M1 Cosmology - 2 - Special Relativity Refresher Course

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Contents

1 Introduction

We will very quickly review some elements of special relativity (SR), with the goal of setting out the notation we will use and giving emphasis to those aspects that are essential in cosmology.

2 Principles of SR and their implications

2.1 The principles

SR is based on two principles. The first is that *there is no absolute frame for $\boxed{\text{linear}}$ motions* (though there is an absolute sense of rotation). This carries over from Newtonian dynamics. The second is that *the speed of light is independent of the state of motion of the observer.*

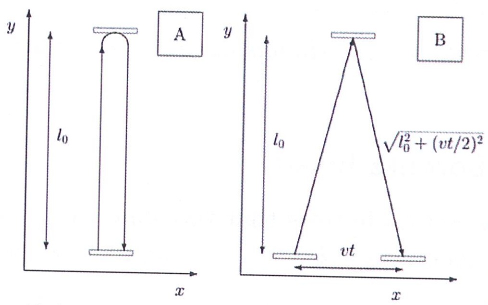
2.2 The implications

As illustrated in figures 1 and $\boxed{3.4}$ these imply that *moving clocks run slow*, and moving rulers are *length contracted*.

(运动的) 尺缩_钟慢.

time-dilation

- moving clocks run slow by a factor $\gamma \equiv 1/\sqrt{1-v^2}$
  - where v is speed ir units of c (or if $c = 1$)
- most easily seen by considering a "light-clock" moving transverse to its length
  - principle 1 says observers have to agree on transverse lengths



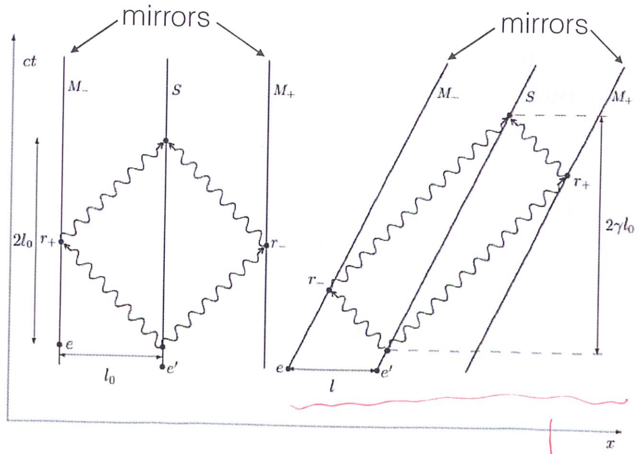
If the clock A is moving relative to me (B) I'll see that the photon has to travel a longer path

So in my frame there is a longer time interval between "ticks"

Figure 1: The time-dilation formula – which says that *moving clocks run slow* can be derived by considering a simple 'light-clock'. From B's perspective, the light has to travel further in making the round-trip, so constancy of c implies a longer elapsed time.

length contraction

- Considering a similar clock but moving parallel to its length shows that it's length must be less in the moving frame by a factor $1/\gamma$



You can derive: $l = \frac{l_0}{\gamma}$

Figure 2: Relativistic length contraction can be derived by considering an identical clock to that used to derive time-dilation, but not moving *parallel* to its length. Here we show the *world-lines* of photons in a *space-time diagram*. In fact we are illustrating a slightly more elaborage clock consisting to a pair of clocks lying end to end. On the left, the wiggly lines are the photon paths in the 'clock-frame' and on the right in the 'lab-frame' (the frame in which the clocks are moving). The solid lines are the world lines of the centre and reflecting ends of the clock; these are vertical in the clock-frame but slope to the right in the lab-frame. Note that principle (2) tells us that the wiggly lines are at 45 degrees to the coordinate axes in both frames.

The *Lorentz factor* is

$$\boxed{\gamma \equiv 1/\sqrt{1 - v^2/c^2}.} \tag{1}$$

A further implication, which is evident from figure 3.4 is that there is *no absolute sense of simultaneity.*

3 Relativistic 4-vectors

3.1 Inertial reference frames

An *inertial reference frame* can be realised as a non-rotating lattice of rulers (to measure spatial coordinates of *events*) carrying clocks (to measure times of said events) and attached to a *non-accelerating observer.*

These frames form a 6-parameter family as they can be rotated or *boosted* with respect to each other.

3.2 The displacement 4-vector $\vec{dx}$

Given two neighbouring *events* with time and space-separations dt and $d\mathbf{x} = \{dx^1, dx^2, dx^3\}$ in some observer O's frame we define the *displacement 4-vector* $\vec{dx}$ to have components $\{dx^\alpha\} = (cdt, dx^1, dx^2, dx^3)$, and we write

$$\boxed{\vec{dx} \underset{\mathrm{O}}{\longrightarrow} dx^\alpha} \tag{2}$$

in order to reinforce the understanding that $\vec{dx}$ is a *frame-independent entity*, whose components depend on the chosen frame of reference.

3.3 Transformation of $\vec{dx}$ under a 'Lorentz boost'

The linearity of time-dilation and length contraction implies that the *same vector* $\vec{dx}$ will have components measured by another observer O', which we denote by $\{dx^{\alpha'}\}$, that are given by a *linear transformation:*

$$dx^{\alpha'} = \sum_\alpha \Lambda^{\alpha'}{}_\alpha dx^\alpha \tag{3}$$

Or, introducing the *Einstein summation convention*, which says that summation is implied in any expression with a pair of identical – or *dummy* – indices, one upstairs and one downstairs,

$$\boxed{dx^{\alpha'} = \Lambda^{\alpha'}{}_\alpha dx^\alpha.} \tag{4}$$

For an observer *boosted along the x^1-axis* at speed v, the components of the *Lorentz boost matrix* are

$$\Lambda^{\alpha'}{}_\alpha = \begin{bmatrix} \gamma & -\gamma v/c & & \\ -\gamma v/c & \gamma & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \tag{5}$$

where the blank entries are zero.

The upstairs (downstairs) index in the boost matrix is taken to label the rows (columns), so $dx^{\alpha'}$ is the column vector obtained by multiplying the above matrix on the right by the column vector with components dx^α.

3.4 The invariant space-time volume element

It is evident from figure that the effect of a Lorentz boost on the original square set of photon paths is to *shear* it: it is stretched by a factor $\sqrt{(c+v)/(c-v)}$ along the upper-right direction and compressed by a factor $\sqrt{(c-v)/(c+v)}$ along the upper-left direction in the $ct - x^1$ plane. The x^2 and x^3 coordinates are unaffected by the boost, so it follows that the *space-time volume element* is invariant under boosts.

逻辑？造了一个洛伦兹不变的度规，定义间隔，发现其不变性，
再用间隔的不变性，发现了度规的不变性

3.5 The invariant squared interval

We define

$$ds^2 \equiv -(dx^0)^2 + (dx^1)^2 + (dx^2)^2 + (dx^3)^2. \tag{6}$$

In the primed frame this is

$$(ds')^2 = -c^2(\gamma dt + \gamma(v/c)dx)^2 + (\gamma dx + \gamma(v/c)dt)^2 + dy^2 + dz^2 \tag{7}$$

multiplying the factors out and using the definition of the Lorentz γ-factor shows that

$$\boxed{(ds')^2 = ds^2} \tag{8}$$

so the squared interval is invariant with respect to Lorentz boosts.

This is similar to the squared length of a vector $dl^2 = dx^2 + dy^2 + dz^2$ in Euclidean space, but with the important difference that here the squared interval can be positive, negative or zero.

We describe 4-vectors as being *space-like, time-like* or *null* depending on whether they lie outside, inside or on the *light-cone*, as illustrated in figure 3.

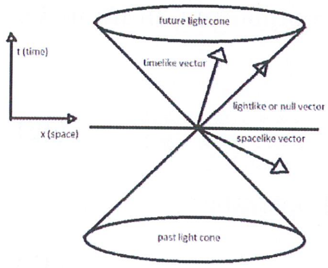


Figure 3: 4-vectors are classified as time-like, space-like or null depending on whether they lie inside, outside, or tangent to the so-called light cones. Time-like or null vectors may be future- or past-directed. Like the sense of rotation, the light-cones are an *absolute* property of *Minkowski space-time*, as is the classification of 4-vectors.

3.6 The Minkowski metric

The squared interval can be written in matrix notation as

$$ds^2 = [dx^0, dx^1, dx^2, dx^3] \begin{bmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} dx^0 \\ dx^1 \\ dx^2 \\ dx^3 \end{bmatrix} = [-dx^0, dx^1, dx^2, dx^3] \begin{bmatrix} dx^0 \\ dx^1 \\ dx^2 \\ dx^3 \end{bmatrix} \tag{9}$$

or, equivalently, as

$$\boxed{ds^2 = \eta_{\alpha\beta} dx^\alpha dx^\beta} \tag{10}$$

where the 4x4 matrix $\eta_{\alpha\beta}$ – known as the *Minkowski metric tensor* – has components

$$\boxed{\eta_{\alpha\beta} = \mathrm{diag}\{-1, 1, 1, 1\}.} \tag{11}$$

Replacing dx^α by $\Lambda^\alpha{}_{\alpha'} x^{\alpha'}$ we see that

$$ds^2 = (ds')^2 = \underbrace{\Lambda^\alpha{}_{\alpha'} \Lambda^\beta{}_{\beta'} \eta_{\alpha\beta}}_{\eta_{\alpha'\beta'}} dx^{\alpha'} dx^{\beta'} \tag{12}$$

so the Minkowski metric evidently transforms as $\eta_{\alpha'\beta'} = \Lambda^\alpha{}_{\alpha'} \Lambda^\beta{}_{\beta'} \eta_{\alpha\beta}$, with a Lorentz boost matrix multiplication for each index. This holds for any tensor; it is the defining property of tensors. But it is easy to show that, when applied to the Minkowski metric, this has no effect: *the components of the Minkowski metric are invariant under boosts.*

Because we are characterizing entities in different frames.

3.7 Covariant and contravariant vectors

The displacement $\vec{dx}$ is the *prototype* for all *contravariant vectors.* Other 4-component entities are contravariant 4-vectors if their components transform under boosts as do those of $\vec{dx}$.

The prototype for a covariant vector – also known as a 1-form – is the gradient of a (Lorentz scalar) function of position. A Lorentz scalar being something that does not change under a boost (an example being the readings on thermometers that measure the temperature of a gas or fluid).

We indicate the components of covariant vectors with downstairs indices, and we write, for the components of the gradient of a function $\phi(\vec{x})$

$$\phi_{,\alpha} \equiv \partial_\alpha \phi = \frac{\partial \phi}{\partial x^\alpha} \tag{13}$$

so

$$\phi_{,\alpha} = (c^{-1}\partial_t \phi, \partial_x \phi, \partial_y \phi, \partial_z \phi). \tag{14}$$

If $\phi(\vec{x})$ is a scalar function then the change $d\phi \equiv \phi(\vec{x} + \vec{dx}) - \phi(\vec{x})$ is also a scalar. Applying the chain rule gives

$$d\phi = dx^\alpha \phi_{,\alpha} \tag{15}$$

which is an example of a contraction of (the components of) a covariant and contravariant pair of vectors.

Analogous to the way we write $\vec{dx} \underset{\mathrm{O}}{\longrightarrow} dx^\alpha$ we can say that the $\{\phi_{,\alpha}\}$ are the components, in frame O, of a frame-independent entity (a 1-form) $\tilde{d}\phi$:

$$\tilde{d}\phi \underset{\mathrm{O}}{\longrightarrow} \partial_\alpha \phi \tag{16}$$

and abstracting away the particular scalar function we have the 1-form differential operator

$$\tilde{d} \underset{\mathrm{O}}{\longrightarrow} \partial_\alpha. \tag{17}$$

We will also often denote the scalar contraction above as the gradient 1-form acting on the displacement vector:

$$\boxed{d\phi = \tilde{d}\phi(\vec{dx}).} \tag{18}$$

It is natural to visualise the 1-form $\tilde{d}\phi$ as a small stack of iso-ϕ surfaces and the vector $\vec{dx}$ as a little arrow. The quantity $\tilde{d}\phi(\vec{dx})$ is equal to the number of surfaces pierced by the arrow.

3.8 Transformation of components of covariant vectors

We can write the invariant squared interval as the contraction

$$ds^2 = dx_\alpha dx^\alpha \tag{19}$$

where

$$dx_\alpha = \eta_{\alpha\beta} dx^\beta \tag{20}$$

where we say we have used the Minkowski metric as a *index lowering operator*, whose effect is simply to flip the sign of the time component.

The invariance of ds^2 implies that the covariant components dx_α transform inversely to the covariant components. The transformation law for the former is

$$\boxed{dx_{\alpha'} = dx_\alpha \Lambda^\alpha{}_{\alpha'}.} \tag{21}$$

So the matrix which effects the transformation of covariant from un-primed to primed frame is the same as that which transforms contravariant components from primed to un-primed, but with the primed and un-primed indices swapped.

That this is correct can be seen by calculating the invariant interval:

$$(ds')^2 = dx_{\alpha'} dx^{\alpha'} = dx_\beta \Lambda^\beta{}_{\alpha'} \Lambda^{\alpha'}{}_\gamma dx^\gamma \tag{22}$$

For this to be the same as $ds^2 = dx_\alpha dx^\alpha$, it must be that the product of matrices here is the identity matrix:

$$\Lambda^\beta{}_{\alpha'} \Lambda^{\alpha'}{}_\gamma = \delta^\beta_\gamma \tag{23}$$

which is true.

Note that in $dx_{\alpha'} = dx_\alpha \Lambda^\alpha{}_{\alpha'}$ the implied sum is over *rows*, whereas in $dx^{\alpha'} = \Lambda^{\alpha'}{}_\alpha dx^\alpha$ the sum is over columns. Thus, in matrix notation, the covariant transformation law gives a *row vector* obtained my multiplying $\Lambda^\alpha{}_{\alpha'}$ on the *left*:

$$\left[\ dx_{\alpha'}\ \right] = \left[\ dx_\alpha\ \right] \left[\ \Lambda^\alpha{}_{\alpha'}\ \right]. \tag{24}$$

3.9 The scalar product of two 4-vectors

The metric can also be used to define the scalar, or dot, product of 4-vectors $\vec{v} \cdot \vec{u}$ for two 4-vectors $\vec{v}$ and $\vec{u}$:

$$\vec{v} \cdot \vec{u} = \eta_{\alpha\beta} v^\alpha u^\beta = v^\alpha u_\alpha = v_\alpha u^\alpha \tag{25}$$

in which notation, the invariant interval is the scalar product of the interval with itself

$$ds^2 = \vec{dx} \cdot \vec{dx}. \tag{26}$$

3.10 The metric as a function

To add a little more formalism, one can think of the metric – whose job is to give the squared length of vectors and scalar products – as a function $\mathbf{g}(\ ,\ )$ which takes two vectors as arguments and returns a scalar. So, for example

$$ds^2 = \mathbf{g}(\vec{dx}, \vec{dx}). \tag{27}$$

If we drop one of the arguments and form $\mathbf{g}(\vec{dx},\ )$ then we have a function of one vector argument that returns a scalar. I.e. it is the 1-form

$$\tilde{dx} = \mathbf{g}(\vec{dx},\ ). \tag{28}$$

This formalism allows us to think about $\mathbf{g}(\ ,\ )$ as a frame-independent geometric entity. It is, of course, somehow determined by the components $\eta_{\alpha,\beta}$ of the Minkowski metric. These numbers may be extracted given by feeding $\mathbf{g}(\ ,\ )$ pairs of unit vectors (often called basis-vectors) that point along the coordinate axes

$$\eta_{\alpha,\beta} = \mathbf{g}(\vec{e}_\alpha, \vec{e}_\beta). \tag{29}$$

Thus, if we feed $\mathbf{g}(\ ,\ )$ two copies of $\vec{e}_0 \longrightarrow (1, 0, 0, 0)$ we get $\eta_{00} = -1$; if we feed it $\vec{e}_1 \longrightarrow (0, 1, 0, 0)$ we get $\eta_{11} = +1$ etc., and if we feed it two different basis vectors it returns zero.

4 The 4-velocity and 4-momentum

4.1 Parameterised particle paths

Consider a particle moving along a parameterised path in space-time $\vec{x}(\lambda)$ where λ is a parameter that increases monotonically along the path.

In some frame – the lab-frame O, for example – this path has coordinates $\vec{x}(\lambda) \underset{\mathrm{O}}{\longrightarrow} x^\alpha(\lambda)$.

A particularly useful parameterisation is to use the proper time τ as registered by a clock that the particle carries. This is an example of what is called, in GR, an affine parameterisation. This would not be well defined for e.g. a photon, for which the proper time is not defined, so we'll assume for now that we are dealing with a material particle.

4.2 The 4-velocity

4.2.1 The definition of the 4-velocity

In an interval of proper time $d\tau$ the position of the particle changes by $\vec{dx} = (d\vec{x}/d\tau)d\tau$, where the derivative is defined in the usual way as a limit:

$$\frac{d\vec{x}}{d\tau} = \lim_{\Delta\tau \to 0} \frac{\vec{x}(\tau + \Delta\tau) - \vec{x}(\tau)}{\Delta\tau}. \tag{30}$$

Since all observers must agree on the readings on the clocks, $d\tau$ is a Lorentz-scalar, while the numerator, being the difference of two 4-vectors is itself a 4-vector it follows that $d\vec{x}/d\tau$ is a 4-vector.

We call it the *4-velocity*, and it is often denoted by $\vec{U}$:

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$$\vec{U} \equiv \frac{d\vec{x}}{d\tau} \tag{31}$$

and its components in the frame O are

$$\vec{U} \underset{\mathrm{O}}{\longrightarrow} U^{\alpha} = \frac{dx^{\alpha}}{d\tau} \tag{32}$$

which transform under a Lorentz transformation as

$$U^{\alpha'} = \Lambda^{\alpha'}{}_{\alpha} U^{\alpha}. \tag{33}$$

The 4-velocity is also known as the *tangent vector* to the path $\vec{x}(\tau)$.

4.2.2 The components of the 4-velocity

At any point along the particle's path, we can make a boost into the *instantaneous rest-frame* of the particle O′ (this is also often called the *momentarily comoving reference frame* or MCRF).

