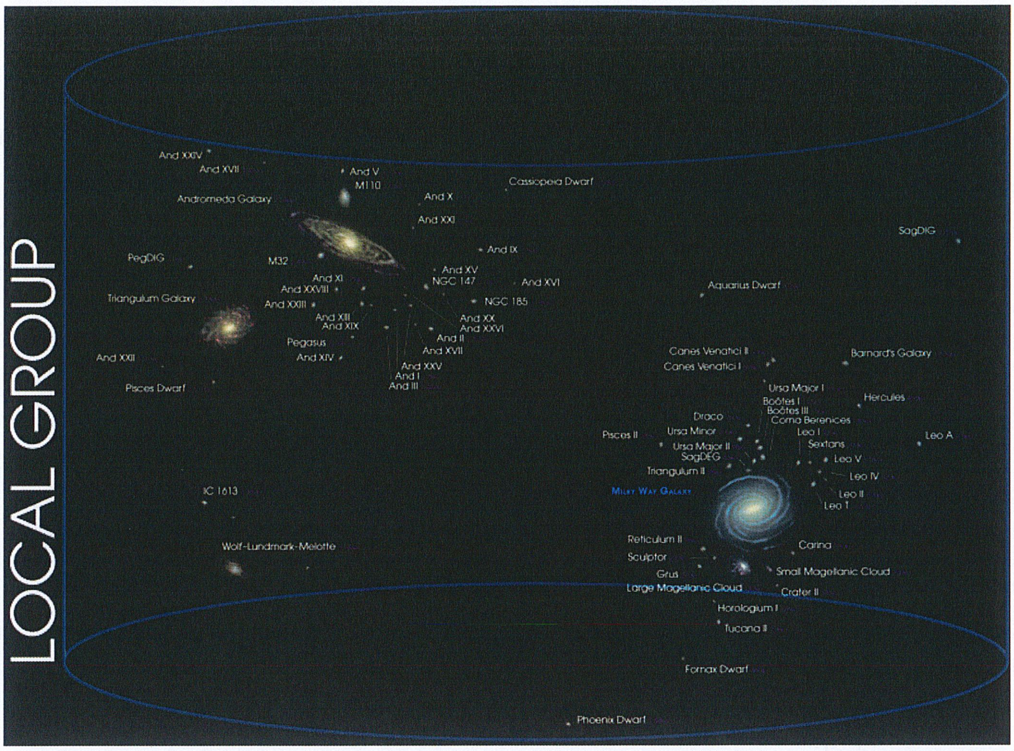

Figure 7: Map of the galaxies in the Local Group (LG). The data on which this is based are angular positions and distances obtained from variable stars and other techniques. The luminosity of the local group is dominated by the MW and Andromeda. There are, in addition, a large number of dwarf galaxies, but, as shown, these are to a large degree concentrated around the two large galaxies. Andromeda is essentially unique among galaxies in having a *blue*-shift; it is moving towards us (though at a low velocity). Observations also indicate that the LG has very little rotation. These observations, together with the age of the universe, allow one to estimate how massive these galaxies have to be – including their extended dark halos – in order for the local group to have just recently 'turned around' and separated from the global expansion.

  - $t(\eta) = B(\eta - \sin\eta)$
  - where A and B are constants.
  - in cosmology η is known as 'conformal time'

- to do this:
  - we let prime denote derivative wrt η so $r' = dr/d\eta$ etc. we then have
  - $\dot{r} = dr/dt = r'/t' = A\sin\eta/B(1 - \cos\eta)$
  - and
  - $\ddot{r} = (\dot{r})'/t' = -A/[B^2(1 - \cos\eta)^2] = -A^3/B^2r^2$
  - so
  - $\ddot{r} = -GM/4r^2$
  - provided the constants A an B satisfy
  - $A^3/B^2 = GM/4$.
  - note that this is like Kepler's law – with $r \sim A$ and $t \sim B$ this says $t^2 \propto r^3$.

