M1 Cosmology - 5 - Scalar Fields in Cosmology

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1 Introduction

Brans
-Dicky

Relativistic scalar fields are used in many ways in cosmology. Such fields appear in *scalar-tensor theories of gravity* and in other *modified theories of gravity*. They are invoked to drive *inflation*, either as the *inflaton field* in the early universe or as the *quintessence field* – one possibility for the *dark energy* – in late time inflation. Scalar fields in the guise of the *axion field* or *ultra-light axion-like fields* are candidates for the dark matter. A complex scalar field with a 'wine-bottle' potential can give rise to *cosmic string networks*, and other types of *cosmological defects* of scalar fields such as domain walls have been widely studied.

All of the above is hypothetical. The only relativistic scalar field known to exist is the Higgs field. But as Zel'dovich comments in his monograph *'My Universe'* [?], *"once the genie was out of the bottle there was no putting it back"*.

In most of the applications above, the field is assumed to be behaving as a *classical field*[1]; these fields are bosonic and so can have large occupation numbers and may therefore exist in *'coherent states'* in which

[1] A notable exception is *inflationary fluctuogenesis*; the idea here being that, while the inflaton field can be assumed to be, in the large, homogeneous, the modes of this field must have, at the very minimum, zero-point fluctuations.

the field has a well defined expectation value and any fluctuations about the expectation value are negligibly small[2] . One can then invoke *Ehrenfest's theorem* which tells us that the field expectation value obeys the classical field equation (the *Klein-Gordon equation*) which is what we consider here.

Below, in §2, we develop the mathematical description of classical scalar fields. We will start, in §2.1 with the Hamiltonian mechanics of a simple mechanical system with one degree of freedom (d.o.f.) and remind ourselves how conservation of energy arises if the Lagrangian has no explicit time dependence (an example of *Noether's theorem*). In §2.2 we consider the dynamics of a simple *scalar-elasticity* field theory consisting of a lattice made of beads on rods that are connected by springs, which, it turns out, is mathematically isomorphic to the relativistic scalar field. If the lattice is spatially homogeneous, this system has an additional conserved quantity; the *wave momentum*. In §2.3 we show how, with an appropriate choice of parameters, this 'scalar elasticity' field theory becomes the relativistic scalar field $\phi(\vec{x})$, whose equation of motion is the *Klein-Gordon equation*[3]. We construct the stress-energy tensor; whose vanishing 4-divergence vanishes expresses continuity of energy and wave-momentum and we compare with the analogous tensor for electromagnetism.

In §3 we develop scalar field theory in an arbitrary coordinate system. We start by showing the form of the action in generalised coordinates in §3.1, and we derive the equations of motion for the field in §3.2. We do this in two ways: First, in §3.2.1, by variation of the action and then, in §3.2.2, using the 'comma becomes semi-colon rule'. We show in §3.2.3 how these seemingly different forms of the KG equation are equivalent. Then, in §3.3 we obtain the equations of continuity of energy and momentum in arbitrary coordinates.

In §4 we discuss how a scalar field can drive inflation and what are the requirements on the form of the potential imposed by the need to obtain sufficiently many e-foldings of accelerated expansion.

Finally, in §5 we turn to the question of the origin of the 'seeds' of structure formation. We show how zero point quantum fluctuations of the modes of the field can generate the so-called 'Harrison-Zel'dovich' spectrum and also how self-interacting scalar fields can, in principle, generate networks of cosmic strings or other cosmological defects.

2 Lagrangian dynamics of classical scalar fields

We shall start, as a warm-up, in §2.1, with the mechanics of a system with one degree of freedom, and remind ourselves how invariance of the Lagrangian with respect to time leads to energy conservation. We generalise this in §2.2 to the scalar elasticity field theory which, in addition to a conserved energy, has conserved wave-momentum, and then choose the parameters of the model, in 2.3, to give the relativistic massive scalar field theory.

2.1 A system with 1 degree of freedom

2.1.1 The Lagrangian and the action

It is useful to start with the Newtonian problem of a mass m on a spring (not necessarily an ideal spring; its potential energy need not be simply quadratic in the dispacement) as illustrated in 1.