In the frame O′, the change in space-time coordinates in proper time interval $d\tau$ is

$$dx^{\alpha'} = (c\,d\tau, 0, 0, 0) \tag{34}$$

so the components of the 4-velocity in the MCRF are

$$\vec{U} \underset{\mathrm{O}'}{\longrightarrow} U^{\alpha'} = (c, 0, 0, 0). \tag{35}$$

Boosting back to the frame O in which the particle has velocity $\mathbf{v}$ – let's call it the 'lab-frame' – we find

$$\boxed{\vec{U} \underset{\mathrm{O}}{\longrightarrow} U^{\alpha} = (\gamma c, \gamma \mathbf{v}).} \tag{36}$$

4.2.3 The norm of the 4-velocity

The *norm of the 4-velocity* is, like the norm of any 4-vector, frame independent. It is most readily computed in the instantaneous rest-frame, where we find

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$$\boxed{\vec{U} \cdot \vec{U} = \mathbf{g}(\vec{U}, \vec{U}) = \eta_{\alpha\beta} U^{\alpha} U^{\beta} = U_{\alpha} U^{\alpha} = -c^2.} \tag{37}$$

4.3 The 4-momentum

The mass of an object can be determined by boosting into the object's rest-frame and firing a sticky reference particle – whose mass can define our unit – at it and measuring how fast the composite particle moves (again in the object's initial rest-frame). The result m is the *inertial mass* in MCRF. It is frame-independent, and is called *proper mass*.

We define the *4-momentum of a particle* – often denoted by $\vec{p}$ – the to be its $\boxed{\text{proper mass}}$ m times its 4-velocity:

$$\boxed{\vec{p} \equiv m\vec{U}.} \tag{38}$$

Frame invariance of m means $\vec{p}$, like $\vec{U}$, is also a 4-vector. The components, in the lab-frame, of the 4-momentum of a particle with lab-frame 3-velocity $\mathbf{v}$ are

$$\boxed{p^{\alpha} = (\gamma m c, \gamma m \mathbf{v}).} \tag{39}$$

The norm of the 4-momentum us

$$\boxed{\vec{p} \cdot \vec{p} = -m^2 c^2} \tag{40}$$

4.4 The relativistic 3-momentum

The spatial part of the 4-momentum, which we will denote by $\mathbf{p}$, is

$$\boxed{\mathbf{p} = \gamma m \mathbf{v}} \tag{41}$$

which is γ times what you would have written down as the normal Newtonian momentum[1]

We will refer to $\mathbf{p}$ as the *relativistic 3-momentum*, or just the (3-)momentum. The reason that we call $\mathbf{p}$ *defined* in this way *the* momentum is, as we shall now show, that it is the quantity that is conserved in collisions

4.4.1 Conservation of 3-momentum

To see why it is the relativistic $\mathbf{p} = \gamma m \mathbf{v}$ that is conserved (and not the *Newtonian momentum* $\mathbf{p} = m\mathbf{v}$) consider two parallel railway lines with separation $2D$ and 2 carriages travelling in opposite directions as illustrated in figure 4.

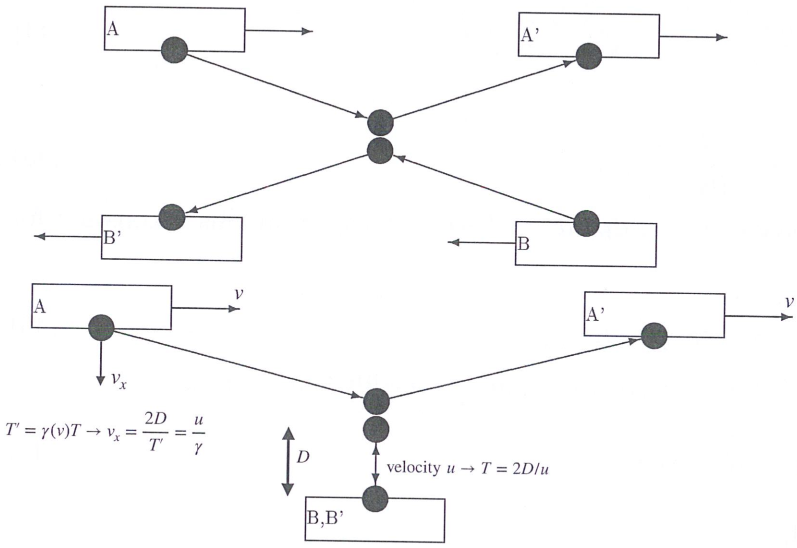

Figure 4: Two cricketers A and B pass each other on rapidly moving trains and as they do so they bounce balls off each other, exchanging momentum. The lower panel shows the situation from B's point of view. In B's frame there is a longer time interval between the pair of events A and A' than B, B'. But transverse distances are invariant so in B's frame A's ball has a lower transverse velocity than that of his own by a factor $1/\gamma(v)$ (where v – assumed to be much greater than u – is A's speed relative to B). Thus, in B's frame mu_x is not conserved, but $\gamma m u_x$ is conserved in the collision.

As they are about to pass each other, two cricketers on the trains throw identical balls (of proper mass m) out of the carriages, in a direction perpendicular to the carriage window frames with some (small) speed u in their frames.

The throws were carefully timed and targeted so that the balls, after bouncing off each other elastically, return to the cricketers entering the carriages again perpendicular to the window frames. So in the frame of a track-side observer all this is symmetrical as shown in the upper diagram.

But now look at this from the frame of one of the cricketers B (lower diagram). Note that both A and B agree on the distance between the trains, since this is perpendicular to their motion. B sees his ball move a distance $2D$ (out and back) in time $T = 2D/u$. And it's moving slowly, so he assigns it a momentum $p = mu$ and so the change of its (Newtonian) momentum when it recoils is $\Delta p_N = 2p = 2mu$

The out and back flight of A's ball takes the same time T in A's frame. But from B's perspective, the time for A's ball's return trip is time dilated: $T' = \gamma T$, where γ is the Lorentz factor for A (in B's frame), and which is also essentially the γ factor for A's ball in B's frame (since the transverse velocities are small compared to the relative motion of the trains).

So in B's frame the transverse component of A's ball's velocity is $v_x = 2D/T' = u/\gamma$. And so, when it bounces, the change $\Delta p_N = 2p = 2mv_x$ of Newtonian momentum of A's ball is *smaller* than that of his ball by a factor $1/\gamma$.

Evidently Newtonian momentum is *not* conserved in collisions. However, the change of the *relativistic* momentum of A's ball is $2m\gamma v_x$ which is the same as $2mu$ which is the same as the relativistic momentum of B's ball (in B's frame) since, by assumption, $u \ll 1$.

Note that we can say that the inertial mass of A's ball – that which we define operationally using of the change of velocity imparted by a given impulse of momentum (minus the change of B's ball's momentum) for this transversely moving mass, is enhanced to γm.

[1] Old books tended to write $\mathbf{p} = m\mathbf{v}$ with $m = \gamma m_0$ where m_0 denotes the proper- or rest-mass.

4.5 The 4-momentum as the 1-form $\tilde{d}S$

Equation (39) above gives the contravariant components of $\vec{p}$. But the 4-momentum is arguably more naturally considered to be a 1-form; the gradient $\tilde{d}S$ of the *action* S.

The action for a free particle is (postulated to be) essentially just the *proper time*:

$$\boxed{S = -mc^2 \int d\tau.} \tag{42}$$

Writing this, instead, in terms of *coordinate time* – in some particular frame – and using $d\tau = dt/\gamma$

$$S = -\int dt \frac{mc^2}{\gamma}. \tag{43}$$

So the *Lagrangian*, defined such that $S = \int dt L(\mathbf{x}, \mathbf{v}, t)$ – generally a function of position $\mathbf{x}$, velocity $\mathbf{v}$ and possibly time t, but here only a function of $\mathbf{v}$ – is

$$L(\mathbf{v}) = -\frac{mc^2}{\gamma(\mathbf{v})} = -mc^2\sqrt{1 - |\mathbf{v}|^2/c^2}. \tag{44}$$

The 3-momentum is

$$\mathbf{p} \equiv \frac{\partial L}{\partial \mathbf{v}} = \gamma m \mathbf{v}, \tag{45}$$

in agreement with (41), and the *Euler-Lagrange* equation $d\mathbf{p}/dt = \partial L/\partial \mathbf{x} = 0$ says that this is constant for a free-particle.

The *Hamiltonian* is

$$H \equiv \mathbf{v} \cdot \mathbf{p} - L \tag{46}$$

which, despite its appearance, is a function of $\mathbf{x}$, $\mathbf{p}$ and possibly t, and is readily shown to be

$$\boxed{H = \gamma mc^2} \tag{47}$$

or c times the time-component p^0 of $\vec{p}$ in (39).

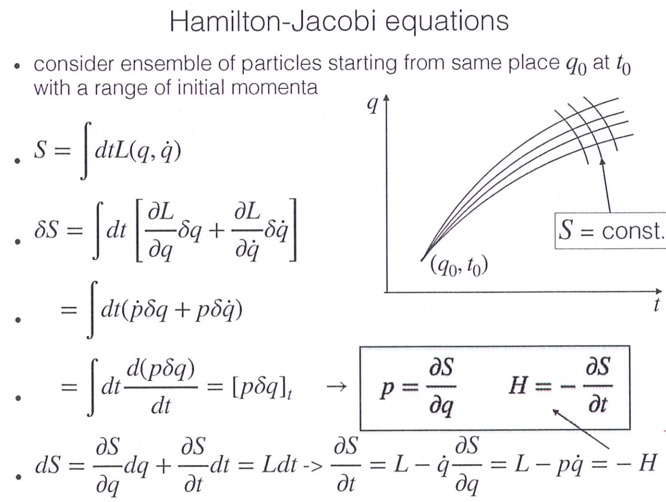

Figure 5: The H-J equations describe the variation of the action $S(q,t)$ for a family of trajectories that start at the same (q_0, t_0). The momentum is the rate of change of S with position and the energy is (minus) the rate of change of S with time. For a charged particle it is the 'canonical' momentum $\mathbf{P} = \mathbf{p} + q\mathbf{A}$ and energy (i.e. the Hamiltonian $H = E + q\varphi$) that appear here. Dirac realised that the quantum mechanical wave function is the exponential of i times the classical action divided by the reduced Planck's constant: $\psi \propto e^{iS/\hbar}$.

If we consider a family of particles emanating from a common initial location with a range of momenta, the *Hamilton-Jacobi* equations (see figure 5) are

$$\boxed{\mathbf{p} = \frac{\partial S}{\partial \mathbf{x}} \qquad H = -\frac{\partial S}{\partial t}} \tag{48}$$

from which we see that

$$\boxed{\tilde{d}S \longrightarrow \partial_\alpha S = (-H/c, \mathbf{p}) = p_\alpha = (-\gamma mc, \gamma m\mathbf{v}).} \tag{49}$$

So the covariant components of the 4-momentum are just the 4-gradient of the action in this situation, and, under the Dirac-Feynman prescription, according to which the quantum mechanical wave function is

$$\psi \sim \exp(iS/\hbar) \tag{50}$$

the 4-momentum operator is

$$\boxed{\tilde{p} = \frac{\hbar}{i}\tilde{d}.} \tag{51}$$

4.6 The time-component of the 4-momentum

As we have just seen, (c times) the time-component of the 4-momentum is the Hamiltonian or the energy:

$$cp^0 = H \equiv \mathbf{v}\cdot\mathbf{p} - L = \gamma mc^2. \tag{52}$$

4.6.1 Non-relativistic limit

For small velocities $|\mathbf{v}| \ll c$ we can Taylor expand the γ factor in cp^0 to give

$$cp^0 = mc^2 + \tfrac{1}{2}m|\mathbf{v}|^2/2 + \ldots. \tag{53}$$

where we see the Newtonian kinetic energy $\frac{1}{2}m|\mathbf{v}|^2/2$ plus a much larger constant term known as the *rest-mass energy*.

4.6.2 The relativistic energy-mass relation

The norm of a particle being fixed to be $-m^2c^2$ means that not all 4 components of $\vec{p}$ can be specified independently; the time component is fixed if we specify the 3-components of $\mathbf{p}$ and, writing $E = H$ (for energy):

$$\boxed{E^2 = c^2|\mathbf{p}|^2 + m^2c^4} \tag{54}$$

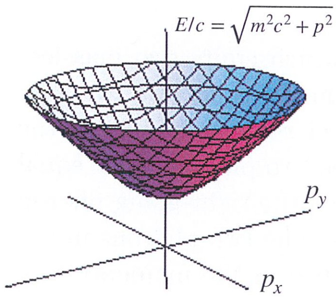

Figure 6: Relativistic energy-momentum relation for a particle of mass m. The energy $E = cp^0$, for a given relativistic momentum $\mathbf{p}$, lies on a hyperboloid $E = \sqrt{|\mathbf{p}|^2c^2 + m^2c^4}$ lying 'over' the 3-momentum 'plane' (only two components of which are shown here). If we expand this for low momenta $\mathbf{p} \ll mc$, we have $E = mc^2 + |\mathbf{p}|^2/2m + \ldots$ so equal to the constant rest-mass energy plus the Newtonian kinetic energy.

4.6.3 Conservation of total 4-momentum and the invariant mass

A free particle moves conserving its 3-momentum and therefore also its 4-momentum. In collisions, the 3-momenta change. For a collection of particles, one can always find a frame in which the sum of the initial 3-momenta vanishes. This is the *centre of momentum* (CM) frame. For elastic collisions, the sum of the time-components is conserved. This is most easily see in the case of a collision between two particles, where their momenta are equal and opposite, both before and after the collision, the collision only changing the direction of $\mathbf{p}$. One then finds that in some other frame, the summed 4 momentum components are the same as one would find from boosting a single massive particle with a mass given by the sum of the CM-frame p^0s that is at rest in the CM-frame.

The summed energy in the CM frame (divided by c^2) is known as the *invariant mass*. It sets a limit on the mass of what can be created in such a collision.

4.6.4 Equivalence of mass and energy

inertial mass

A corollary of the foregoing is that if an object loses (gains) energy, its mass must decrease (increase) by an amount $\Delta M = \Delta E/c^2$. So, according to Einstein, mass and energy are interchangeable, and that a mass M can be annihilated to create energy

$$E = Mc^2 \tag{55}$$

which is perhaps the most celebrated equation in science.

A simple argument to support this is sketched in figure 7 in which a mass M emits two highly relativistic particles with momenta, in the rest frame of the mass, $\mathbf{p} = \pm\mathbf{p}_0$. If we look at this from a (slowly) moving reference frame, two momenta are found, on boosting, to be $\mathbf{p}'_\pm = \pm\mathbf{p}_0(1 \pm v/c)$ so their sum is non-zero: $\mathbf{p}'_+ + \mathbf{p}'_- = 2\mathbf{p}_0 v/c$. The primed frame momentum of the residual mass must therefore be reduced by the same amount. But its velocity is simply the reflex of the observer's velocity, and is unaffected by the emission process. It follows that the mass of the emitter must be reduced by an amount ΔM such that $\Delta M v = 2\mathbf{p}_0 v/c$. Given that the energy emitted in the two particles is $\Delta E = 2\mathbf{p}_0 c$, it follows that

$$\Delta M = \Delta E/c^2. \tag{56}$$

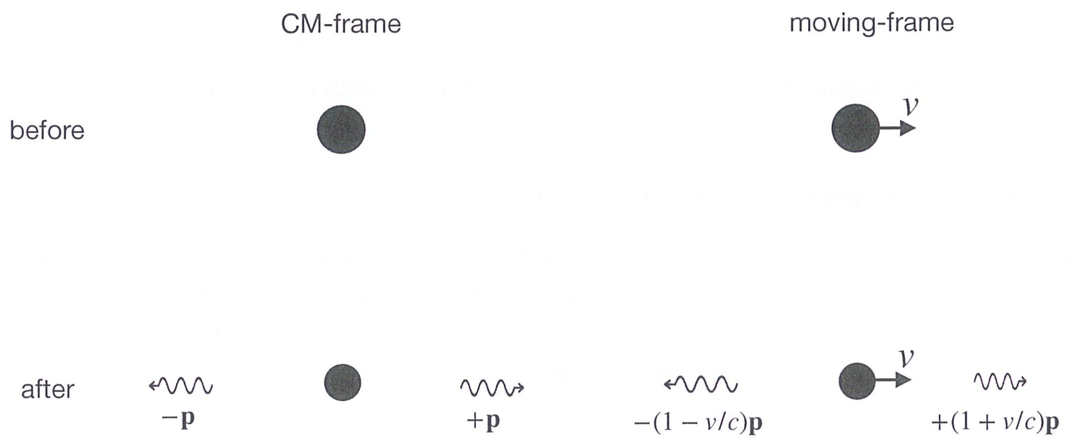

Figure 7: On the left is shown an object of a certain mass m in its rest frame which emits two massless particles (photons) with equal and opposite momenta. The same thing is shown on the right in a frame in which the object is initially moving to the right with speed v. Since the the object remains at rest in the CM frame, neither does its speed change in the moving frame. In the CM frame the two photons have equal and opposite momentum, so total momentum is conserved. But in the moving frame the right-going photon is blue-shifted and has a greater (absolute) momentum than the left-going photon. The two photons in the moving frame have a non-zero net momentum, which must be balanced by a change in the momentum of the object. But as its speed did not change this implies its mass must have decreased.

4.7 What particle energy does an observer measure?

If we know the components of the 4-momentum of some particle in the lab-frame, what is the energy that would be measured by some observer?

The answer depends on how the observer is moving. If the observer is at rest in the lab frame – i.e. the observer has 4-velocity $\vec{U} \longrightarrow (c, \mathbf{0})$ – the answer is simply $E = cp^0$.

In terms of $\vec{U}$, this is

$$\boxed{E = -\vec{U} \cdot \vec{p}} \tag{57}$$

or, equivalently, $E = -\tilde{p}(\vec{U})$ or $E = -\mathbf{g}(\vec{U}, \vec{p}) = -U^\mu p_\mu = -U_\mu p^\mu = -\eta_{\mu\nu} U^\mu p^\nu$, since, with $\vec{U} \longrightarrow (c, \mathbf{0})$ all of these evaluate to cp^0.

But (57) gives the energy the observer measures regardless of how the observer is moving. This is simply because it is Lorentz invariant; it measures cp^0 in the rest-frame of the observer.

4.8 Massless particles

4.8.1 4-momentum for a massless particle

For highly relativistic particles the speed tends to c, so the coordinate time to cover a displacement $\Delta\mathbf{x}$ is finite, while the Lorentz factor blows up, with the result that $\Delta\tau \to 0$. The result is that the the 4-velocity is poorly-defined for a particle of zero mass.

The 4-momentum, on the other hand, remains well defined, if we think of a massless particle as being the limit, as $m \to 0$, of a massive particle. Put another way, the general 4-momentum

$$\vec{p} \longrightarrow (\sqrt{|\mathbf{p}|^2 + m^2c^2}, \mathbf{p}) \tag{58}$$

tends to a well-defined limit (for fixed $\mathbf{p}$) as $m \to 0$:

质能

$$\boxed{\vec{p} \longrightarrow (|\mathbf{p}|, \mathbf{p}) = |\mathbf{p}|(1, \hat{\mathbf{p}}).} \tag{59}$$

4.8.2 Affine distance

The proper time remains ill-defined, but one can use, as an alternative *affine parameterisation*

$$d\lambda = \lim_{m\to 0} \frac{d\tau}{m} \tag{60}$$

in terms of which the 4-momentum is

$$\vec{p} = \lim_{m\to 0} m\frac{d\vec{x}}{d\tau} = \frac{d\vec{x}}{d\lambda}. \tag{61}$$

Considering the modulus of the 3-momentum we see that in a displacement $\mathbf{dx}$ the affine distance changes by an amount

$$d\lambda = |\mathbf{dx}|/|\mathbf{p}| \tag{62}$$

so we can say that *affine distance is physical distance travelled per unit 3-momentum.* The values of $|\mathbf{dx}|$ and $|\mathbf{p}|$ are frame dependent, but they transform in the same way, so their ratio is frame independent.