- Next we want to relate the constants A and B to $d_{\max}$ and $t_{\max}$.
  - from $r(\eta)$ we have $r_{\max} = 2A$, while $r_{\max} = d_{\max}/2$
  - and from $t(\eta)$ we have $t_{\max} = \pi B$, so
    - $A = d_{\max}/4$

    * and
    * $B = t_{\rm max}/\pi$

- This allows us to obtain M in terms of $d_{\rm max}$ and $t_{\rm max}$:
  - $M = \pi^2 d_{\rm max}^3/16Gt_{\rm max}^2 \simeq 1.38 \times 10^{12} M_\odot$
- Q: How does this result compare with the mass profile of the MW halo $M(r)$ that one would obtain assuming a spherical dark-matter dominated galaxy with rotation velocity $v_{\rm rot} = 220{\rm km/s}$? $M(r) = v^2 r/G \simeq 10^{11} M_\odot (r/10{\rm kpc})$.
- Q: To what radius would such a flat rotation curve halo need to extend to give a mass equal to that obtained from the kinematics of the LG? $r \simeq 140{\rm kpc}$.

5 Peebles's dynamical analysis of the Local Group using the principle of least action

- The model of the LG in the 'timing argument' above is extremely crude
- Figure 7 shows a number of 'satellite galaxies' in addition to the MW and M31
- Jim Peebles realised in 1989 that there is an elegant method to extend the timing argument to model these as well using the *principle of least action*:
  - if we have a collection of point masses m_i
  - then the trajectories $\mathbf{x}_i(t)$ are those that extremise the *action* $S[\mathbf{x}_i(t)]$ – a *functional* of the paths:
  - $\boxed{S[\mathbf{x}_i(t)] = \int dt L(\mathbf{x}_i, \dot{\mathbf{x}}_i)}$
  - where $L(\mathbf{x}_i, \dot{\mathbf{x}}_i)$ is the *Lagrangian*
  - $L(\mathbf{x}_i, \dot{\mathbf{x}}_i) = {\rm KE} - {\rm PE} = \frac{1}{2}\sum_i m_i \dot{\mathbf{x}}_i^2 + \frac{1}{2}\sum_i \sum_j G m_i m_j / |\mathbf{x}_i - \mathbf{x}_j|$
  - it is easy to show that the condition that $\delta S = 0$
    * given some initial and final positions $\mathbf{x}_i(t_{\rm i})$ and $\mathbf{x}_i(t_{\rm f})$
  - gives Newton's equations of motion:
    * since
    * $\delta S = \int dt \sum_i \delta\mathbf{x}_i(t) \cdot [d/dt(\partial L/\partial \dot{\mathbf{x}}_i) - \partial L/\partial \mathbf{x}_i]$
    * so vanishing of δS for an arbitrary variation $\delta \mathbf{x}_i(t)$ requires $[\ldots] = 0$, or
    * $m_i \ddot{\mathbf{x}}_i = -\boldsymbol{\nabla}_{\mathbf{x}_i} V = \boldsymbol{\nabla}_{\mathbf{x}_i} \sum_j G m_i m_j / |\mathbf{x}_i - \mathbf{x}_j|$
- rather than solve these equations of motion what Peebles did was to solve numerically for a set of particle trajectories $\{\mathbf{x}_i(t)\}$ that minimised the action subject to the measured current positions and, in essence, assuming that the particles all started at $\mathbf{x} = 0$ at $t = 0$ (about 10 billion years ago)
  - it is a little more complicated than that
  - he worked in so-called 'comoving coordinates' $\mathbf{r} \equiv \mathbf{x}/a(t)$ where $a(t)$ is the 'expansion factor of the universe'
    * though that is of no real significance as it is just a 'book-keeping' exercise
  - and actually assumed for the initial conditions that the particles were initially stationary in $\mathbf{r}$-coordinates
- for the masses of the particles m_i, he assumed, quite reasonably, that they were proportional to the luminosities of the galaxies L_i
  - but with an overall scaling factor Γ: so $m_i = \Gamma L_i$
- that gives a solution for *any* assumed value of the *'mass-to-light ratio'* Γ