Letting the displacement be ϕ , the *Lagrangian* is the kinetic energy minus the potential energy

$$L(\phi, \dot{\phi}, t) = \tfrac{1}{2} m\dot{\phi}^2 - V(\phi) \tag{1}$$

where $\dot{\phi} \equiv d\phi/dt$, and we are allowing for a possibly explicit dependence of the Lagrangian on time (e.g. if the spring or mass were time dependent).

The *action* is a functional of the path $S[\phi(t)] = \int dt L$.

2.1.2 The Euler-Lagrange equations:

The equation of motion or *Euler-Lagrange equation* is obtained by asserting that the trajectory of the particle $\phi(t)$ – that goes from ϕ_1 at t_1 to ϕ_2 at t_2 – is such as to extremise the action: $\delta S = 0$. Considering

[2]It is by no means necessary for the field to behave classically even if the occupation number is large. A counter-example would be a field in thermal equilibrium where the occupation number is large for modes in the Rayleigh-Jeans regime, but the quantum state of such modes is incoherent – the density matrix being diagonal – and the expectation value for the field vanishes.

[3]This is the same as the relativistic version of Schrödinger's equation that he came up with by replacing E and $\mathbf{p}$ in the relativistic energy momentum relation $E^2 = p^2c^2 + m^2c^4$ with the quantum operators $i\hbar\partial_t$ and $-i\hbar\nabla$.

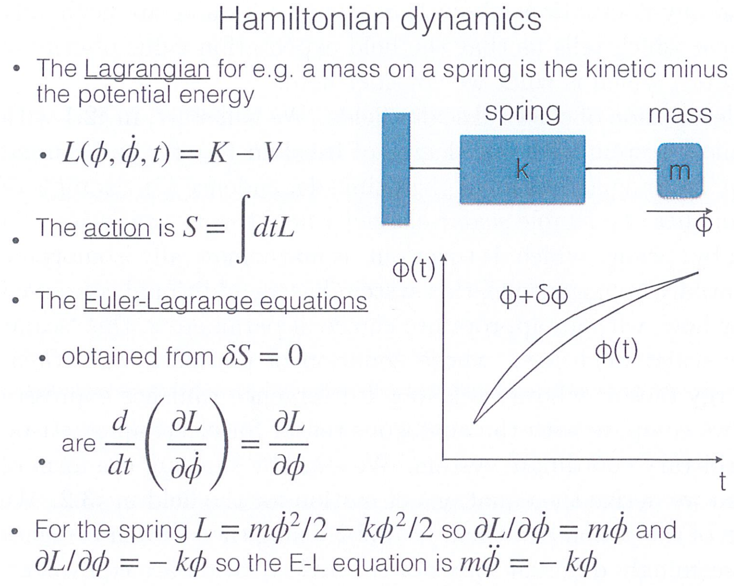


Figure 1: Illustration of Hamiltonian dynamics for a system with 1 degree of freedom (dof). The Lagrangian is the kinetic energy K of the mass minus the potential energy V of the spring. The action $S[\phi(t)]$ is a *functional* of the path: $S = \int Ldt$. Requiring that the action for the actual trajectory $\phi(t)$ – for some given starting and end points – gives the equations of motion, just as as one would obtain from Newton's law. For an ideal spring, with potential energy $V = \frac{1}{2}k\phi^2$ the force is $F = \partial L/\partial\phi = -k\phi$.

the change in the action between a path $\phi(t)$ and a neighbouring path $\phi(t) + \delta\phi(t)$ we have

$$
\begin{aligned}
0 = \delta S = \int_{t_1}^{t_2} dt\, \delta L &= \int_{t_1}^{t_2} dt \left(\frac{\partial L}{\partial \dot{\phi}}\delta\dot{\phi} + \frac{\partial L}{\partial \phi}\delta\phi\right) \\
&= \int_{t_1}^{t_2} dt \left(\overbrace{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\phi}}\delta\phi\right) \underbrace{- \delta\phi\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\phi}}\right)}}\underbrace{+ \frac{\partial L}{\partial \phi}\delta\phi}\right) \\
&= \overbrace{\cancel{\left[\frac{\partial L}{\partial \dot{\phi}}\delta\phi\right]_{t_1}^{t_2}}} + \int_{t_1}^{t_2} dt\, \delta\phi \left(\frac{\partial L}{\partial \phi} - \frac{d}{dt}\frac{\partial L}{\partial \dot{\phi}}\right)
\end{aligned}
\tag{2}
$$