The same parameterisation can be used for a massive particle, for which $d\lambda = d\tau/m$.

5 Transformation of volumes and densities

5.1 Spatial volumes and space-density of cold particles

Consider a cubical 1m^3 volume in some reference frame O that contains, say, 1 million particles. So the density of particles in that frame is $n = 1$ per cm^3. If we observe that volume from a relatively moving frame O$'$ we will find it to be length contracted, so the density of particles in the frame O$'$ is larger (by a factor γ), right?

But consider instead a cubical volume at rest in the primed frame O$'$. Would not the un-primed observers find this to be length contracted in *their* frame. And wouldn't they conclude – by the same line of argument – that the density of particles is *higher* in the un-primed frame, not lower.

We seem to have an analogue of the famous barn and pole paradox. Which, if either, of these contradictory conclusions is correct?

The answer depends on how the *particles* in question are moving: If the particles are all at rest in the frame O then the former conclusion is correct.

To see why, consider the 4 pairs of events that define the corners of the cube in O' at a certain time in that frame, where the separation of each pair lies along the direction of relative motion.

Applying a boost to these, the spatial separation between these events in O will be larger by a factor γ. Now they will also have a non-vanishing temporal separation, but as the particles are at rest in O – so they have world-lines that are vertical – this is irrelevant. This is illustrated in figure 8.

We can conclude that any observer that is moving with respect to a 'cold dust' of particles – i.e. particles with no velocity *dispersion* – will see their density to be *higher* than the density that would be measured by an observer moving with the particles.

And that the density – which we shall denote by n – is enhanced by exactly the same amount as the *energy* of one of these particles is enhanced in the relatively moving frame.

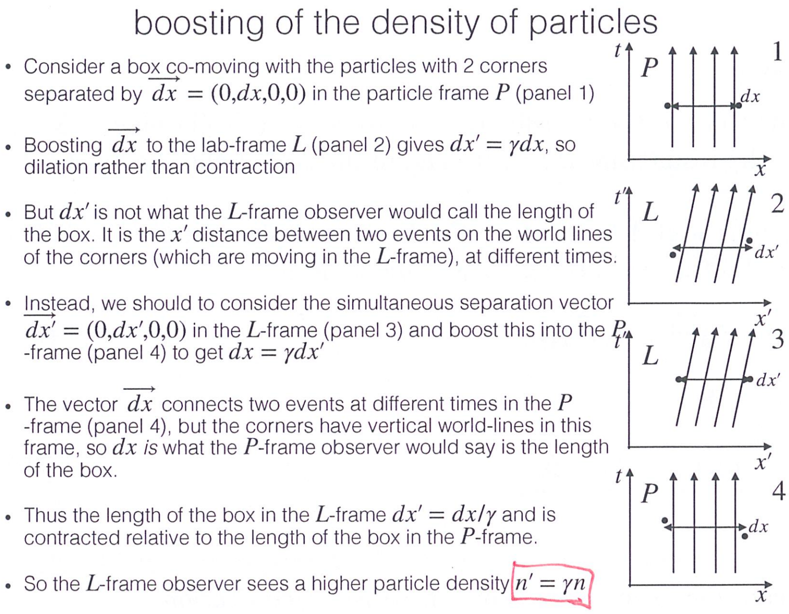

Figure 8: Illustration of transformation of volumes under boosts. The barn-and-pole paradox is at play here. The top panels shows how one might mistakenly conclude that the density of particles seen from a moving frame would be decreased, rather than enhanced. To get the right answer we need to consider the transformation of a pair of events that are simultaneous in the observer-frame O′.

- from which it follows that n/E is a *Lorentz-invariant*
- as is $E \times d^3r$

the latter following from the fact that $n \times d^3r$ is the number of particles, which is automatically invariant.

Thus the density of cold particles transforms under boosts in the same way as E, which is the time component of a 4-vector. You might want to pause and ask yourself: what would the spatial parts of such a 4-vector represent?

What about particles that aren't cold? I.e. particles with a range of velocities. Obviously we need to consider this as a superposition of different streams and compute an appropriate average of the compression factor. We will return to this presently. First we will look at how momentum space volumes transform.

5.2 Transformation of momentum space volume

Consider now a set of particles (as illustrated in figure 9) with a certain space-density in the frame O that have a very small range of velocities so they have (relativistic) 3-momenta $\mathbf{p} = (dp_x, dp_y, dp_z)$ that lie in an infinitesimal cubical volume of 3-momentum space volume Δp^3 centred on a fiducial particle with $\mathbf{p} = \mathbf{p}_0 = (0,0,0)$

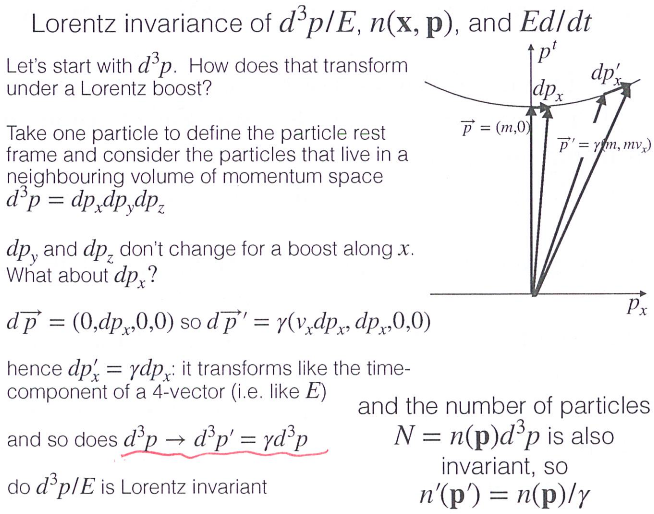

Figure 9: Transformation of 3-momentum volume elements. We consider here particles that have a range of 3-momenta d^3p centred on the origin $\mathbf{p} = 0$ in the frame of the particles. The difference of momentum between such particles is evidently purely space-like. In the lab-frame these particles lie on the hyperboloidal 'mass-shell' and occupy a 3-momentum volume that is larger by a factor γ: i.e. $d^3p' = \gamma d^3p$. That implies that the *density* in 3-momentum-space is decreased.

Recalling the relativistic hyperbolic energy momentum surface $E^2 = p^2c^2 + m^2c^4$ it is evident that these have (to first order in $|\mathbf{p}|$, which is an infinitesimal) the same energy $E = cp^0 = mc^2$.

An observer in the lab-frame O$'$ moving in the (minus) x-direction will find the fiducial particle (subscript 0), with $\vec{p}_0 \longrightarrow (mc, 0, 0, 0)$, to have 4-momentum components

$$\begin{bmatrix} p_0^{0'} \\ p_0^{1'} \\ p_0^{2'} \\ p_0^{3'} \end{bmatrix} = \begin{bmatrix} \gamma & \gamma v & & \\ \gamma v & \gamma & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} mc \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \gamma mc \\ \gamma mv \\ 0 \\ 0 \end{bmatrix} \tag{63}$$

and will see a neighbouring particle, with $\vec{p} \longrightarrow (mc, dp_x, dp_y, dp_z)$, to have 4-momentum

$$\begin{bmatrix} p^{0'} \\ p^{1'} \\ p^{2'} \\ p^{3'} \end{bmatrix} = \begin{bmatrix} \gamma & \gamma v & & \\ \gamma v & \gamma & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} mc \\ dp_x \\ dp_y \\ dp_z \end{bmatrix} = \begin{bmatrix} \gamma(mc + v dp_x/c) \\ \gamma(mv + dp_x) \\ dp_y \\ dp_z \end{bmatrix}. \tag{64}$$

Taking the difference of the spatial momenta we have

$$d\mathbf{p}' = \mathbf{p}' - \mathbf{p}'_0 \longrightarrow (\gamma dp_x, dp_y, dp_z) \tag{65}$$

$dr'^3 = \frac{1}{\gamma} dr^3$

$dp'^3 = \gamma dp^3$

so they inhabit a volume in 3D momentum space

$$\boxed{dp'^3 = \gamma dp^3} \tag{66}$$

$n(r') = \gamma n(r)$

$n(p') = \frac{1}{\gamma} n(p)$

And, using the argument that $n(\mathbf{p})d^3p$ is a number of particles and is automatically Lorentz invariant, the density of particles $n(\mathbf{p})$ per unit momentum-space volume is correspondingly *decreased* (by a factor $1/\gamma$) as compared to that in the rest-frame of the particles:

$$\boxed{n'(\mathbf{p}') = n(\mathbf{p})/\gamma}. \tag{67}$$

This is in contrast to the *space-density*, where, as we saw, a moving observer sees the particles to have a density *enhanced* by a factor γ

$$\boxed{n'(\mathbf{r}') = \gamma n(\mathbf{r})}. \tag{68}$$

5.3 Phase-space density and phase-space volume invariance

A useful description for particulate matter is in terms of the phase-space density defined by

$$\boxed{d^6N = f(\mathbf{r}, \mathbf{p})d^3rd^3p} \tag{69}$$

where d^3r is a spatial volume element, d^3p is a (relativistic) momentum-space volume element and d^6N the number of particles in the 6D phase-space volume element d^3rd^3p

The results of the two above sub-sections furnish the remarkable result

$$\boxed{d^3r'd^3p' = d^3rd^3p} \tag{70}$$

so the *phase-space volume is Lorentz invariant*!

And, yet again using that argument that $d^6N = f(\mathbf{r}, \mathbf{p})d^3rd^3p$ is a number of particles – albeit an infinitesimal one – and therefore also something everyone has to agree on regardless of their reference frame we find

- *the phase-space density $f(\mathbf{r}, \mathbf{p})$ is Lorentz invariant also*

Working with Lorentz invariant quantities is very useful. As an example, we consider in appendix B the collisional Boltzmann equation (CBE) describing the evolution the distribution function for particles undergoing 2-body scattering. The CBE is used extensively in cosmology and particle physics. As we show, this can be written in a way such that the integrations over momenta of the particles involved can be expressed as integrals over d^3p/E (times other quantities that are also Lorentz invariant). And we calculate the Lorentz invariant Edf/dt rather than df/dt which would be frame-dependent. Note, however, that numerical calculations are usually performed in a specific reference frame.

5.4 Invariance of power

While we will not need it much here, another useful Lorentz invariant quantity is *radiated power*; for example the power radiated by a collection of moving charges.

This is Lorentz invariant as it is the ratio of the energy emitted in some period of time divided by that time – both of which transform as the 0th components of 4-vectors.

This is useful in radio astronomy if one is observing radiation from charges being accelerated in a magnetic field. We can use Larmor's (non-relativistic) formula to calculate the total power radiated in the instantaneous rest-frame of the accelerated particles, and use this to give the total power in the observer or lab-frame.[2]

6 Continuity of particle number, energy and momentum

6.1 Introductory remarks

The goal of this section is to familiarise ourselves with the *stress-energy 4-tensor* $T_{\mu\nu}$ for particulate matter. This is a generalisation and extension of the the *3-dimensional stress-tensor* T_{ij} from elastics and fluid mechanics that describes the transport of 3-momentum.

The definition of T_{ij} is that it gives the flux density (amount per unit time per unit area) of the jth component of momentum travelling in the ith direction.

The stress-energy 4-tensor generalises this to include the transport of energy (the 0th component of the 4-momentum) and to include the flux in the time-direction. This concept may seem a little strange at first sight but relativistically it is quite natural. Particles 'carry' quantities like their mass, charge and momentum and a moving particle 'transports' those quantities in space as it moves around. It is the flux of momentum, for example, that appears in the stress T_{ij} for a gas of particles. For a particle at rest, it doesn't have momentum, but it has mass, and therefore energy, and can be thought of as transporting that in the time direction as it 'moves' along its world-line (i.e. as coordinate time evolves as a function of is proper time).

The stress-energy tensor plays a critical role in GR as it is how matter appears in the Einstein field equations since, in the words of John Wheeler, it is through the stress-energy tensor that *matter tells space-time how to curve*. The reason for this is that the component T^{00} is the energy density $\mathcal{E}$ and, for slowly moving matter, this is much larger than the other components. Thus, with the equivalence of mass and energy: $\mathcal{E} = \rho c^2$, the tensor $T^{\mu\nu}$ provides the natural relativistic generalisation of the density ρ that appears as the source of Newtonian gravity in Poisson's equation.

An important feature of $T_{\mu\nu}$ is that it has vanishing *4-divergence*: i.e. $T^{\mu\nu}{}_{,\mu} = 0$ for $\nu = 0, 1, 2, 3$. Thus there are four identities of the form $X^{\mu}{}_{,\mu} = 0$. Such an equation is a *conservation law* as it says that the rate of change with time of X^0 is minus the 3-divergence $X^{i}{}_{,i} = \nabla \cdot \mathbf{X}$. Taking the integral of $X^{\mu}{}_{,\mu} = 0$ over all space, and applying sensible boundary conditions, results in $d/dt(\int d^3x X^0) = 0$ so the integral here is a *globally conserved quantity.*

The equations $T^{\mu\nu}{}_{,\mu} = 0$ express the conservation – or, more generally, the continuity – of energy and the 3-components of momentum. and they follow fundamentally, and rather directly, from the invariance of space-time under displacement in time and in the 3-components of space.

Many GR texts consider primarily – and many exclusively – the case of the stress-energy for an *ideal fluid*. This is a historical throwback to the early days of GR where the focus was on relativistic stars. But many current applications of GR are to matter that is not very well described as a fluid; examples are the dark matter, neutrinos and stars in stellar systems – which are all described by a phase-space density (or 'distribution function') $f(\mathbf{r}, \mathbf{p})$ – or the dark energy (and perhaps the dark matter if it is the axion or an axion-like field) which, if it does indeed appear on the 'matter-side' of Einstein's equation.

Fluids are still important in cosmology; for instance the hot plasma in the radiation era can be well approximated as being an ideal fluid. We will see how the stress-energy tensor for a fluid emerges as a limiting case from the more general description in terms of phase-space distribution functions, when scattering is highly efficient and establishes local thermal equilibrium and the macroscopic behaviour can be described in terms of a space-density and a temperature and velocity field.

We will start, as something of a warm-up exercise, with the derivation of the law of continuity of particle density $\nabla \cdot \vec{n} \longrightarrow n^{\alpha}{}_{,\alpha} = 0$ where the particle flux (or current) 4-vector is $\vec{n} \longrightarrow n^{\alpha}$. This is the vector whose

[2] though if we want to describe the angular distribution of the power we need to worry about aberration.

time component is (c times) the number space-density of particles.

We will then move on to construction of $T_{\mu\nu}$ for a distribution of particles in phase-space and demonstrate its conservation laws; followed by the specialisation to a fluid. In a later chapter we will consider $T_{\mu\nu}$ for radiation and for a scalar field – arguably the simplest type of matter. The latter finds several extremely important applications in cosmology (early- and late-time inflation).

6.2 Particle number continuity equation

Conservation of particles in phase space is expressed in the *Vlasov equation*:

$$\boxed{\partial f/\partial t + \nabla^{(6)} \cdot (f\dot{\mathbf{x}}^{(6)}) = 0} \tag{71}$$

where

- $\mathbf{x}^{(6)} \equiv (\mathbf{r}, \mathbf{p})$ denotes particle coordinates in phase-space
- $f(\mathbf{r}, \mathbf{p}, t)$ is the phase-space density
- $\nabla^{(6)} \equiv (\nabla_{\mathbf{r}}, \nabla_{\mathbf{p}})$ is the 6D partial derivative,
- $\dot{\mathbf{x}}^{(6)}$ denotes the rate of change of $\mathbf{x}^{(6)}$ with respect to coordinate time (*not* proper time)

The Vlasov equation comes from the fact that the rate of change of number of particles in a 6-dimensional phase-space volume element comes from the sum over all 6 directions of the difference between the fluxes of particles across the two opposite faces, as illustrated in figure 10.

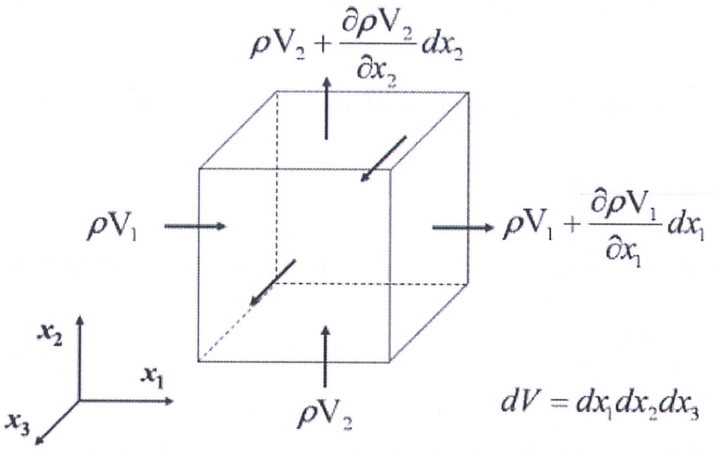

Figure 10: Conservation of mass for a fluid. The flux density is the product of the mass density $\rho(\mathbf{r})$ – though it could be the number density $n(\mathbf{r})$ if we think of the fluid as composed by a large number of identical mass particles – and the velocity flow-field $\mathbf{V}(\mathbf{r})$. The rate of change of the amount of mass in the cubical volume is obtained by differencing the flux across the 3 pairs of surfaces. Dividing by the volume gives the rate of change of ρ as minus the divergence $\nabla \cdot (\rho\mathbf{V})$ of the flux density. The continuity equation for particles in 6-D phase-space is directly analogous.

Implicit in the Vlasov equation is the idea that at any point $(\mathbf{r}, \mathbf{p})$ in 6D phase space there is a unique 6-velocity $(\dot{\mathbf{r}}, \dot{\mathbf{p}})$. So particles in 6-dimensional phase-space behave like a *fluid* in 3D where, at a macroscopic level at least, there is a *velocity field* $\mathbf{v}(\mathbf{r})$. A fluid in 3D, however, is quite different from a *collisionless gas*, where at any point in 3-space there is a a *distribution* of 3-momenta, and therefore velocities.