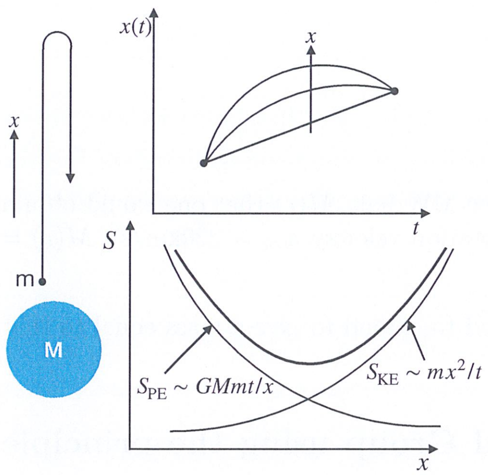

Figure 8: Schematic illustration of least action applied to orbits in a gravitational field. Consider a particle of mass m launched from a body of mass M. The principle of least action says that, of all the possible paths from the specified beginning and end positions (and times), the one that nature chooses is the one that extremises the action $S = \int dt(\mathrm{KE} - \mathrm{PE})$. If the mass of the body M were zero then $\mathrm{PE} = 0$ and the path chosen is a straight line. But if $M \neq 0$ the potential term will be non-vanishing, and the action can be made smaller than the straight-line value by rising more rapidly at the beginning – at the cost of increasing S_{KE} – so as to decrease S_{PE} (middle curve). But that only works up to a point since for too large a 'detour' the kinetic term dominates. The 'sweet spot', clearly, is when $S_{\mathrm{KE}} \sim S_{\mathrm{PE}}$ which implies $GM/x^3 \sim 1/t^2$ – or that the 'dynamical time' be on the order of $1/\sqrt{G\rho}$.

- but requiring that the solution give the correct relative approach speed for M31 fixes Γ
  - and hence the masses of all the galaxies (subject to the $m \propto L$ scaling)
- and predicts the relative approach or recession speeds of all of the other galaxies
  - which serves as a test of the method and the assumptions

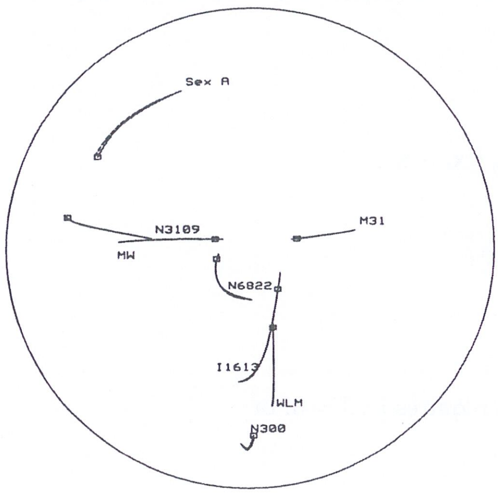

Figure 9: Jim Peebles' solution for the trajectories of the galaxies in the Local Group – the MW, M31 and their 6 brightest companions – found by requiring that they minimize the action. The title of his paper was *Tracing Galaxy Orbits Back in Time*. The lines show the orbits (with boxes showing the present positions) in 're-scaled' coordinates $\mathbf{r}(t) = \mathbf{x}(t)/a(t)$, where $a(t)$ is the expansion factor for the rest of the universe. So this is like what we would see if we were moving away with the overall expansion. The mass-to-light ratio in the model is fixed by requiring that M31 be moving towards us at the observed speed. The mass of M31 (Andromeda), which contains= about 70% of the total luminosity, comes out to be $m_{\mathrm{M31}} \simeq 3 \times 10^{12} M_\odot$ and that of the MW is $m_{\mathrm{M31}} \simeq 1.8 \times 10^{12} M_\odot$. The model does a good, but not perfect, job of predicting the recession and/or approach speeds of the other galaxies.