The first brace here indicates how we eliminate $\delta\dot{\phi}$ by 'hiding' it in a total derivative term, which then drops out in the final line as we assume that the variation of the path vanishes at the end points.

For δS to vanish for arbitrary $\delta\phi$ requires that the quantity in parentheses in the integral vanish, giving the *Euler-Lagrange equation*

$$\boxed{\frac{d}{dt}\frac{\partial L}{\partial \dot{\phi}} = \frac{\partial L}{\partial \phi}.} \tag{3}$$

That is quite general and applies even with a time-dependent system. For the mass on a spring, where $L = K - V$ with $K = m\dot{\phi}^2/2$, so $\partial L/\partial\dot{\phi} = m\dot{\phi}$, and $\partial L/\partial\phi = -\partial V/\partial\phi$ we have

$$m\ddot{\phi} = -dV/d\phi \tag{4}$$

i.e. Newton's law $F = -dV/d\phi = ma$. For an ideal spring – where the energy is quadratic: $V = \frac{1}{2}k\phi^2$, with k the spring constant, this gives a simple harmonic oscillator equation:

$$m\ddot{\phi} = -k\phi \tag{5}$$

which is linear in ϕ and has solutions like $\phi = \phi_0 \cos\omega t$ where $\omega^2 = k/m$.

This is all quite readily generalised to variables (or *multiple degrees of freedom*) $\phi \Rightarrow \phi_i$, with i labelling the particles we obtain one equation per degree of freedom

$$\boxed{\frac{d}{dt}\frac{\partial L}{\partial \dot{\phi}_i} = \frac{\partial L}{\partial \phi_i}} \tag{6}$$

and if K is just the sum of the individual kinetic energies and V some function of all of the positions, we would have, for each i,

$$m_i\ddot{\phi}_i = -\partial V/\partial\phi_i. \tag{7}$$

2.1.3 Energy conservation

As is well known, for such a system we can define the *Hamiltonian:* $H = p\dot{\phi} - L$, where $p \equiv \partial L/\partial\dot{\phi}$ is the generalised – or 'canonical' – momentum, in terms of which the E-L equation is $\dot{p} = \partial L/\partial\phi$. The differential of H is

$$dH = \overbrace{p\,d\dot{\phi} + \dot{\phi}dp - \frac{\partial L}{\partial\phi}d\phi - \frac{\partial L}{\partial\dot{\phi}}d\dot{\phi}}^{\text{these cancel because } p\equiv\partial L/\partial\dot{\phi}} - \frac{\partial L}{\partial t}dt, \tag{8}$$

where, as indicated, the 1st and 4th terms cancel by virtue of the definition of p. Since the only differentials remaining are dp, $d\phi$ and dt, H is a function only of ϕ, p and t, and, from the coefficients of dp and $d\phi$, we can infer that $\partial H/\partial p = \dot{\phi}$ and $\partial H/\partial\phi = -\partial L/\partial\phi = -\dot{p}$, which are *Hamilton's equations*.

Of more interest to us here is the time variation of the Hamiltonian. Considering the remaining terms in (8), and considering the displacements dp and $d\phi$ for a possible trajectory, for which the E-L equation implies $\dot{\phi}dp = \dot{\phi}\dot{p}dt = \dot{p}d\phi = (\partial L/\partial\phi)d\phi$, it is apparent that the 2nd and 3rd terms then also cancel, and, dividing by dt, we have

$$\boxed{\frac{dH}{dt} = -\frac{\partial L}{\partial t}.} \tag{9}$$

Thus, if the Lagrangian has no *explicit* dependence on time (which, here, would mean that the spring constant k and mass m are time independent) the Hamiltonian, is conserved. This is an example of *Noether's therem*, which says that a *symmetry* – here the Lagrangian being independent of time – implies a *conservation law* – in this case constancy of H. For the particle on a spring with Lagrangian (1), $p = m\dot{\phi}$, and therefore $H = m\dot{\phi}^2/2 + V$, which is the total (kinetic plus potential) energy.