The reason for this stark difference is straightforward but fundamental: The equations of motion for particles moving in space are second order in time $m\ddot{\mathbf{x}} = \mathbf{F}$, so the trajectory of a particle depends not just on its initial position $\mathbf{x}$ but on its initial velocity $\dot{\mathbf{x}}$. The equations of motion for a particle in phase-space are *Hamilton's equations*, which are six 1st order differential equations giving $\dot{\mathbf{r}}$ and $\dot{\mathbf{p}}$ as functions of $\mathbf{r}$ and $\mathbf{p}$. So at any point $(\mathbf{r}, \mathbf{p})$ the *phase-space-velocity* $(\dot{\mathbf{r}}, \dot{\mathbf{p}})$ is fully determined by the position $(\mathbf{r}, \mathbf{p})$.

Note that this is not strictly true for a plasma composed of charged particles, for instance, where different species are subject to different Lorentz forces. But the Vlasov equation is obeyed for each species of particles separately.

The Vlasov equation is nice, but our goal here is the continuity equation for the density of particles in 3D. To obtain this we simply integrate $\partial f/\partial t + \nabla^{(6)} \cdot (f\dot{\mathbf{x}}^{(6)}) = 0$ over momenta $\mathbf{p}$. Splitting the 6 terms in $\nabla^{(6)} \cdot (f\dot{\mathbf{x}}^{(6)})$ into space and momentum parts, we have

$$\int d^3p \left[\frac{\partial f}{\partial t} + \nabla_{\mathbf{r}} \cdot (f\dot{\mathbf{r}}) + \nabla_{\mathbf{p}} \cdot (f\dot{\mathbf{p}})\right] = 0 \tag{72}$$

Noting that the last term, involving $\int d^3p \boldsymbol{\nabla}_{\mathbf{p}} \ldots$, integrates to zero assuming $f \to 0$ as $|\mathbf{p}| \to \infty$, we have

$$\boxed{\partial n/\partial t + \boldsymbol{\nabla} \cdot (n\overline{\mathbf{v}}) = 0} \tag{73}$$

where we have dropped the subscript $\mathbf{r}$ on $\boldsymbol{\nabla}_{\mathbf{r}}$ and where we have defined the *space number density* n and the *mean velocity* as

$$\begin{aligned} n &\equiv \int d^3p\, f \\ \overline{\mathbf{v}} &\equiv \int d^3p\, \dot{\mathbf{r}} f / \int d^3p\, f \end{aligned} \tag{74}$$

and we have used the commutativity of integrating over $\mathbf{p}$ and taking time or space derivatives:

- so $\int d^3p\, \partial_t f = \partial_t \int d^3p f$
- and $\int d^3p\, \boldsymbol{\nabla} \cdot (f\dot{\mathbf{r}}) = \boldsymbol{\nabla} \cdot \int d^3p\, f\dot{\mathbf{r}}$

Equation (73) says that the time rate of change of the particle number density n at fixed position $\partial n/\partial t$ is minus the 3-divergence of the particle 3-current $n\overline{\mathbf{v}}$.

Equivalently, and more compactly, we can write this as the vanishing of a single *4-divergence*

$$\boxed{n^\alpha{}_{,\alpha} = 0} \tag{75}$$

this being shorthand for $c^{-1}\partial_t n^0 + \boldsymbol{\nabla} \cdot \mathbf{n} = 0$ where the $\{n^\alpha\}$ are the components of a 4-vector

?!

$$\vec{n} \longrightarrow n^\alpha = (n^0, \mathbf{n}) = (nc, n\overline{\mathbf{v}}) \tag{76}$$

and which we call the particle 4-current.

Equivalently, invoking the definitions of n and $\overline{\mathbf{v}}$ in (74), which imply that $n\overline{\mathbf{v}} = \int d^3p \dot{\mathbf{r}} f$, we can write

$$n^\alpha = \int d^3p\, (c, c^2\mathbf{p}/E)\, f \tag{77}$$

since $\mathbf{p} = \gamma m \mathbf{v}$ and $E = \gamma m c^2$ implies $\mathbf{v} = \dot{\mathbf{r}} = c^2\mathbf{p}/E$. Or again, equally, but a little more transparently, as

$$\vec{v} = \frac{c^2\vec{p}}{E} = \frac{c\vec{p}}{E/c}$$

$$\boxed{\vec{n} = c\int \frac{d^3p}{p^0} f\vec{p}} \tag{78}$$

where this last expression makes it clear that $\vec{n}$ does indeed transform as a 4-vector since, as we have seen, both $d^3p/p^0 = cd^3p/E$ and $f(\mathbf{r}, \mathbf{p}, t)$ are Lorentz invariants. Note that, as we have integrated over 3-momentum, $\vec{n}$ here is a 4-vector field $\vec{n}(\vec{x})$.

The vanishing 4-divergence $n^\alpha{}_{,\alpha} = 0$ is a compact and concise expression of particle conservation. Schutz derives this for a 'dust': i.e. a collection of particles where all the particles in a given region of space have the same velocity. I.e. the special case where the velocity dispersion vanishes – where we have a pressureless fluid. That is rather restrictive. The derivation above is a little more involved, but allows an arbitrary distribution of velocities at each point in space.

Finally, defining the 4-dimensional covariant derivative operator

$$\nabla \longrightarrow \partial/\partial x^\alpha \tag{79}$$

we can write this continuity or conservation law as

$$\boxed{\nabla \cdot \vec{n} = 0.} \tag{80}$$

6.3 The stress tensor and continuity of energy and momentum

6.3.1 Stress energy tensor for a collisionless gas of particles

Let's now multiply the fundamental equation expressing conservation of particles (Vlasov) by the components of p^α and again integrate over all 3-momenta $\mathbf{p}$. This is called 'taking the first moment' *first moment of the Vlasov equation* – what we did before was take the zeroth moment.

And, for simplicity, (we'll relax this presently) let's also assume that there are no forces acting on the particles, so $\dot{\mathbf{p}} = 0$ and hence the Vlasov equation becomes $\partial f/\partial t + \nabla \cdot (f\dot{\mathbf{r}}) = 0$.

Multiplying by the energy $cp^0 = E$ and integrating gives

$$\int d^3p\, E(\partial f/\partial t + \nabla \cdot (f\dot{\mathbf{r}})) = 0 \tag{81}$$

or, since we can take the space- and time-derivatives out of the integral over momentum,

$$\frac{\partial}{\partial t}\int d^3p\, Ef(\mathbf{r},\mathbf{p}) + \nabla \cdot \int d^3p\, E\mathbf{v}f(\mathbf{r},\mathbf{p}) = 0 \tag{82}$$

which we may write as

$$\boxed{\partial(n\langle E\rangle)/\partial t + \nabla \cdot (n\langle E\mathbf{v}\rangle) = 0} \tag{83}$$

where $n = \int d^3p\, f$ is the mean density of particles as before and

$$\langle X\rangle \equiv \frac{\int d^3p\, fX}{\int d^3pf} \tag{84}$$

defines the *number weighted average* of X over particles.

This says that the rate of change of energy density $\mathcal{E} \equiv n\langle E\rangle$ – the number density times the *mean energy per particle* – is minus the 3-divergence of the *energy flux density* $n\langle E\mathbf{v}\rangle$. This is analogous to the charge current $n\langle q\mathbf{v}\rangle$, which is equal to the number density of the particles times the average of their charge times their velocity and gives the rate per unit area at which charge is being transported.

Here $n\langle E\mathbf{v}\rangle$ gives the rate (per unit time per unit area) of transport of energy. So (83) expresses conservation (or continuity) of energy.

Similarly, multiplying Vlasov by the 3-momentum $\mathbf{p}$ and integrating gives

$$\int d^3p\, \mathbf{p}(\partial f/\partial t + \nabla \cdot (f\dot{\mathbf{r}})) = 0 \tag{85}$$

or

$$\frac{\partial}{\partial t}\int d^3p\, \mathbf{p}f(\mathbf{r},\mathbf{p}) + \nabla \cdot \int d^3p\, \mathbf{v}\mathbf{p}f(\mathbf{r},\mathbf{p}) = 0 \tag{86}$$

or

$$\boxed{\partial(n\langle \mathbf{p}\rangle)/\partial t + \nabla \cdot (n\langle \mathbf{v}\mathbf{p}\rangle) = 0} \tag{87}$$ 压强（动量流密度）

This is the relativistic generalisation of Newton's $\mathbf{F} = m\mathbf{a}$, since it says the rate of change of the *space density of relativistic momentum* $n\langle\mathbf{p}\rangle$ is minus the 3-divergence of the *momentum flux density* $n\langle\mathbf{v}\mathbf{p}\rangle$.

Note that there is no 'dot' between the vectors $\mathbf{v}$ and $\mathbf{p}$. What we have here is the *3-stress-tensor*: $n\langle v_i p_j\rangle$ which is the rate (per unit area per unit time) at which the j^{th} component of 3-momentum is being transported in the i^{th} direction. It is also the *definition* of the kinetic pressure of a gas (note that if the distribution of velocities of the gas particles is isotropic then the pressure tensor is diagonal and therefore characterised by a single number P, being the value of any of the diagonal components). This may seem at odds with the idea that pressure is force per unit area, but they are equivalent in the sense that if one were to insert an object into a gas, the particles of the gas will bounce off the surface and in doing so transfer momentum (as they change their momenta when they bounce yet momentum is conserved). If you are new to the idea of pressure as being *defined* as the flux density of momentum – and force density being its 3-divergence – you might want to ask yourself: In what direction is momentum flowing in a pressurised balloon? Or in a compressed spring?

Replacing $\mathbf{v}$ by $c\mathbf{p}/p^0 = c\gamma m\mathbf{v}/\gamma mc$ we can combine these four conservation laws into the vanishing of a 4-divergence

$$\boxed{T^{\alpha\beta}{}_{,\alpha} = 0} \tag{88}$$

where the *stress-energy 4-tensor* is

$$\boxed{T^{\alpha\beta} \equiv c \int \frac{d^3p}{p^0} p^\alpha p^\beta f(\mathbf{r}, \mathbf{p})} \tag{89}$$

which can also be expressed symbolically as

$$\boxed{\mathbf{T} \equiv c \int \frac{d^3p}{p^0} \vec{p} \otimes \vec{p} f(\mathbf{r}, \mathbf{p})} \tag{90}$$

The stress-energy tensor has profound significance in GR as it turns out that there is a 4-tensor $G^{\alpha\beta}$ – which is of a purely geometrical nature and is a measure of the curvature of space-time – that obeys an identical continuity equation. Einstein's insight was to equate these as the prescription for how matter tells space-time how to curve.

6.3.2 Stress-energy tensor for a collisional gas

It is restrictive to impose $\dot{\mathbf{p}} = 0$, and also to impose conservation of particles – since high energy collisions can create and annihilate particles and transmute particles from one type to another.

But the conservation law $T^{\alpha\beta}{}_{,\alpha} = 0$ is of much more general applicability. For example, consider the effect of *two-body collisions* as illustrated in figure 11.

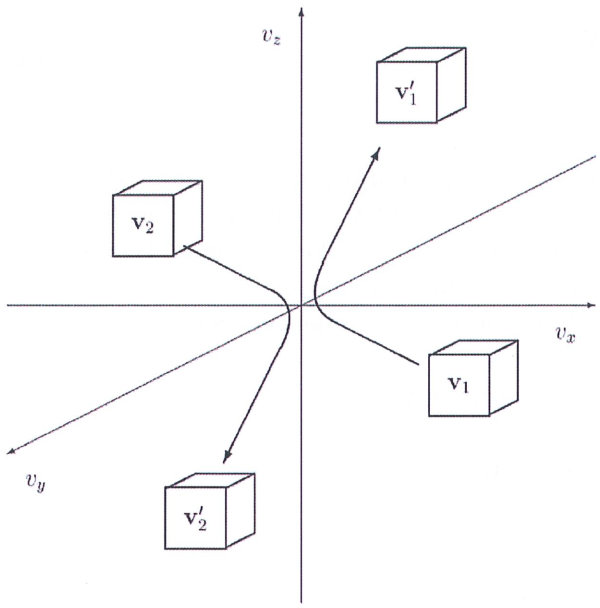

Figure 11: Illustration of a collision between two particles where initial particles are in momentum space cells labeled $\mathbf{v}_1$, $\mathbf{v}_2$ and end up in cells labeled $\mathbf{v}_{1'}$, $\mathbf{v}_{2'}$. In calculating the rate of change of occupation number for cell $\mathbf{v}_1$ say, we have a negative term corresponding to the 'forward' reactions as shown, but we also have a positive term arising from 'inverse' reactions, so the net rate is proportional to $-(f(\mathbf{v}_1)f(\mathbf{v}_2) - f(\mathbf{v}_{1'})f(\mathbf{v}_{2'}))$. That is for the particular combination of momenta shown. To obtain the total rate of change of f_1 we need to integrate over all possible values of $\mathbf{v}_2$ and over the direction for one of the outgoing particles (e.g. $\hat{\mathbf{v}}_{2'}$). That gives a 5-dimensional integral to perform. Several important consequences – the form for the equilibrium distribution function and Boltzmann's H-theorem – can be understood just using the fact that the net rate involves this combination of occupation numbers.

If we allow for collisions between the particles then we get a 'collision term', so

$$\boxed{\partial f/\partial t + \boldsymbol{\nabla}^{(6)} \cdot (f\dot{\mathbf{x}}^{(6)}) = (\partial f/\partial t)_{\rm coll}} \tag{91}$$

but if the collisions individually conserve 4-momentum then $\int d^3p\, p^\alpha (\partial f/\partial t)_{\rm coll} = 0$.

It may be helpful to think of 3-momentum space as divided into a fine grid of cubical cells with label $\mathbf{p}$. A collision where two particles with 3-momenta $\mathbf{p}_1$ and $\mathbf{p}_2$ scatter off one another and emerge with 4-momenta $\mathbf{p}'_1$ and $\mathbf{p}'_2$ depletes f in the cells $\mathbf{p}_1$ and $\mathbf{p}_2$ and enhances f in the cells $\mathbf{p}'_1$ and $\mathbf{p}'_2$.

If the rate at which collisions of this kind is $R_{12\to1'2'}$ then this gives a contribution to the collision term

$$\delta(\partial f/\partial t)_{\rm coll}(\mathbf{p}) = R_{12\to1'2'}(\delta(\mathbf{p}-\mathbf{p}'_1) + \delta(\mathbf{p}-\mathbf{p}'_2) - \delta(\mathbf{p}-\mathbf{p}_1) - \delta(\mathbf{p}-\mathbf{p}_2)) \tag{92}$$

where $\delta(\mathbf{p}-\mathbf{p}')$ is zero unless $\mathbf{p}$ and $\mathbf{p}'$ refer to the same cell.

Multiplying $\delta(\partial f/\partial t)_{\rm coll}(\mathbf{p})$ by $\mathbf{p}$ and integrating over all $\mathbf{p}$ one finds that this has no effect since $\mathbf{p}'_1 + \mathbf{p}'_2 = \mathbf{p}_1 + \mathbf{p}_2$. And similarly for the energy.

So collisions of this type have no effect on the conservation law $T^{\alpha\beta}{}_{,\alpha} = 0$. The total collision term is, of course, the sum over all possible types of collisions, but the conclusion that $T^{\alpha\beta}{}_{,\alpha} = 0$ is unchanged remains valid.

Moreover, this is not restricted to number conserving collisions; the same holds true for number violating reactions and for reactions involving different types of particles. In such case the number conservation law is violated, of course, but energy and momentum conservation – being more fundamental (their conservation stemming from the invariance of the laws of physics under translations in time and space) – still hold.

Also, if we have macroscopic forces such as electromagnetic forces acting on the particles, then energy and momentum for the particles alone is naturally no longer conserved. But there is also a stress-energy tensor for the electromagnetic field, and the *total stress-energy tensor* – that of particles *plus* fields – obeys the continuity equation. This guarantees that the work equation (which says that the rate at which particles gain energy from the electromagnetic field is balanced by the loss of electromagnetic field energy) and Newton's 3rd law (equal and opposite reaction) are obeyed.

6.3.3 Summary and useful expressions for the stress-energy tensor

We assumed above a *phase-space fluid* with *phase-space density* $f(\mathbf{r}, \mathbf{p})$ where at each point in phase-space there is a unique 6-velocity $(\dot{\mathbf{r}}, \dot{\mathbf{p}})$. As already discussed, if one has particles with different charge-to-mass ratios moving under the influence of EM fields then each 'species' will have a unique velocity and the total stress-energy is the sum over the different 'species' of particles.

To summarise, we can write the stress-energy tensor (for each species) in various different but equivalent ways:

$$
\begin{aligned}
T^{\alpha\beta} = \left[\begin{array}{c|c} T^{00} & T^{0i} \\ \hline T^{i0} & T^{ij} \end{array}\right] &= \int d^3p\, f v^\alpha p^\beta = c\int \frac{d^3p}{p^0}\, f p^\alpha p^\beta \\
&= \int \frac{d^3p}{E} f \left[\begin{array}{c|c} E^2 & cE\mathbf{p} \\ \hline cE\mathbf{p} & c^2\mathbf{pp} \end{array}\right] = \left[\begin{array}{c|c} n\langle E\rangle & nc\langle\mathbf{p}\rangle \\ \hline nc\langle\mathbf{p}\rangle & n\langle\mathbf{vp}\rangle \end{array}\right]
\end{aligned}
\tag{93}
$$

In the top row we have the *densities*, of energy and momentum respectively, and below we have the energy and momentum *flux densities*.

Terminology: We normally name 4-quantities by the spatial part. For example, we call $\vec{p} = (E/c, \mathbf{p})$ the *4-momentum* and, as we shall see, in electromagnetism we call the 4-vector j^α whose spatial parts are the vector $\mathbf{j}$ the 4-current. If we were to follow that convention we would call $T^{\alpha\beta}$ the 4-stress since the 3×3 spatial part T_{ij} in the lower-right segment is conventionally called the stress. However, it is more often called the stress-energy tensor, which, while more cumbersome, is what we will adopt. The stress-energy tensor is alse sometimes called the energy-momentum tensor but that is becoming less common.

6.4 Transformation of the components of the stress tensor under a boost

From the definition $T^{\alpha\beta} \equiv \int (d^3p/E)p^\alpha p^\beta f$ and from the invariance of both d^3p/E and f we have

$$
T^{\alpha'\beta'} \equiv \int (d^3p/E)p^{\alpha'}p^{\beta'} f = \int (d^3p/E)(\Lambda^{\alpha'}{}_\alpha p^\alpha)(\Lambda^{\beta'}{}_\beta p^\beta) f = \Lambda^{\alpha'}{}_\alpha \Lambda^{\beta'}{}_\beta T^{\alpha\beta}
\tag{94}
$$

so we simply apply (matrix multiply by) a Lorentz transformation matrix for each index.

The same is true for any other tensor – the *definition of a tensor* being that it transforms in this manner. We saw, for example, that this was the law for the transformation of the Minkowski metric, which was rather special in that its components are frame-independent. The stress-energy tensor is an example of a what is called a *rank-2 tensor* (it having 2 indices).