- the orbit solutions are shown in figure 9
  - where the fact that the MW and M31 have move in towards the centre in the 're-scaled' coordinates simply indicates that the LG has not expanded as the universe as a whole
- the masses obtained from this modelling are in broad agreement with the simple estimates from the timing argument
  - and the accord with other velocities supports the idea that the LG came into being by the process of *'gravitational instability'*
  - it having been, initially, slightly overdense as compared to the mean density of the universe
  - so its expansion was slowed and it has now turned around and 'de-coupled' from the universal expansion
- a weak point of the model is that the mass is assumed to be concentrated in points, whereas in reality the halos are extended.

- another concern (noted in Peebles's paper) is that, in general, there is no unique solution
  - the principle only states that the action is extremised
  - there are, in general, multiple solutions
  - there is a unique *global minimum* of the action, but it is not at all clear that this is the correct solution in general
  - consider for example if we were to apply this to the motion of the Earth around the Sun, with the end position being our current location and the initial position say 10 years ago (i.e. the same position)
  - what would be the orbit with the absolute minimum action?
- it would be dangerous to apply this modelling using motions of objects that have had multiple passes
- but for the LG – which has only just turned around – it is probably not a big issue

6 What is the dark matter?

- The *composition* of the DM is unknown – this is one of the major outstanding questions in physics.
  - another is the nature of the 'dark energy' – we'll address that later
- One possibility is that the DM is composed entirely, or in part, of dark stellar remnants
  - sometimes known as 'massive astrophysical compact halo objects' or MACHOS
  - but these would cause 'micro-lensing' events – transient enhancements of the flux density of background stars
  - searches have put strong upper limits on the fraction of DM that is in objects of $\sim$ solar mass
  - another constraint on more massive MACHOS comes from weakly gravitationally bound binary star systems. This will be disrupted by massive MACHOS and this rules out MACHOS of mass more than a few tens of solar masses
  - so it may be that *some* of the DM might be e.g. 'primordial' black-holes (PBHS), but Occam's razor makes that unattractive
  - NEED PLOT OF UPPER LIMITS
- Another clue is that the DM does not seem to be enhanced in the disk
  - this lends further credence to the idea that the DM is quite distinct from the normal matter we see in stars
- Another important result is that the total density in DM is about 30% of the so-called 'critical density' calculated from the rate at which the Universe is expanding
- While 'Big-bang nucleosynthesis' (BBN) calculations can only match the observed abundance of light elements in un-evolved stars if the density of 'baryons' – i.e. protons and neutrons – is only about 5% of critical.
- These lead one to suspect that the DM is 'non-baryonic'. The hypothesis favoured by many physicists is that the DM is a relic weakly interacting massive particle WIMP from the big-bang.
  - the idea here is that there is some weakly interacting particle with mass $\sim$ 100GeV as are predicted in e.g. super-symmetric extensions of the standard model
  - if so the relic abundance would be in the right general order of magnitude – this is called the 'WIMP-miracle'
  - WIMP DM might be detectable in 3 ways:
    - creating in particle accelerators
    - direct detection of galactic halo WIMPS by nuclear recoil

    * indirect detection via annihilation

  - but attempts to directly detect these particles have so far not been successful

- another quite well motivated candidate for the DM is that it is the 'axion'; a scalar field – whose bosonic excitations have mass on the order of maybe 10^{-5}eV – and which is invoked to explain why CP violation in the standard model is weak
  - one can determine, given the size and internal velocities of galaxies or clusters that the phase-space density of the DM must be $f \sim n/p^3 \sim n/(m^3v^3)$ but $G\rho \sim (v/R)^2$ and $\rho = mn$ so $f \sim G^{-1}R^{-2}v^{-1}m^{-4}$. But for fermions, f cannot exceed $\hbar^{-3}$. That gives a lower bound on the mass of fermionic DM of on the order of 10eV, (the 'Tremaine and Gunn' bound) so particles of mass as low as the axion would have to be bosons to evade this limit (bosons can have much larger occupation numbers)
  - such fields/particles interact electromagnetically, albeit weakly, and there are various experiments trying to directly detect them

- less well motivated, but still interesting, are variations on the axionic DM theme such as the 'ultra-light axion-like DM' or so-called 'fuzzy' DM with $mc^2 \sim 10^{-22}$eV (this mass being chosen so that the de Broglie scale for such fields in galaxies would be astronomically relevant).