Just as we think of things like $\vec{V}$ or $\mathbf{g}(\ ,\ )$ or $\tilde{V} = \mathbf{g}(\vec{V},\ )$ as geometric entities which may be described either in terms of its contravariant or covariant components, the same is true for tensors, where we will generally use bold-face to denote the frame-invariant quantity and write e.g.

$$
\mathbf{T} \underset{\mathrm{O}}{\longrightarrow} T^{\alpha\beta}
\tag{95}
$$

to say that $\mathbf{T}$ is the tensor that, in frame O, has the indicated contravariant components.

Just as for 4-vectors we can lower (or raise) indices with the Minkowski metric. We can make, for example, the *mixed components* of the stress-energy tensor

$$T^{\alpha}{}_{\beta} = \eta_{\gamma\beta} T^{\gamma\alpha} \tag{96}$$

which has the effect of changing the sign of the components in top row and the *fully covariant rank-two tensor* components

$$T_{\alpha\beta} = \eta_{\gamma\beta} T^{\gamma}{}_{\alpha}. \tag{97}$$

Note that since the matrix representing the Minkowski metric is its own inverse, so $\eta^{\alpha\gamma}\eta_{\gamma\beta} = \delta^{\alpha}_{\beta}$, and $\eta^{\alpha\gamma}$ is the index raising operator then we can say that the mixed components of Minkowski are $\eta^{\alpha}{}_{\beta} = \delta^{\alpha}_{\beta}$.

6.5 Ideal fluids

6.5.1 Stress-energy tensor for a ideal fluid

We can always boost from the lab-frame into the frame where the momentum density $n\langle \mathbf{p} \rangle \longrightarrow T^{0i}/c$ (and therefore also the energy flux density $n\langle \mathbf{v}E \rangle \longrightarrow cT^{i0}$) vanish. This is the the so-called co-moving frame, with stress-energy tensor determined by the energy density T^{00} and the 3D stress or pressure tensor T^{ij}.

If, moreover, collisions between the particles are sufficient to render the momentum distribution function isotropic in this frame, so $f(\mathbf{p}) = f(|\mathbf{p}|)$ then the 3D pressure tensor $T^{ij} = n\langle v_i p_j \rangle$ will be isotropic, with equal diagonal components, which we will denote by P. Thus

$$T^{ij} = P\delta^{ij} \tag{98}$$

and the 4D stress tensor is then

$$T^{\alpha\beta} = \mathrm{diag}\{\mathcal{E}, P, P, P\} \tag{99}$$

where $\mathcal{E} \equiv n\langle E \rangle$ is the energy density.[3]

That is in a specific frame; the comoving frame. We can also write this in a frame independent form as

$$\mathbf{T} = (\mathcal{E} + P)\vec{u} \otimes \vec{u}/c^2 + P\mathbf{g} \tag{100}$$

by which we mean the rank-two tensor – considered as a geometric entity – with contravariant components

$$T^{\alpha\beta} = (\mathcal{E} + P)u^{\alpha}u^{\beta}/c^2 + P\eta^{\alpha\beta}. \tag{101}$$

Here $\vec{u}$ is the 4-velocity of an observer co-moving with the fluid element, i.e. the observer who measures zero momentum density locally, and where ρ and P are the energy density and pressure as measured by such an observer – i.e. the *proper energy density and pressure*.

That (101) and (99) are equivalent can easily be checked, since, in the comoving frame, $\vec{u} \to (c, 0, 0, 0)$, so $T^{00} = \mathcal{E} + P + \eta^{00}P = \mathcal{E}$ and $T^{ij} = P\eta^{ij} = P\delta^{ij}$.

If we boost this into the lab-frame in which the comoving observer has velocity $\mathbf{v} = (\beta c, 0, 0)$ – i.e. so the fluid is moving in the x-direction in the lab-frame – the components of the stress-energy tensor are readily found to be

$$T^{\mu\nu} = \begin{bmatrix} \gamma^2(\mathcal{E} + \beta^2 P) & \beta\gamma^2(\mathcal{E} + P) & & \\ \beta\gamma^2(\mathcal{E} + P) & \gamma^2(\beta^2\mathcal{E} + P) & & \\ & & P & \\ & & & P \end{bmatrix} \tag{102}$$

and in general $T^{0i} = \beta^i\gamma^2(\mathcal{E} + P)$.

6.5.2 Why $T^{\mu 0}{}_{,\mu} = 0$ is the first law of thermodynamics

One of the key equations of cosmology is

$$\dot{\rho} = -3H(\rho + P/c^2) \tag{103}$$

[3]Note that the energy density $\mathcal{E}$ is *not* c^2 times the *proper* mass density; since it contains the kinetic energy in addition to rest-mass. Note also that the $\mathcal{E} + P$ appearing repeatedly here is called the enthalpy.

where ρ is the mass density and $H = \dot{a}/a$ is the expansion rate. If $P = 0$ this has a solution $\rho \propto a^{-3}$, which expresses conservation of mass. If, on the other hand, $P \neq 0$ there is an extra rate of decrease of the mass density (or increase if $P < 0$). It is not at all difficult to see why; the energy in a volume element V is $E = \mathcal{E}V$ so $\dot{E} = V\dot{\mathcal{E}} + \mathcal{E}\dot{V}$. But according to the 1st law of thermodynamics $dE = -PdV$ so $\dot{E} = -P\dot{V}$. Eliminating $\dot{E}$ from these two expressions give

$$\dot{\mathcal{E}} = -\frac{\dot{V}}{V}(\mathcal{E} + P) \tag{104}$$

this is equivalent to the continuity equation for the density since mass-energy equivalence implies $\mathcal{E} = \rho c^2$ and, if we consider a sphere of radius a, with volume $V = (4/3)\pi a^3$, $\dot{V}/V = 3\dot{a}/a = 3H$.

Let's look in a little more detail how that emerges from $T^{\mu 0}{}_{,\mu} = 0$ with $T^{\mu 0}$ for an ideal fluid where $\mathcal{E}$ and P are smoothly varying in space and time.

What we have to do is work in the rest-frame of the fluid. At first sight that seems nonsensical, since in the rest-frame $T^{\mu 0} = (\mathcal{E}, 0, 0, 0)$, with no sign of the pressure, which is clearly an essential ingredient. But if we consider a small neighbourhood of a point $\mathbf{x}_0$ where, in our chosen inertial reference frame, $\vec{u} = (c, 0, 0, 0)$ we can use (102), the left column of which is

$$T^{\mu 0} = (\gamma^2(\mathcal{E} + \beta^2 P), \boldsymbol{\beta}\gamma^2(\mathcal{E} + P)) \tag{105}$$

and if we work to 1st order in $|\boldsymbol{\beta}|$ and/or distance from $\mathbf{x}_0$ we can set $\gamma = 1$ and neglect the pressure term in T^{00} to obtain

$$T^{\mu 0} = (\mathcal{E}, \boldsymbol{\beta}(\mathcal{E} + P)) \tag{106}$$

so the continuity equation is

$$0 = T^{\mu 0}{}_{,\mu} = \partial_0 \mathcal{E} + \boldsymbol{\nabla} \cdot (\boldsymbol{\beta}(\mathcal{E} + P)) \tag{107}$$

or, with $\partial_0 \mathcal{E} = \dot{\mathcal{E}}/c$ and $\boldsymbol{\beta} = \mathbf{v}/c$,

$$\boxed{\dot{\mathcal{E}} = -(\mathcal{E} + P)\boldsymbol{\nabla} \cdot \mathbf{v}} \tag{108}$$

where we have recognised that $\boldsymbol{\beta} \cdot \boldsymbol{\nabla}(\mathcal{E} + P)$ vanishes at $\mathbf{x}_0$.

All that remains now is to show (if it is not obvious) that $\boldsymbol{\nabla} \cdot \mathbf{v} = \dot{V}/V$ for a small volume element that is expanding with the fluid. To do this we simply do a Taylor series expansion $\mathbf{v}(\mathbf{x}) = \mathbf{0} + \mathbf{H} \cdot \mathbf{x}$ (setting $\mathbf{x}_0 = \mathbf{0}$ for simplicity) where $\mathbf{H} = \partial \mathbf{v}/\partial \mathbf{x}$ is the 3×3 symmetric *expansion rate tensor*. Then consider the rate of change of the volume of a small sphere, of radius $|\mathbf{x}| = a$, with outward normal $d\mathbf{A} = a^2 d\Omega\, \hat{\mathbf{x}}$, which is

$$\dot{V} = \int d\mathbf{A} \cdot \mathbf{v} = \int d\mathbf{A} \cdot \mathbf{H} \cdot \mathbf{x} = a^3 \int d\Omega\, \hat{\mathbf{x}} \cdot \mathbf{H} \cdot \hat{\mathbf{x}} = a^3 H_{ij} \int d\Omega\, \hat{x}_i \hat{x}_j = \tfrac{4}{3}\pi a^3 H_{ii} = V H_{ii} \tag{109}$$

since $\int d\Omega\, \hat{x}_i \hat{x}_j = \delta_{ij} \int d\Omega\, \mu^2 = 2\pi \int_{-1}^{1} d\mu \mu^2$. But the divergence of $\mathbf{v}(\mathbf{x}) = \mathbf{H} \cdot \mathbf{x}$ is

$$\boldsymbol{\nabla} \cdot \mathbf{v} = \boldsymbol{\nabla} \cdot \mathbf{H} \cdot \mathbf{x} = H_{ij} \partial x_i / \partial x_j = H_{ij} \delta_{ij} = H_{ii}. \tag{110}$$

7 Acceleration

7.1 The 4-acceleration

The 4-velocity is the derivative of the position of a particle or observer with respect to proper time: $\vec{U} = d\vec{x}(\tau)/d\tau$. It is a 4-vector as $\vec{x}$ is (i.e. transforms as) a 4-vector while $d\tau$ is Lorentz invariant.

Similarly, we can define the *4-acceleration* as

$$\vec{a} \equiv \frac{d\vec{U}}{d\tau} \tag{111}$$

which is also a 4-vector.

It has an interesting property that stems from the normalisation condition for the 4-velocity $\vec{U} \cdot \vec{U} = -c^2$ (which in turn stems from the fact that the invariant $ds^2 = d\vec{x} \cdot d\vec{x}$ for a time-like $d\vec{x}$ is equal to $-c^2 d\tau^2$ – evidently true in the frame such that $d\mathbf{x} = 0$). This normalisation implies

$$\frac{d}{d\tau} \vec{U} \cdot \vec{U} = 2\vec{U} \cdot \frac{d\vec{U}}{d\tau} = 2\vec{U} \cdot \vec{a} = 0 \tag{112}$$

so $\vec{a}$ is orthogonal to $\vec{U}$ (in the relativistic sense of the word).

This means that, in the instantaneous frame of rest (or MCRF) of the particle, in which frame $\vec{U} \longrightarrow (c, 0, 0, 0)$, $\vec{a} \longrightarrow (0, \mathbf{a})$ and is purely space-like.

The 3-acceleration in the MCRF is the *proper acceleration*. It is the weight (divided by the proper mass) of a massive observer suffering this acceleration (as would be measured by a weighing scale) and it is (minus) the acceleration that observer would see for a test-particle that he releases.

7.2 Lab-frame equations of motion for an accelerating particle

Consider a particle subject to an acceleration $\mathbf{a}$. In the frame of reference O of the particle at some proper time τ_0 its 3-velocity will change from zero to $\mathbf{v} = \mathbf{a}\Delta\tau + \mathcal{O}(\Delta\tau)^2$ during a small interval of proper time $\Delta\tau$. It will develop a $\boxed{\text{non-zero}}$ γ factor, but $\gamma = 1 + \mathcal{O}(\Delta\tau)^2$, so, to 1st order in Δt, its 4-velocity in this frame is

$$\vec{U}(\tau_0 + \Delta\tau) \underset{\mathrm{O}}{\longrightarrow} (c, \mathbf{a}\Delta\tau). \tag{113}$$

If we orient our coordinate system so that, at τ_0, the particle is moving in the x-direction relative to the lab-frame, with speed $v = \beta c$ then, after this short interval, the 4-velocity in the lab-frame O$'$ will be

$\dfrac{d^2 x_\perp}{dt'^2} = \dfrac{a_\perp}{\gamma^2}$

$\dfrac{d^2 x_\parallel}{dt'^2} = \dfrac{a_\parallel}{\gamma^3}$

$$\vec{U}(\tau_0 + \Delta\tau) \underset{\mathrm{O}'}{\longrightarrow} \gamma' \begin{bmatrix} c \\ \dot{x}' \\ \dot{y}' \\ \dot{z}' \end{bmatrix} = \begin{bmatrix} \gamma & \gamma\beta & & \\ \gamma\beta & \gamma & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} c \\ a_x\Delta\tau \\ a_y\Delta\tau \\ a_z\Delta\tau \end{bmatrix} = \begin{bmatrix} \gamma(c + \beta a_x\Delta\tau) \\ \gamma(c\beta + a_x\Delta\tau) \\ a_y\Delta\tau \\ a_z\Delta\tau \end{bmatrix} \tag{114}$$

where $\dot{x}' = dx'/dt'$ etc. are the components of the lab-frame 3-velocity and $\gamma' = 1/\sqrt{1 - |\dot{\mathbf{x}}'|^2/c^2}$.

If we take the ratio of the 2nd to 1st lines in (114) we see that

$$\dot{x}' = c(c\beta + a_x\Delta\tau)/(c + \beta a_x\Delta\tau) = v(1 + (a_x/v - va_x/c^2)\Delta\tau + \mathcal{O}(\Delta\tau)^2). \tag{115}$$

But, since $v = \dot{x}'(\tau_0)$, we can read off that the change in parallel component of the velocity in the lab-frame is, to linear order in $\Delta\tau$,

$$\Delta\dot{x}' = a_x(1 - v^2/c^2)\Delta\tau = a_x\Delta\tau/\gamma^2 = \boxed{a_x\Delta t'/\gamma^3} \tag{116}$$

so the lab-frame acceleration in the direction parallel to the direction of motion is diminished with respect to the parallel component of the proper acceleration by three powers of $1/\gamma$.

7.3 Trajectory of a uniformly accelerating particle

If we consider a particle with constant proper acceleration a parallel to the x-axis (and now denote lab-frame coordinates by un-primed symbols) the equation of motion is

$$\ddot{x} = a/\gamma^3 = a(1 - \dot{x}^2/c^2)^{3/2} \tag{117}$$

a solution to which is the hyperbola

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$$x(t) = \sqrt{X^2 + c^2t^2} \tag{118}$$

with

$$X \equiv c^2/a. \tag{119}$$

This is easily verified since differentiating the above gives $\dot{x} = c^2t/x$ so $\gamma^{-2} = 1 - \dot{x}^2/c^2 = 1 - c^2t^2/x^2$, and differentiating again yields $\ddot{x} = c^2/x - c^4t^2/x^3 = c^2/x\gamma^2$. But $X^2 = x^2 - c^2t^2 = x^2(1 - c^2t^2/x^2) = x^2/\gamma^2$ or $x = \gamma X$. Hence $\ddot{x} = c^2/X\gamma^3$, which accords with the equation of motion if $X \equiv c^2/a$.

This is a specific solution. The general solution, involving two constants of integration, is obtained by replacing $x \to x - x_0$ and $t \to t - t_0$ for an arbitrary constant t_0 and x_0.

One may note that one physical significance of the distance $X = c^2/a$ is that if you travel this distance you will reach a velocity $v \sim c$. Another interesting property of X, and of these trajectories, is that the proper length of the vector $\vec{x} \longrightarrow (c(t - t_0), x - x_0, 0, 0)$ is

$$s^2 = \vec{x} \cdot \vec{x} = -c^2(t - t_0)^2 + (x - x_0)^2 = X^2 \tag{120}$$

so a particle moving with constant acceleration maintains a constant proper distant from the point (ct_0, x_0).

. .4 Rindler space-time

We now introduce a fascinating model invented by Wolfgang Rindler, which is the ordinary empty *Minkowski space-time* of special relativity but with the *line element* $ds^2 = \eta_{\alpha\beta} dx^\alpha dx^\beta$ expressed in terms of coordinates tied to observers who are being uniformly accelerated in the manner described above.

Rindler invented this in order to elucidate some of the peculiar properties of the Schwarzschild and other black-hole solutions of general relativity, but it also proves useful to show how the laws of physics appear to an accelerating observer (such as an astronaut in a rocket or, according to Einstein, us standing here on the Earth).

In this model, we consider a family of particles accelerating in the x-direction, each having constant y and z, and all with $(t_0, x_0) = (0, 0)$ but with different accelerations (and therefore different minimum x-coordinates – the distance $X = c^2/a$ being the minimum value of $x(t)$; the intercept of the trajectory with the x-axis) . These trajectories therefore foliate a part of the full Minkowski space-time – what is known as the *Rindler wedge* – as illustrated in figure 12.

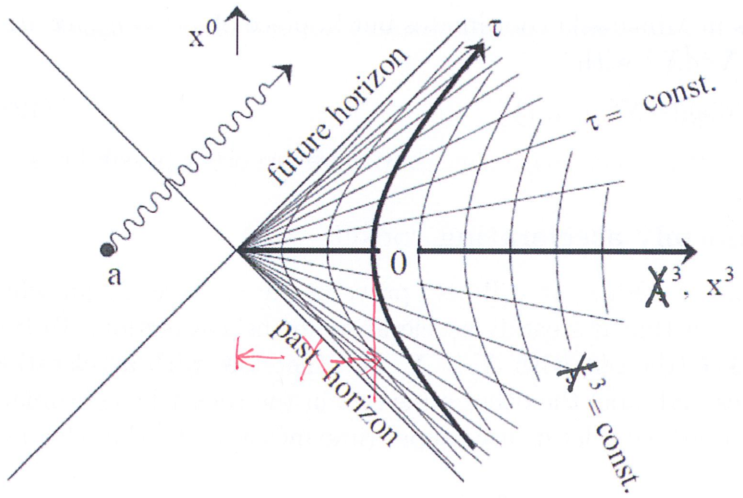

Figure 12: Rindler space-time - we use X and ν rather than ξ and τ used here. This shows, in flat Minkowskian space-time, the trajectories of a set of observers each undergoing constant acceleration. Their world-lines are a set of hyperbolae $x^2 = c^2t^2 + X^2$ where $X = c^2/a$ labels the particles. So they have the same asymptotic trajectories $\pm x \to \pm ct$ as $t \to \pm\infty$. These world-lines foliate Minkowski space, but only part of it. There are what are, in a sense, *apparent horizons* as indicated. A photon emitted from an event such as a will never reach any of the accelerated observers.

The trajectories of the *Rindler observers* can also be expressed parametrically as $\cosh^2 - \sinh^2 = 1$

$$\begin{aligned} t &= (c/a)\sinh(a\tau/c) = (X/c)\sinh(c\tau/X) \\ x &= (c^2/a)\cosh(a\tau/c) = X\cosh(c\tau/X) \end{aligned} \tag{121}$$

where the parameter τ is actually the proper time since, for fixed X, $dt = \cosh(c\tau/X)d\tau$ and $dx = c\sinh(c\tau/X)d\tau$ so $ds^2 = -c^2dt^2 + dx^2 = c^2(-\cosh^2(c\tau/X) + \sinh^2(c\tau/X))d\tau^2$. But $\cosh^2 - \sinh^2 = 1$, so this says $ds^2 = -c^2d\tau^2$. The parameterisation chosen here implies that the clocks of the observers all read the same time $\tau = 0$ as the particles cross the hypersurface $t = 0$.

We will express the line element in terms of *Rindler coordinates*

$$X^\alpha(\vec{x}) = (X^0, X, Y, Z) = (X^0(t,x), X(t,x), y, z) \tag{122}$$

so the observers maintain constant spatial coordinates $\mathbf{X} = (X, y, z)$ and these coincide with the Minkowski spatial coordinates at $t = 0$.

There are different versions of the *Rindler metric* as there are various choices of 'time' coordinate X^0. All are interesting, and reveal different things.

7.5 The Rindler-space metric

In Rindler coordinates, the 'time' is taken to be

$$X^0 = T \equiv c\tau/X \tag{123}$$

i.e. the (dimensionless) argument of the hyperbolic functions in the parametric solutions.

This increases along the world-lines of the observers in proportion to the proper time of clocks that they carry (these being assumed to be set to $\tau = 0$ at $t = 0$), which is sensible for a 'time' coordinate, but it does so at a rate that is different for the different observers.

not GR $\left(g = -\nabla\phi\right)$ $\Delta \ddot{x} = \Delta x \nabla\nabla\phi$

An explicit expression for X^0 in terms of Minkowski t, x coordinates is readily seen to be

$$X^0(t,x) = T(t,x) = \tanh^{-1}(ct/x). \tag{124}$$

The *Rindler metric* is obtained by differentiating the parametric expressions for t and x (121). This gives

$$\begin{aligned} cdt &= d(X\sinh T) = SdX + XCdT \\ dx &= d(X\cosh T) = CdX + XSdT \end{aligned} \tag{125}$$

where $C \equiv \cosh T$ and $S \equiv \sinh T$. The other differentials are trivially $dy = dY$ and $dz = dZ$, and squaring and combining these gives

$$\begin{aligned} ds^2 &= -c^2dt^2 + dx^2 + dy^2 + dz^2 \\ &= -(SdX + XCdT)^2 + (CdX + XSdT)^2 + dY^2 + dZ^2 \end{aligned} \tag{126}$$

from which we readily obtain

$$\boxed{ds^2 = -X^2dT^2 + dX^2 + dY^2 + dZ^2} \tag{127}$$

so the metric in these coordinates is diagonal, as in Minkowski coordinates but in place of $ds^2 = \eta_{\alpha\beta}dx^\alpha dx^\beta$ with $\eta_{\alpha\beta} = \text{diag}(-1,1,1,1)$ we have $ds^2 = g_{\alpha\beta}dX^\alpha dX^\beta$ with

$$g_{\alpha\beta} = \text{diag}(-X^2,1,1,1). \tag{128}$$

This is what is most commonly called the metric of Rindler space-time, but there are other possibilities.

Tidal: $\frac{\partial^2\phi}{\partial x_i \partial x_j}$

7.6 The metric of space-time in a uniformly accelerating rocket

The metric above is valid within the entire Rindler wedge. It will also prove useful to have a somewhat simplified version of this that is valid inside a rocket that is steadily accelerating in the x-direction. To this end, let us take a *reference observer* in the rocket (the one with $X = X_o$ and therefore with acceleration $a = c^2/X_o$) to be the origin of spatial coordinates and label the other observers in the rocket by coordinate $\mathbf{x} \equiv (x,y,z) = (X - X_o, YZ)$. And for the time coordinate, let us use proper time measured by the reference observer, $dt = X_o dT$.

The metric coefficient $g_{TT} = X^2$ is therefore equal to $(X_o + x)^2$ and we therefore have for the time part of the invariant squared interval $g_{TT}dT^2 = X^2dT^2 = (X/X_o)^2dt^2 = (1 + x/X_o)^2dt^2$ so the line-element, in rocket coordinates $x^\alpha = (ct,x,y,z)$ is

$$ds^2 = -(1 + ax/c^2)^2c^2dt^2 + dx^2 + dy^2 + dz^2 \tag{129}$$

where $a = c^2/X_o$ is the acceleration of the reference observer (or any of the observers with $x = 0$). Writing $ds^2 = g_{\alpha\beta}dx^\alpha dx^\beta$ the metric in rocket coordinates is

$$ds^2 = \text{diag}(-(1 + ax/c^2)^2, 1, 1, 1). \tag{130}$$

As we will discuss in more detail later, *Einstein's equivalence principle* (EEP) states that the physics seen by an observer in a gravitational field is the same – aside from tidal effects, which will be small if the field is nearly constant; as for a small observer standing on a large planet – as that which would be seen by an observer in a rocket with the same acceleration. Thus the metric we have obtained above can be used to calculate things – like the trajectories of freely falling particles and pressure gradients in gas in hydrostatic equilibrium – in a gravitational field.

In Einsteinian gravity, the gravitational field is the curvature of space-time. His field equations give, for instance, the curvature created by a given matter distribution. And the curvature is encoded in the metric. These statements might lead one to think that what we have constructed above is a metric for a curved space-time. But that is not the case; we have expressed the metric in coordinates – very natural ones from the perspective of accelerated observers – for which the metric does not have the simple Minkowskian form. But it is still the metric of flat space-time. This is like using polar coordinates in planar geometry, where the metric is $dl^2 = dr^2 + r^2d\phi^2$ where the space remains flat. That is closely analogous to Rindler space-time where on a surface of constant Y and Z the metric is $ds^2 = -X^2dT^2 + dX^2$, so, aside from the extra minus sign to make the geometry locally Lorentzian rather than Euclidian, this is like polar coordinates with $r \Rightarrow X$ and $T \Rightarrow \phi$. The analogy can be made closer still if we replace time by ict, in which coordinates the Lorentz boost matrix looks very much like a rotation matrix. But that is deprecated.

We now turn to consider what the equations of motion of particles look like using Rindler and other coordinate systems.

8 Free particle trajectories as extremal paths

An unaccelerated particle in Minkowski space time that goes from one event $\vec{x}_{\rm A}$ to another $\vec{x}_{\rm B}$ (these being assumed to have a time-like separation) takes a 'straight-line' path (in Minkowski coordinates) $x^\alpha(\tau) = x^\alpha_{\rm A} + (\tau - \tau_{\rm A})(x^\alpha_{\rm B} - x^\alpha_{\rm A})/(\tau_{\rm B} - \tau_{\rm A})$. An accelerated particle will – as is well known from the *twin paradox* – take a shorter proper time. That's because, in the frame where $\mathbf{x}_{\rm B} = \mathbf{x}_{\rm A}$, the accelerating observer will have some finite velocity for at least some of its path so, relative to the unaccelerated observer, his time is dilated (his clock runs slow).

Unaccelerated – or 'inertial' – particles are therefore those for which

$$\delta \int d\tau = 0 \tag{131}$$

i.e. they follow *paths of extremal proper time*, also known as *geodesics*. This is a bit like *Fermat's principle of least time* for photons – though here the proper time is maximised.

What would the path be in rocket (or Rindler) coordinates? One way to answer this would be able to simply use $T = \tanh^{-1}(ct/x)$ and $X = x/\cosh(T)$. A more interesting way is to develop and solve the equations of motion for a particle in rocket coordinates.

8.1 Equation of motion for inertial particles

To obtain the equations of motion, let's parameterise the path by λ, so $x^\alpha = x^\alpha(\lambda)$. The path whose equations of motion we seek satisfies (131). Since $d\tau^2 = -ds^2$, the interval of proper time corresponding to the interval of path parameter $d\lambda$ is

$$d\tau = d\lambda \sqrt{-g_{\alpha\beta}\dot{x}^\alpha \dot{x}^\beta} \tag{132}$$

where $\dot{x}^\alpha \equiv dx^\alpha/d\lambda$, so we can replace (131) by

$$\delta \int d\lambda L(x^\alpha, \dot{x}^\alpha) = 0. \tag{133}$$

where we are defining

$$L(x^\alpha, \dot{x}^\alpha) \equiv \sqrt{-g_{\alpha\beta}(\vec{x})\dot{x}^\alpha \dot{x}^\beta} = \frac{d\tau}{d\lambda}. \tag{134}$$

This looks like a problem in non-relativistic classical mechanics where, for example for a particle moving under the influence of a potential $V(\mathbf{x})$, the action is $S = \int dt L(\mathbf{x}, \dot{\mathbf{x}})$ where the Lagrangian is $L = K - V$ where the kinetic energy is $K = \frac{1}{2}m\dot{\mathbf{x}}^2$.

Just as in non-relativistic mechanics, the equations of motion are obtained by requiring that if we vary the actual path $\vec{x}(\lambda)$ as illustrated in figure 13, there is no change, to first order, in S.

Here the change in the 'Lagrangian' is

$$\begin{aligned} \delta L &= \frac{\partial L}{\partial x^\alpha}\delta x^\alpha + \frac{\partial L}{\partial \dot{x}^\alpha}\delta \dot{x}^\alpha \\ &= \frac{\partial L}{\partial x^\alpha}\delta x^\alpha + \frac{d}{d\lambda}\left(\frac{\partial L}{\partial \dot{x}^\alpha}\delta x^\alpha\right) - \delta x^\alpha \frac{d}{d\lambda}\left(\frac{\partial L}{\partial \dot{x}^\alpha}\right) \end{aligned} \tag{135}$$

where we have performed a trick we will use frequently: we eliminate $\delta\dot{x}^\alpha$ in favour of δx^α by 'hiding' it in a term that is a total derivative, and which will, one line from now, magically disappear.

The variation of the proper time is

$$\delta \int d\tau = \int d\lambda \delta L = \left[\frac{\partial L}{\partial \dot{x}^\alpha}\delta x^\alpha\right]_{\lambda_{\rm A}}^{\lambda_{\rm B}} + \int d\lambda \delta x^\alpha \left[\frac{\partial L}{\partial x^\alpha} - \frac{d}{d\lambda}\left(\frac{\partial L}{\partial \dot{x}^\alpha}\right)\right] \tag{136}$$

but δx^α vanishes at the end points, so the first term above vanishes, and requiring that $\delta \int d\tau$ vanishes for otherwise arbitrary $\delta x^\alpha(\lambda)$ gives the equations of motion (what are called the *Euler-Lagrange equations* in mechanics) $[\ldots] = 0$:

$$\frac{d}{d\lambda}\left(\frac{\partial L}{\partial \dot{x}^\alpha}\right) = \frac{\partial L}{\partial x^\alpha}. \tag{137}$$

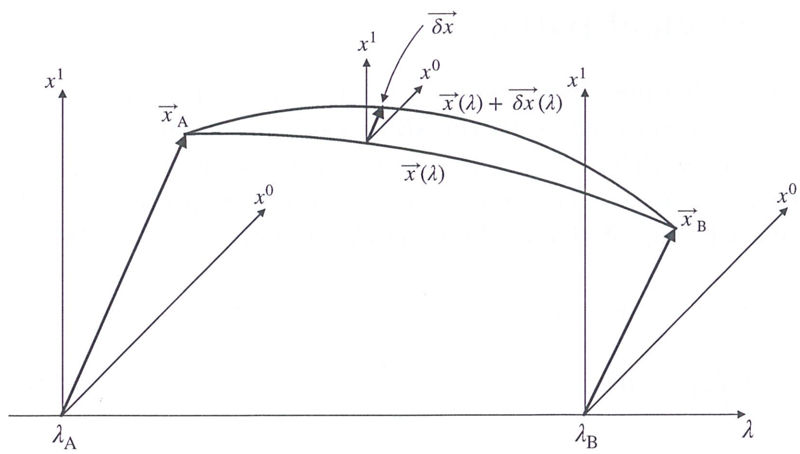

Figure 13: To obtain the equations of motion for a path that extremises $\int d\tau = \int L d\lambda$ we consider a small perturbation to the path $\vec{x}(\lambda) \Rightarrow \vec{x}(\lambda) + \vec{\delta x}(\lambda)$ and calculate the 1st order change in L: $L \Rightarrow L + \delta L$ and hence the 1st order change in $\delta \int d\tau = \int d\lambda \delta x^\alpha [\ldots]$. Requiring this vanishes gives the Euler-Lagrange equations.

Using the definition of L in (134) the right hand side of this (what is called the *generalised force* in mechanics) is

$$\frac{\partial L}{\partial x^\alpha} = -\frac{g_{\mu\nu,\alpha}\dot{x}^\mu \dot{x}^\nu}{2L} \tag{138}$$

while the left hand side is

$$\frac{d}{d\lambda}\left(\frac{\partial L}{\partial \dot{x}^\alpha}\right) = -\frac{d}{d\lambda}\frac{g_{\alpha\beta}\dot{x}^\beta}{L} = -\frac{g_{\alpha\beta,\mu}\dot{x}^\mu \dot{x}^\beta + g_{\alpha\beta}\ddot{x}^\beta}{L} + \frac{g_{\alpha\beta}\dot{x}^\beta}{L^2}\frac{dL}{d\lambda} \tag{139}$$

where we have used $dg_{\alpha\beta}/d\lambda = (\partial g_{\alpha\beta}/\partial x^\mu)(dx^\mu/d\lambda) = g_{\alpha\beta,\mu}\dot{x}^\mu$.

This is a bit complicated, but can be simplified by exploiting the freedom of parameterisation of the path – which was only constrained by the coordinates at the end-points being $x^\alpha(\lambda_A) = x^\alpha_A$ and $x^\alpha(\lambda_B) = x^\alpha_B$ – by demanding that $dL/d\lambda = 0$ or, equivalently, since $L = d\tau/d\lambda$,

choose affine parameter

$$d^2\tau/d\lambda^2 = 0 \tag{140}$$

so we are demanding that the path parameter λ be the same, up to a constant multiplicative factor and a shift of zero-point, as the proper time. I.e. λ is an affine parameter.

That gets rid of the second term on the left hand side and we obtain (replacing $g_{\alpha\beta,\mu}\dot{x}^\mu \dot{x}^\beta$ by $\frac{1}{2}(g_{\alpha\nu,\mu} + g_{\alpha\mu,\nu})\dot{x}^\mu \dot{x}^\nu$)

$$g_{\alpha\beta}\ddot{x}^\beta = -\tfrac{1}{2}(g_{\alpha\nu,\mu} + g_{\alpha\mu,\nu} - g_{\mu\nu,\alpha})\dot{x}^\mu \dot{x}^\nu \tag{141}$$

or, finally, matrix multiplying by $g^{\gamma\alpha}$, the matrix with components that are the inverse of the metric, so $g^{\gamma\alpha}g_{\alpha\beta} = \delta^\gamma_\beta$,

$$\boxed{\frac{d^2 x^\gamma}{d\lambda^2} = -\tfrac{1}{2}g^{\gamma\alpha}(g_{\alpha\nu,\mu} + g_{\alpha\mu,\nu} - g_{\mu\nu,\alpha})\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}} \tag{142}$$

which is called the *geodesic equation*.

8.1.1 Geodesic equation for the contravariant components of the 4-momentum

The geodesic equation (142) is a 2nd order equation for the the contravariant coordinate x^α. It can also be thought of as a 1st order equation for the contravariant components of the 4-momentum. If we let $\lambda = \tau/m$ (or $\lambda = \lim_{m\to 0} \tau/m$ for an ultra-relativistic or massless particle) it says that the 4-momentum components $p^\alpha = dx^\alpha/d\lambda$ obey

$$\boxed{dp^\gamma/d\lambda = \underbrace{-\tfrac{1}{2}g^{\gamma\alpha}(g_{\alpha\nu,\mu} + g_{\alpha\mu,\nu} - g_{\mu\nu,\alpha})}_{\Gamma^\gamma_{\alpha\beta}}p^\mu p^\nu} \tag{143}$$

8.1.2 Geodesic equation for the covariant components of the 4-momentum

The *covariant version of the geodesic equation* is somewhat simpler. We define the covariant 4-momentum components by

$$p_\beta = g_{\beta\gamma}p^\gamma \tag{144}$$

L.I. $f(x,p)$, $\frac{d^3p}{p^0}$, $E\frac{d}{dt}$

so we are using the metric just as the Minkowski metric is used in Minkowski coordinates as an index lowering operator. Taking the derivative with respect to λ gives

$$
\begin{aligned}
dp_\beta/d\lambda &= p^\gamma dg_{\beta\gamma}/d\lambda + g_{\beta\gamma}dp^\gamma/d\lambda \\
&= g_{\beta\gamma,\mu}p^\mu p^\gamma - \tfrac{1}{2}g_{\beta\gamma}g^{\gamma\alpha}(g_{\alpha\nu,\mu} + g_{\alpha\mu,\nu} - g_{\mu\nu,\alpha})p^\mu p^\nu
\end{aligned}
\tag{145}
$$

where we have used $dg_{\beta\gamma}/d\lambda = g_{\beta\gamma,\mu}dx^\mu/d\lambda = g_{\beta\gamma,\mu}p^\mu$ and, in the second term, used (143) for $dp^\gamma/d\lambda$. But $g_{\beta\gamma}g^{\gamma\alpha} = \delta^\alpha_\beta$ and we can rewrite the first term as $g_{\beta\gamma,\mu}p^\mu p^\gamma = \frac{1}{2}(g_{\beta\nu,\mu} + g_{\beta\nu,\mu})p^\mu p^\nu$, so this cancels the first two terms in the second expression to give

$$
\boxed{dp_\gamma/d\lambda = \tfrac{1}{2}g_{\mu\nu,\gamma}p^\mu p^\nu.}
\tag{146}
$$

This is very useful. It tells us that if all of the components of the metric are independent of the γ^{th} coordinate then p_γ is constant along the trajectory. We will use this in appendix C to calculate the turning point for particles fired upwards in an accelerating frame.

CMB 300 km/s

Radio flux density Catalogue ≈ 1000 km/s

A Problems involving photons Conflict/Tension

A.1 The Doppler Shift

Consider a photon with energy in the 'lab' or 'observer' frame $E_{\text{obs}} = cp^0$ and moving in the direction $\hat{\mathbf{p}} = \hat{\mathbf{n}}(\theta, \phi)$ where θ and ϕ are the usual polar angles.