A The general Jeans (or Euler) equation

The form of Jeans's equation used above was quite simple as

- it was 1-dimensional
- it assumed that there is no net streaming motion – so it assumed the distribution of stars in the disk is 'static' and all the pressure providing the support is 'kinetic'

More generally, we need consider motions in 3-dimensions and we would also want to allow for the possibility of motions over and above random kinetic motions

- we would certainly want to allow for the possibility of 'streaming' motions – such as galactic rotation, for instance – in systems that are 'stationary' (but not static)
- or, more generally, for systems that are not in equilibrium

If there is *not* balance between the increase in momentum density from kinetic motion and that from gravity the matter in the shell or slab will have a changing momentum.

- Jeans's equation is then $\partial_t(n\langle v\rangle) + \partial_z(n\langle v^2\rangle) = ng$.
- where we have defined $\partial_t \equiv \partial/\partial t$ and $\partial_z \equiv \partial/\partial z$

This is sometimes written in a slightly different way by writing $\langle v\rangle \equiv \overline{v}$ – the *mean streaming velocity* – and defining the *peculiar velocity* to be $u \equiv v - \overline{v}$ (which implies $\langle u\rangle = 0$) and defining the velocity dispersion to be the dispersion relative to the mean flow: $\sigma^2(z) = \langle u^2\rangle$ (rather than as $\sigma^2(z) \equiv \langle v^2\rangle$).

- With those definitions, Jeans equation becomes
  - $\partial_t n\overline{v} + \partial_z n(\overline{v}^2 + \sigma^2) = ng$
- or, using the chain rule,
  - $n\partial_t\overline{v} + \overline{v}\partial_t n + \overline{v}\partial_z n\overline{v} + n\overline{v}\partial_z\overline{v} + \partial_z n\sigma^2 = ng$
- but the 2nd and 3rd terms vanish by virtue of the *equation of conservation of number density of particles:*
  - $\partial_t n + \partial_z n\overline{v} = 0$
- while the 1st and 4th terms are n times $(\partial_t + \overline{v}\partial_z)\overline{v}$

- the differential operator here being the *convective time derivative*
- i.e. the rate of change with time d/dt as seen by an observer moving with velocity $\overline{v}$
- so $(\partial_t + \overline{v}\partial_z)\overline{v} = d\overline{v}/dt$ is the rate at which the local streaming velocity is changing as seen by an observer who happens to have $u = 0$

- Hence, on dividing through by n, Jeans equation becomes
  - $\boxed{d\overline{v}/dt = -n^{-1}\partial_z(n\sigma^2) + g}$
- The 3-dimensional generalisation of this is fairly straightforward:
  - the equation of particle conservation is
  - $\boxed{\partial_t n + \boldsymbol{\nabla} \cdot n\langle \mathbf{v} \rangle = 0}$
  - and Jeans (or Euler's) equation is
  - $\boxed{d\overline{\mathbf{v}}/dt = -n^{-1}\boldsymbol{\nabla} \cdot (n\langle \mathbf{uu} \rangle) + \mathbf{g}}$
  - where $n\langle \mathbf{uu} \rangle$ is the 'pressure tensor' – really the pressure divided by the mass of the tracer particle – and $\langle \mathbf{uu} \rangle$ is the velocity dispersion tensor
  - Euler's equation says that if you want to move in such a way as to 'go with the flow' (i.e. move so that the particles around you never have any net relative streaming motion) then you need to have, in addition to the gravitational acceleration, an extra non-gravitational acceleration equal to the '(minus) pressure gradient' term.
- The time dependent Jeans equation could be used, for instance, to determine the gravity (and hence, by taking its divergence, the mass density) in a model with some prescribed streaming velocity field, velocity dispersion tensor and tracer density.
- Alternatively, people sometimes talk about 'solving' Jeans equation to get the velocity flow and dispersion compatible with some observed tracer and mass density field.