《最初三分钟》 (核合成)

From $\vec{p} = |\mathbf{p}|(1, \hat{\mathbf{p}})$ the lab-frame 4-momentum is

$$
cp^\alpha = E_{\text{obs}}(1, \cos\theta, \sin\theta\cos\phi, \sin\theta\sin\phi).
\tag{147}
$$

Let the emitter be moving with respect to the lab-frame with velocity v in the $+x$ direction. Applying a boost show that the cp^0 in the emitter frame (i.e. the energy in the emitter frame) is

R. Wagoner Les Houches

$$
E_{\text{em}} = \gamma(1 - (v/c)\cos\theta)E_{\text{obs}}
\tag{148}
$$

so the *red-shift* is

$$
1 + z \equiv \lambda_{\text{obs}}/\lambda_{\text{em}} = \nu_{\text{em}}/\nu_{\text{obs}} = E_{\text{em}}/E_{\text{obs}} = \gamma(1 - (v/c)\cos\theta).
\tag{149}
$$

$E\frac{df(P_1)}{dt} = \int\frac{d^3P_2}{P_2^0}\int\frac{d^3P_3}{P_3^0}\int\frac{d^3P_4}{P_4^0}$

A.1.1 The radial Doppler shift

If the source is moving in the same direction as the photon then $\cos\theta = 1$ and we have, for the *radial Doppler shift*

$$
1 + z = \sqrt{(c - v)/(c + v)}
\tag{150}
$$

which is greater than unity (red-shift) for a source moving away from the observer.

This is a non-linear function of the relative speed of the emitter and observer. For low velocities, there is a component

$$
z \simeq v/c
\tag{151}
$$

which is first order in the speed.

A.1.2 The transverse Doppler shift

For a source moving perpendicular to the direction the observer sees the photon coming from ($\cos\theta = 0$) this is also a redshift:

$$
1 + z = \gamma = 1/\sqrt{1 - |\mathbf{v}|^2/c^2}
\tag{152}
$$

known as the *transverse Doppler redshift*, and which for small $|\mathbf{v}| \ll c$ is quadratic

$$
z = \gamma - 1 \simeq \tfrac{1}{2}|\mathbf{v}|^2/c^2.
\tag{153}
$$

This is therefore much smaller – at low speeds – than the typical 1st order effect if the relative velocity has any appreciable line of sight components.

However, if we observe a source composed of many individual sources that are randomly moving there can be statistical cancellation of the 1st order effect and the transverse effect may come to dominate.

A.1.3 Why is the transverse Doppler effect a red-shift?

Imagine a rocket travelling along the $y-$axis which emits a photon as it passes $y = 0$ whose energy is measured by an observer who is sitting somewhere on the $+x$ axis (as illustrated in figure 14)

The above formula says the observed energy is less than that emitted: $E_{\rm obs} = E_{\rm em}/\gamma$. But the rocket had to destroy some rest-mass to create that radiation. And that mass was moving relative to the observer, and so had lab-frame energy γmc^2: i.e. *greater* than mc^2. Thus, if energy is conserved shouldn't we see a *transverse Doppler blue-shift*?

What is going on? Similarly, one might consider the (rather fanciful) thought experiment illustrated in figure ??.

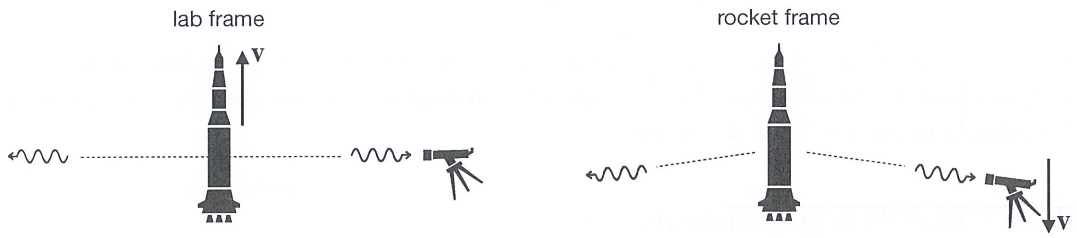

Figure 14: On the left is shown, in the 'lab-frame' two photons created by the conversion of some mass into energy in a rocket. The rocket is moving, so in the lab-frame whatever rest-mass was converted had a higher energy by a factor γ. So the energy of the photons, as compared to their energy in the rocket frame, ought to be *boosted* by a factor γ, right? But the transverse Doppler shift formula says that, in this situtation, where the photon is moving perpendicular to the velocity $\mathbf{v}$ of the source, the photon energies should be *decreased.*

A.2 Aberration and relativistic beaming

Consider a source emitting unit momentum photons with angles (polar and azimuthal) θ, ϕ, so

$$p^\alpha = (1, \cos\theta, \sin\theta\cos\phi, \sin\theta\sin\phi) \tag{154}$$

If the source is moving at speed $v = \beta c$ along the $+x$ axis, boosting p^α into the lab-frame (primed-frame) yields

$$p^{\alpha'} = (\gamma(1 + \beta\cos\theta), \gamma(\beta + \cos\theta), \sin\theta\cos\phi, \sin\theta\sin\phi) \tag{155}$$

but this is equally

$$p^{\alpha'} = |\mathbf{p}'|(1, \cos\theta', \sin\theta'\cos\phi', \sin\theta'\sin\phi') \tag{156}$$

Thus, comparing the time components we have

$$|\mathbf{p}'| = \gamma(1 + \beta\cos\theta) \tag{157}$$

while comparing the expressions for $p^{1'}$ gives

$$\cos\theta' = \gamma(\beta + \cos\theta)/|\mathbf{p}'| = (\beta + \cos\theta)/(1 + \beta\cos\theta) \tag{158}$$

Consider photons emitted in the equatorial plane in the emitter frame (i.e. with $\cos\theta = 0$). These have lab-frame direction

$$\cos\theta' = \beta \tag{159}$$

If the source is rapidly moving, $\beta \simeq 1$ and these photons therefore emerge with very small angle θ' from the direction of motion. For $\theta' \ll 1$ we have $\cos\theta' \simeq 1 - (\theta')^2/2$ so $(\theta')^2 \simeq 2(1 - \beta)$ while using the definition of $\gamma = 1/\sqrt{1-\beta^2} = 1/\sqrt{(1+\beta)(1-\beta)} \simeq 1/\sqrt{2(1-\beta)}$ and hence

$S_\nu = I_\nu d\Omega$

$\left[\frac{I_\nu}{\nu^3}\right] = \frac{I_\nu'}{\nu'^3} \Rightarrow$ L. I.

$$\theta' \simeq \sqrt{2(1-\beta)} = 1/\gamma \tag{160}$$

$S_{\nu'} = I_\nu \left(\frac{\nu'^3}{\nu^3}\right) d\Omega \frac{1}{(1+2\beta\mu)} \approx I_\nu d\Omega (1+\beta\mu)$

Since half of the photons emerge in the forward-moving hemisphere in the emitter frame ($\cos\theta > 0$) that means that, for $\gamma \gg 1$, half of the photons will be at $\theta' < 1/\gamma$; i.e. very strongly beamed.

$\nu' = \gamma(1+\beta\mu)\nu$

d

$\beta = \cos\theta$

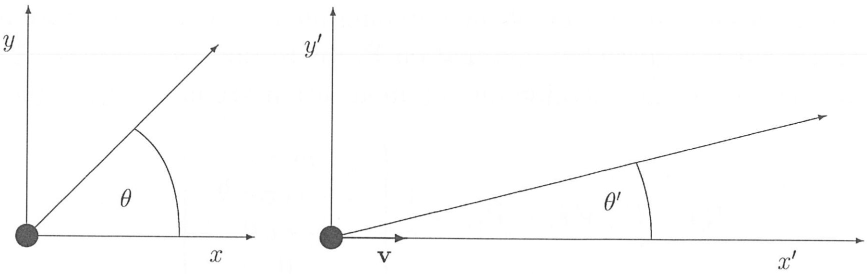

Figure 15: A photon (arrow) emitted from a source (moving to the right) in the source-frame (left) and observer-frame (right). For highly relativistic particles the photons tend to emerge tightly 'beamed' in the forward direction

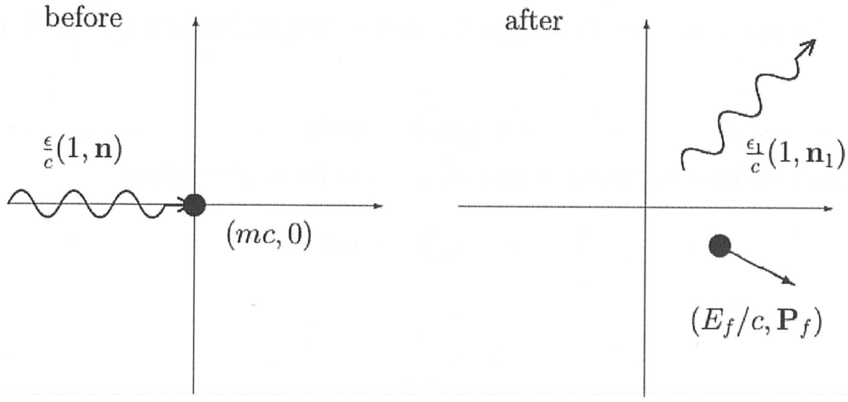

Figure 16: Four-momenta of particles involved in a Compton scattering event, working in a frame such that the electron is initially at rest and the initial photon direction is $\mathbf{n} = (1, 0, 0)$. The final photon direction is $\mathbf{n}_1 = (\cos\theta, \sin\theta\cos\phi, \sin\theta\sin\phi)$.

A.3 Kinematics of Compton Scattering

Consider the scattering of a photon by an electron, as illustrated in figure 16.

Suitable null 4-vectors to represent the initial and final photon 4-momenta are

$$\vec{P}_{\gamma i} = \frac{\epsilon}{c}\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \quad \text{and} \quad \vec{P}_{\gamma f} = \frac{\epsilon_1}{c}\begin{bmatrix} 1 \\ \cos\theta \\ \sin\theta\cos\phi \\ \sin\theta\sin\phi \end{bmatrix} \tag{161}$$

where ϵ denotes the energy, the subscript 1 denotes the outgoing photon state, (ie after one scattering) and we have chosen the initial photon have momentum parallel to the x-axis.

Similarly, the 4-momenta for the initial and final electron states are

$$\vec{P}_{\mathrm{e}i} = \begin{bmatrix} mc \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{and} \quad \vec{P}_{\mathrm{e}f} = \begin{bmatrix} E/c \\ P_x \\ P_y \\ P_z \end{bmatrix} \tag{162}$$

where we are working in the rest-frame of the initial electron. These 4-momenta are illustrated in figure 16.

Conservation of the total 4-momentum is

$$\vec{P}_{\gamma i} + \vec{P}_{\mathrm{e}i} = \vec{P}_{\gamma f} + \vec{P}_{\mathrm{e}f}. \tag{163}$$

If we specify the incoming momenta $\vec{P}_{\mathrm{e}i}$ and $\vec{P}_{\gamma i}$ then the outgoing 4-momenta contain six free parameters, ϵ_1, θ and ϕ for the photon and $\mathbf{P}_{\mathrm{e}f}$ for the electron (with the electron energy then fixed by the mass-shell condition $E^2 = p^2c^2 + m^2c^4$). If we specify the direction θ, ϕ of the outgoing photon say, then equation (163) provides us with the necessary four constraints to fully determine the collision (ie the energy of the photon and the 3-momentum of the outgoing electron).

If we simply want to determine the energy of the outgoing photon ϵ_1, then we only need one equation. A convenient way to throw out the unwanted information $\mathbf{P}_{ef}$ is to take the norm of $\vec{P}_{ef}$. If we orient our spatial coordinate system $\phi = 0$, so the outgoing photon momentum lies in the $x - y$ plane, then $\vec{P}_{ef}$ is

$$\vec{P}_{ef} = P_{\gamma i} + \vec{P}_e - \vec{P}_{\gamma f} = \frac{1}{c} \begin{bmatrix} \epsilon + mc^2 - \epsilon_1 \\ \epsilon - \epsilon_1 \cos\theta \\ \epsilon_1 \sin\theta \\ 0 \end{bmatrix} \tag{164}$$

and the mass-shell requirement $E_{ef}^2 = c^2 |\mathbf{P}_{ef}|^2 + m^2 c^4$ becomes

$$(\epsilon + mc^2 - \epsilon_1)^2 = (\epsilon - \epsilon_1 \cos\theta)^2 + (\epsilon_1 \sin\theta)^2 + m^2 c^4. \tag{165}$$

Which is a single equation one can solve for ϵ_1 given ϵ and θ. Expanding out the products and reordering gives

$$\epsilon_1 = \frac{\epsilon}{1 + \frac{\epsilon}{mc^2}(1 - \cos\theta)} \tag{166}$$

and expressing the photon energies in terms of wavelength $\epsilon = h\nu = hc/\lambda$ gives

$$\lambda_1 - \lambda = \lambda_c (1 - \cos\theta) \tag{167}$$

where the parameter

$$\lambda_c \equiv \frac{h}{mc} \tag{168}$$

is the *Compton wavelength*.

Equations (166,167) describe the energy loss for photons scattering off stationary electrons. They show that the collision is effectively elastic (ie $\epsilon_1 \simeq \epsilon$) if $\epsilon \ll mc^2$.

B The Collisional Boltzmann equation

The *collisional Boltzmann equation* is widely used in cosmology (for example in *big-bang nucleosynthesis* and in calculation of the evolution of density perturbations during the all important epoch when the universe is *decoupling*). It provides a nice example of how one can construct relativistically covariant (and quantum mechanically correct) equations in the *3+1 formalism* described earlier.

This can be obtained, in the context of a 'toy' model of a self-interacting scalar field as follows:

- we start with a non-interacting field, which can be decomposed as a sum of Fourier modes $\phi = \sum_{\mathbf{k}} \phi_{\mathbf{k}} e^{i\vec{k}\cdot\vec{x}}$, each of which obeys a simple harmonic oscillator equation

- this has *occupation number eigenstates* $|\ldots n_{\mathbf{k}} \ldots\rangle$ where $n_{\mathbf{k}}$ is the occupation number for the $\mathbf{k}^{\text{th}}$ mode

- we then add interactions to the *free-field Lagrangian density* (see chapter ??) – for this toy model $\mathcal{L}_{\text{int}} = -\lambda \phi^4$

- and integrate the *non-relativistic Schrödinger equation* to calculate the *quantum mechanical amplitude* $\langle \ldots n'_{\mathbf{k}} \ldots | \ldots n_{\mathbf{k}} \ldots \rangle$ for the system to transition to a state with a different set of occupation numbers (where e.g. a pair of particles have scattered out of states $\mathbf{k}_1$ and $\mathbf{k}_2$ into the states $\mathbf{k}_3$ and $\mathbf{k}_4$

- squaring this gives the probability (per unit time) for the reaction (or its inverse) which is a Lorentz covariant entity where the rate depends not only on the density of 'reactants' but also contains factors $1 \pm n_{\mathbf{k}}$ (with minus sign for Fermions) for the final state. The rate also includes a energy and momentum conserving Dirac δ-function $\delta(\mathbf{k}_1 + \mathbf{k}_2 - \mathbf{k}_3 - \mathbf{k}_4)$

- the resulting collisional Boltzmann equation is a relativistically covariant, quantum mechanically correct, deterministic equation giving the rate of change of the occupation of the mode $\mathbf{k}_1$ (equivalent to what we have been calling the phase-space density $f(\mathbf{k})$ above) as an integral over the other three 4-momenta.

The result is shown in figure 17 where we see, on the left hand side, the Lorentz invariant operator $E_1\partial/\partial t$ acting on $n_{\mathbf{k}_1}$, being the phase-space density for particles with momentum $\mathbf{p} = \hbar\mathbf{k}$. On the right is a manifestly Lorentz invariant – if ugly – 9-dimensional integral involving the phase-space densities for the other momenta involved. After integrating out the delta-function, we get a 5-dimensional integration. This is reasonable; the rate at which particles are being scattered out of state $\mathbf{p}_1$ depends on the three components of the other incoming particle ($\mathbf{p}_2$) and on the *direction* of one of the other particles (say $\hat{\mathbf{p}}_3$); the value of its modulus $|\mathbf{p}_3|$ and the 3 components of the 3-momentum $\mathbf{p}_4$ of the 4th particle being set by conservation of total 4-momentum.

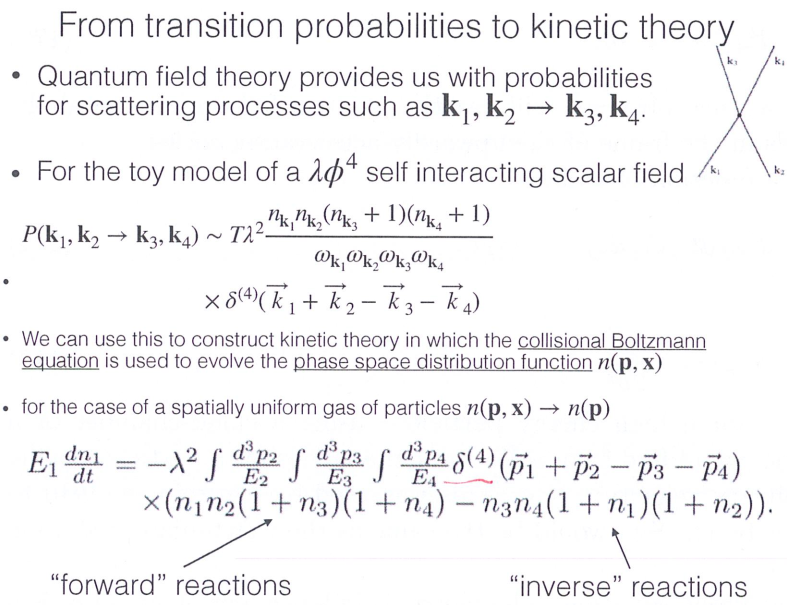

Figure 17: Quantum field theory gives Lorentz invariant differential scattering cross-section – here illustrated for a 'toy' model of a self-interacting scalar field – containing an energy and momentum conserving Dirac δ-function. Stimulated emission and Fermi blocking factors also emerge naturally from field theory. This results in the collisional Boltzmann equation: a deterministic equation for evolving the phase-space density (here for the case that the space-density is uniform, so $f(\mathbf{x}, \mathbf{p}) = n_\mathbf{k}$). This is, by construction, manifestly Lorentz-invariant.

cross section : ← 场论

The collisional Boltzmann equation can be used, as well as in the cosmological applications mentioned above, to obtain e.g. the Bose-Einstein distribution and Fermi-Dirac distribution; the equilibrium distributions functions $f(\mathbf{p})$ for which the net rate of reactions vanishes.

C Free particle trajectories in rocket coordinates

The geodesic equation above is very powerful, and applies for any metric. Let's apply it to that which describes space-time in our rocket coordinates, in which the only metric component that has any dependence on position is $g_{00} = -(1+ax/c^2)^2$ and it depends only on $x = x^1$. The metric is also diagonal, which simplifies matters, since its inverse is also diagonal, which greatly facilitates finding terms on the right hand side of the geodesic equation that are non-zero.

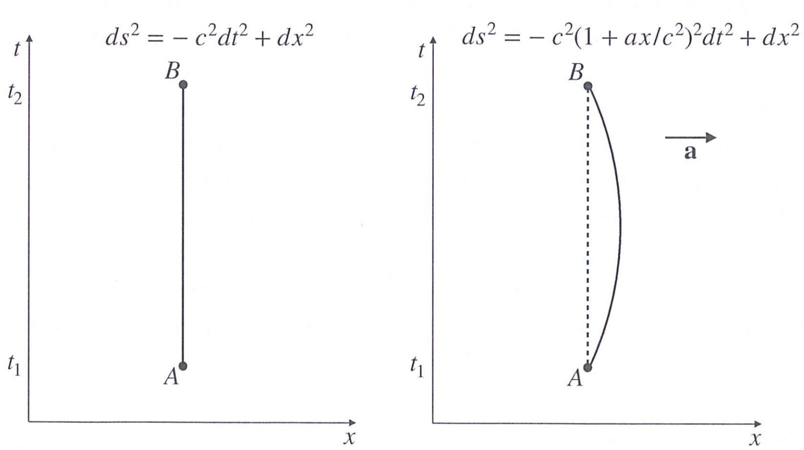

Figure 18: Extremal paths in a rocket. On the left is shown a space-time diagram in Minkowski coordinates, for which a particle trajectory – world-line of maximal proper time – would be a straight line (any other trajectory would have shorter elapsed proper time). On the right is indicated the metric in a frame such that the spatial coordinates are tied to an accelerated frame (such as in a rocket). In this case the extremal paths are displaced; the particle spends some time closer to the nose of the rocket, or higher up in a gravitational field, where time runs faster.

The situation is sketched in figure 18. Our goal is to make this quantitative using the geodesic equation.

Let's consider first a particle that is moving along the $x^3 = z$ axis, with some coordinate speed $v = dz/dt$ (which would be the physical speed measured by the reference observer as our time coordinate is proper time for that observer). For this particle we have then initially $\dot{x}^\alpha = (c\dot{t}, 0, 0, \dot{z})$ and the geodesic equation says that $\ddot{x}^0 = 0$, so $dx^0/d\lambda = c dt/d\lambda =$ constant. Similarly, $\ddot{x}^3 = 0$, so $dz/d\lambda =$ constant also. But $\ddot{x}^1 \neq 0$; the third term on the right hand side is non-zero for $\mu = \nu = 0$ and we have (using $g^{1\alpha} g_{\mu\nu,\alpha} = \delta^{\alpha 1} g_{\mu\nu,\alpha} = g_{\mu\nu}{}^{,1} = g_{\mu\nu,1}$)

$$d^2x/d\lambda^2 = \tfrac{1}{2} g_{00,x} (dx^0/d\lambda)^2 = -(1 + ax/c^2)(a/c^2)(dx^0/d\lambda)^2 \tag{169}$$

or, since $dx^0/d\lambda =$ constant, for a particle that is initially moving past the reference observer ($x = 0$)

$$d^2x/dt^2 = -a. \tag{170}$$

This is a not at all surprising result; the reference observer will see the particle following a parabolic trajectory $x = -at^2/2$; accelerating downwards in the frame of the upwardly accelerating rocket.

Similarly, the trajectory of the particle as a function of z obeys

$$d^2x/dz^2 = d^2x/dt^2 (dt/dz)^2 = -a/v^2 \tag{171}$$

giving the parabolic trajectory

$$x(z) = -\frac{az^2}{2v^2} \tag{172}$$

which is something that could be measured – for a high energy particle – using a cloud-chamber or a photographic emulsion. This is the same as one would find from a Newtonian analysis for a particle moving in a potential $\phi(x) = ax$, but applies for arbitrary velocity $v \leq c$. This was used by Einstein ca. 1910 to predict (incorrectly) that the deflection of light by the Sun would be the same as the Newtonian prediction for a test particle moving at $v = c$.

Now let's use the covariant version of the geodesic equation. The metric is independent of x^0, so p_0 is a constant of the motion of the particle. That means we can calculate the x-momentum p^x (or $p_x = p^x$) at any point on the trajectory using

$$-m^2c^2 = \vec{p} \cdot \vec{p} = g^{\mu\nu} p_\mu p_\nu = -(1 + ax/c^2)^{-2} p_0^2 + p_x^2 \tag{173}$$

where we have used the fact that the inverse metric is simply $g^{\alpha\beta} = \mathrm{diag}(-(1 + ax/c^2)^{-2}, 1, 1, 1)$.

We can use this to calculate the turning point, where $dx/d\lambda = p^x = p_x = 0$. Since $p_0^2 = m^2c^2 + |\mathbf{p}_i|^2$, where $\mathbf{p}_i$ is the initial 3-momentum, this occurs, if at all, at x such that

$$\sqrt{1 + |\mathbf{p}_i|^2/m^2c^2} = 1 + ax/c^2 \tag{174}$$

or

$$ax = c^2(\sqrt{1 + |\mathbf{p}_i|^2/m^2c^2} - 1) \tag{175}$$

for a non-relativistic particle (one with $|\mathbf{p}_i|^2/m \ll mc^2$), this says the particle turns around at $ax = |\mathbf{p}_i|^2/2m^2$, just as one would find in a gravitational field with $a = d\phi/dx$. If we hold the momentum fixed but let the mass become small, the turning point increases without limit. So an accelerated observer can never out-run a photon he emits.

The machinery developed here therefore nicely describes the kinematics of free particles as observed in an accelerating frame using the metric. The results are not particularly surprising, and could have been deduced in other ways. What is different is the viewpoint. If a rocketeer tosses a pebble upwards, an external observer would say that the accelerating rocketeer catches up with it. The alternative viewpoint, according to the metric in rocket coordinates, is that time is running faster higher up in the rocket. The path of the tossed pebble takes it upward from the observer and back again because, by spending some time higher up, its elapsed proper time is increased as compared to what (counterfactually) would have elapsed if it had simply stayed put. It shouldn't overdo it, however, because if it were to travel very fast going up and back it would suffer time-dilation caused by its velocity, and that would overwhelm any gain from being at higher altitude. The actual path is a compromise between these two competing effects.

D Energy and momentum continuity for a ideal fluid

The continuity equations

$$T^{\mu\nu}{}_{,\mu} = 0, \tag{176}$$

while succinct, rather obscure the physical content. Expressed in 3+1 form they are more revealing as they tell us firstly how the energy density of a 'parcel' of fluid changes in response to changes of its volume – as considered above, but in more generality – and secondly how the 3-velocity of the parcel changes in response to the pressure gradients.

Conservation of energy – the $\nu = 0$ component of this set of equations – is (multiplied by c^2 for convenience)

$$0 = c^2 T^{\mu 0}{}_{,\mu} = \frac{\partial}{\partial x^\mu}[(\mathcal{E} + P)u^0 u^\mu + c^2 P \eta^{0\mu}] = \frac{\partial}{\partial x^\mu}[(\mathcal{E} + P)u^0 u^\mu] - c\frac{\partial P}{\partial t} \tag{177}$$

where we have replaced x^0 by ct.

Conservation of the i^{th} component of the 3-momentum is expressed by setting $\nu = i$:

$$\begin{aligned} 0 = c^2 T^{\mu i}{}_{,\mu} &= \frac{\partial}{\partial x^\mu}[(\mathcal{E} + P)u^\mu u^i + c^2 P \eta^{\mu i}] \\ &= \frac{\partial}{\partial x^\mu}[(\mathcal{E} + P)u^\mu u^0 \beta^i] + c^2 \frac{\partial P}{\partial x^i} \\ &= \beta^i \frac{\partial}{\partial x^\mu}[(\mathcal{E} + P)u^0 u^\mu] + (\mathcal{E} + P)u^0 u^\mu \frac{\partial \beta^i}{\partial x^\mu} + c^2 \frac{\partial P}{\partial x^i} \end{aligned} \tag{178}$$

where, to obtain the second line we have used $u^i = u^0 \beta^i$ and $\eta^{i\mu} = \delta^{i\mu}$, and in the last step we have simply used the rule for differentiating a product.

From (177) we see that the first term on the right hand side here can be written as $c\beta^i \partial P/\partial t = v^i \partial P/\partial t$ and, with $u^0 = \gamma c$, (178) becomes

$$(\mathcal{E} + P)\gamma u^\nu \frac{\partial v^i}{\partial x^\nu} + c^2 \frac{\partial P}{\partial x^i} + v^i \frac{\partial P}{\partial t} = 0. \tag{179}$$

But $u^\nu \partial/\partial x^\nu = \gamma \dot{x}^\nu \partial/\partial x^\nu = \gamma(\partial/\partial t + (\mathbf{v} \cdot \boldsymbol{\nabla}))$ and therefore this becomes

$$\boxed{\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \boldsymbol{\nabla})\mathbf{v} = -\frac{c^2}{\gamma^2(\mathcal{E} + P)}\left[\boldsymbol{\nabla} P + \frac{\mathbf{v}}{c^2}\frac{\partial P}{\partial t}\right]} \tag{180}$$

where we recognize, on the left hand side, the convective derivative $d\mathbf{v}/dt$: the rate of change with respect to lab-time t of the velocity of an element of the fluid. Equation (180) is the *relativistic form of the Euler equation*.

A more useful form for the energy conservation law is obtained if we contract $T^{\mu\nu}{}_{,\mu}$ with u_ν:

$$\begin{aligned} 0 = u_\nu T^{\mu\nu}{}_{,\mu} &= \frac{1}{c^2} u_\nu \frac{\partial}{\partial x^\mu}[(\mathcal{E} + P)u^\mu u^\nu] + u^\mu \frac{\partial P}{\partial x^\mu} \\ &= \frac{u_\nu u^\nu}{c^2}\frac{\partial[(\mathcal{E} + P)u^\mu]}{\partial x^\mu} + (\mathcal{E} + P)\frac{u^\mu u_\nu}{c^2}\frac{\partial u^\nu}{\partial x^\mu} + u^\mu \frac{\partial P}{\partial x^\mu} \\ &= -\frac{\partial[(\mathcal{E} + P)u^\mu]}{\partial x^\mu} + 0 + u^\mu \frac{\partial P}{\partial x^\mu} \\ &= -(\mathcal{E} + P)\frac{\partial u^\mu}{\partial x^\mu} - u^\mu \frac{\partial \mathcal{E}}{\partial x^\mu} \end{aligned} \tag{181}$$

where we have used $u_\mu \partial u^\mu/\partial x^\nu = \frac{1}{2}\partial(\vec{u} \cdot \vec{u})/\partial x^\nu = 0$.

Using again $u^\mu/\partial x^\mu = \gamma(\partial/\partial t + (\mathbf{v} \cdot \boldsymbol{\nabla}))$ in the second term, and in the first, $\partial u^\mu/\partial x^\mu = \partial\gamma/\partial t + \boldsymbol{\nabla} \cdot (\gamma\mathbf{v})$, and dividing by γ, the above equation then provides an expression for the convective derivative of the energy density:

$$\boxed{\frac{\partial \mathcal{E}}{\partial t} + (\mathbf{v} \cdot \boldsymbol{\nabla})\mathcal{E} = -\frac{\mathcal{E} + P}{\gamma}\left[\frac{\partial \gamma}{\partial t} + \boldsymbol{\nabla} \cdot (\gamma \mathbf{v})\right].} \tag{182}$$

Equations (182) and (180) take a particularly simple form in the vicinity of a point where the momentum density, and therefore also $\mathbf{v}$, vanishes, since we can then take $\gamma = 1$ to obtain

$$\frac{d\mathcal{E}}{dt} = -(\mathcal{E} + P)\boldsymbol{\nabla} \cdot \mathbf{v} \tag{183}$$

as found above, and

$$\frac{d\mathbf{v}}{dt} = -\frac{c^2 \nabla P}{\mathcal{E} + P}. \tag{184}$$

Equations (180) and (182) provide four equations for five unknowns $\mathcal{E}$, P and $\mathbf{v}$ (note that $\gamma = 1/\sqrt{1 - v^2/c^2}$ is not an independent variable). To close this system of equations we need an *equation of state*; a rule giving, for example, the pressure as a function of $\mathcal{E}$. In the following sections we find this relation for the two limiting cases of a fluid with energy density and pressure dominate by highly relativistic particles, and the opposite case of a non-relativistic gas.

E What causes length contraction?

The picture developed thus far regarding the frame dependence in special relativity is very much like that for passive rotations (i.e. rotations of the observer) in Euclidean space.

- I.e. the world consists of objects like events, 4-momenta of particles etc..
- these are real and frame independent entities
- but the *coordinates* that we assign to events and the *components* of 4-momenta depend on the frame from which we view them

This encourages the view that special relativity is 'just' geometry, and, in that world-view, the question *what causes length contraction?* may seem ill posed. One might rather be inclined to say that nothing *causes* length contraction. Or perhaps one would say that it is just a consequence of the postulate that the speed of light is the same in all frames. But a different view emerges if we think about what happens in a physical object while it is *in the process of becoming length contracted.* I.e. while it is being accelerated.

For example, consider a train composed of carriages that accelerates out of a station, with identical thrust being applied to each carriage, as illustrated in figure 19. You might want to think of it as being a mag-lev train with the thrust being applied by electro-magnets, or perhaps imagine that the carriages are propelled by rocket motors (this is, in fact, a slightly re-worded version of what is known as *Bell's rocket paradox*). The details aren't important; what is is that the thrust applied is identical.

We would surely all agree – and this would be correct – that a track-side observer would perceive the train as a whole to be length contracted.

But what if the train were actually two trains that were initially lying nose-to-tail? As they accelerate, each train will become length contracted, but what about the mid-point of the combined system? Do the two trains remain touching each other? Or does a gap develop? Many physicists, when presented with this problem, seem to feel intuitively that no gap would develop.

But an alternative view leads one to question this: Consider a space-time diagram of the paths of the two trains in the track-side observer frame. By symmetry, it would seem that the centres of the two trains – being identical – would move along identical paths.

But if so, the distance between the centres of the trains would remain the same in the frame of the track-side observer. Thus if, as is surely correct, the trains *individually* contract, a gap must develop.

Similarly, if the 'train' consisted of de-coupled carriages, each carriage would length contract, but, since the carriages move along identical paths that are simply displaced from one another by their initial separation, the overall length of the train would not contract.

From this it would seem that the contraction of the train when the carriages *are* coupled, is, in fact, caused in some way by the couplings between the carriages.

And the question posed above is not at all ill posed; its answer is that there are stresses in the couplings between the carriages – and in the material of the carriages also, that is causing the length of the train as a whole to become contracted.

Note that if *we* are accelerated then, in our frame, the train will become contracted in a purely passive manner – in line with the common view that it is 'just geometry'. But if we are not accelerated and the train is, then in the process of becoming contracted forces must inevitably have been brought into play that resulted in the contraction; the rearmost carriage must have been accelerated slightly more than the front-most carriage, and it is stress within the train that has resulted in the transfer of momentum from the front of the train to the back.

This is a very small effect, for a realistic train, but it is present nonetheless.

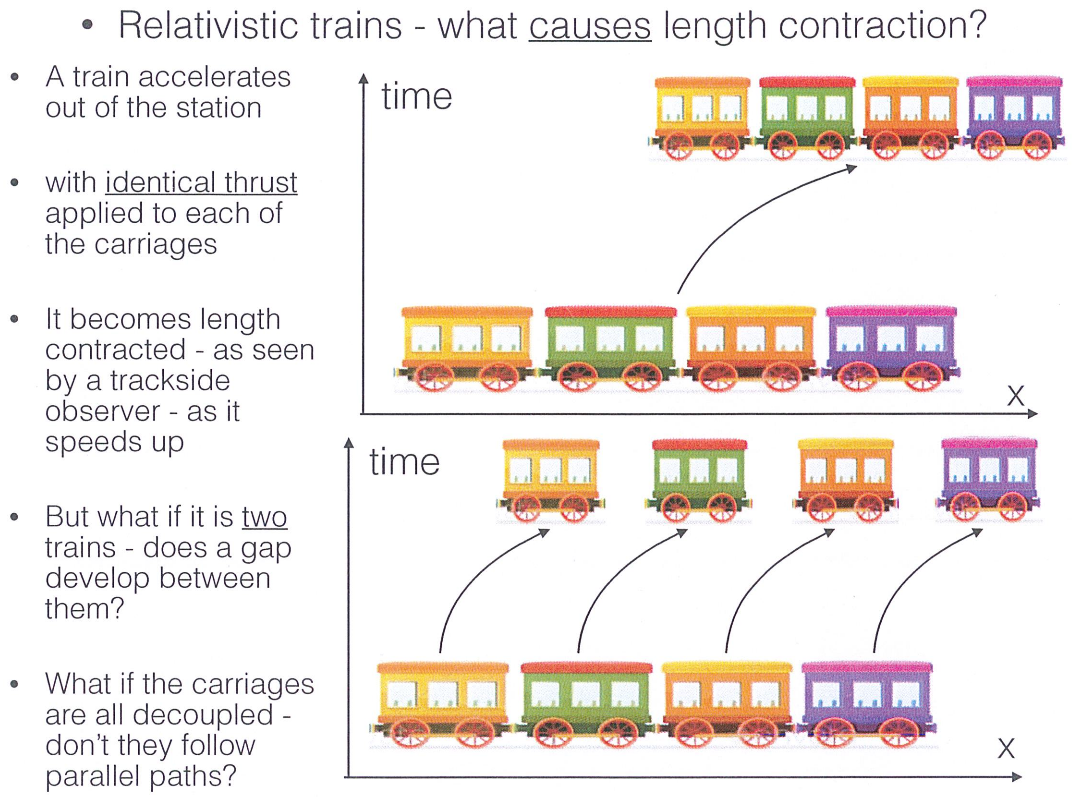

Figure 19: The *accelerating trains paradox*. A train being accelerated out of a station will become length contracted (as seen in the frame of the track). But what about a train made of uncoupled carriages all being accelerated identically? The view that the phenomena of SR – time dilation and length contraction – are 'just' geometry might lead one to think that the train as a whole would still become length contracted. But consider the lower diagram: this suggests that the carriages follow paths in space-time that are identical, aside from their initial spatial displacement. This would suggest that gaps would develop between the carriages. If correct, this would say that the coupling between the carriages (in the train drawn in the upper diagram) plays a critical role in the contraction of the train.

Here is an alternative way to think about this. Put yourself in the position of the driver of the trailing train. You see the the leading train get ahead of you. So from your perspective, the leading train has a greater acceleration. Let's imagine we are dealing with rocket driven trains. This means that from your perspective, the leading train must be burning its fuel more rapidly? And if you think of the fuel gauge of the leading train as a 'clock', it must be the time is running faster in the leading train. Do you see why?

Minkowsk SR is just geometry. X

AE

Bell's rocket paradox.

Speakable + Unsp