This is all very nice, but we started by pulling $H \equiv p\dot{\phi} - L$ out of a hat, so to speak. It will prove useful to have a procedure for constructing those quantities – not just energy – that are conserved by virtue of the symmetries of space and time.

One approach is to take the total derivative of the Lagrangian, which, for any particular solution of the E-L equation, can be considered a function of time alone: $L(t) = L(\phi(t), \dot{\phi}(t), t)$. Applying the chain rule, its (total) time derivative is

$$\begin{aligned} \frac{dL(t)}{dt} &= \frac{\partial L}{\partial\phi}\frac{d\phi}{dt} + \frac{\partial L}{\partial\dot{\phi}}\frac{d\dot{\phi}}{dt} + \frac{\partial L}{\partial t} \\ &= \underbrace{\dot{\phi}\frac{d}{dt}\frac{\partial L}{\partial\dot{\phi}} + \frac{\partial L}{\partial\dot{\phi}}\frac{d\dot{\phi}}{dt}} + \frac{\partial L}{\partial t} \\ &= \quad\frac{d}{dt}\left(\dot{\phi}\frac{\partial L}{\partial\dot{\phi}}\right) + \frac{\partial L}{\partial t} \end{aligned} \tag{10}$$

where, in the first step, we have invoked the equation of motion to eliminate $\partial L/\partial\phi$ and where $\partial L/\partial t$ means the derivative of L, considered as a function of ϕ, $\dot{\phi}$ and t holding ϕ and $\dot{\phi}$ constant. Rearranging and combining the total time derivative terms gives

$$\frac{d}{dt}\left(\dot{\phi}\frac{\partial L}{\partial\dot{\phi}} - L\right) = -\frac{\partial L}{\partial t} \tag{11}$$

where we see, on the left, the rate of change of the Hamiltonian as before.

The generalisation to a system consisting of multiple particles is straightforward. Each term in the differential dL becomes a sum – so e.g. $(\partial L/\partial\phi)d\phi/dt$ becomes $\sum_i(\partial L/\partial\phi_i)d\phi_i/dt$ – and we have

$$\boxed{\frac{d}{dt}\left(\sum_i\dot{\phi}_i\frac{\partial L}{\partial\dot{\phi}_i} - L\right) = -\frac{\partial L}{\partial t}} \tag{12}$$

so there is a single conserved quantity; the total energy.

2.2 The 'scalar elasticity' model for a scalar field

The mathematics of a relativistic scalar field $\phi(\vec{x})$ is, it turns out, isomorphic to that of (the continuum limit of) a simple mechanical model. In one spatial dimension it is illustrated on the left hand panel of figure 2; it is an array of particles tethered to a base by springs and with additional springs coupling neighbouring particles.

2.2.1 The discrete model

The kinetic energy is simply the sum of the kinetic energies of the particles $\frac{1}{2}M\dot{\phi}_i^2$, where M is the bead-mass. And the potential energy V contains the sum of squares of the differences of the displacements between neighbouring particles $K'(\phi_{i+1} - \phi_i)^2/2$, where K' is the spring constant for the coupling springs. To this we need to add the potential energy stored in the base-springs $= K\phi_i^2/2$. From this we can obtain the Lagrangian

$$L(\phi_i, \dot{\phi}_i) = \frac{1}{2}\sum_i \left[M\dot{\phi}_i^2 - K'(\phi_{i+1} - \phi_i)^2 - K\phi_i^2\right]. \tag{13}$$

We can make the system finite by having it be a closed ring of N units[4], and thus ignore boundary effects. We would then obtain N E-L equations and we would find that time-invariance ($\partial L/\partial t = 0$) implies conservation of the total energy as in (12).

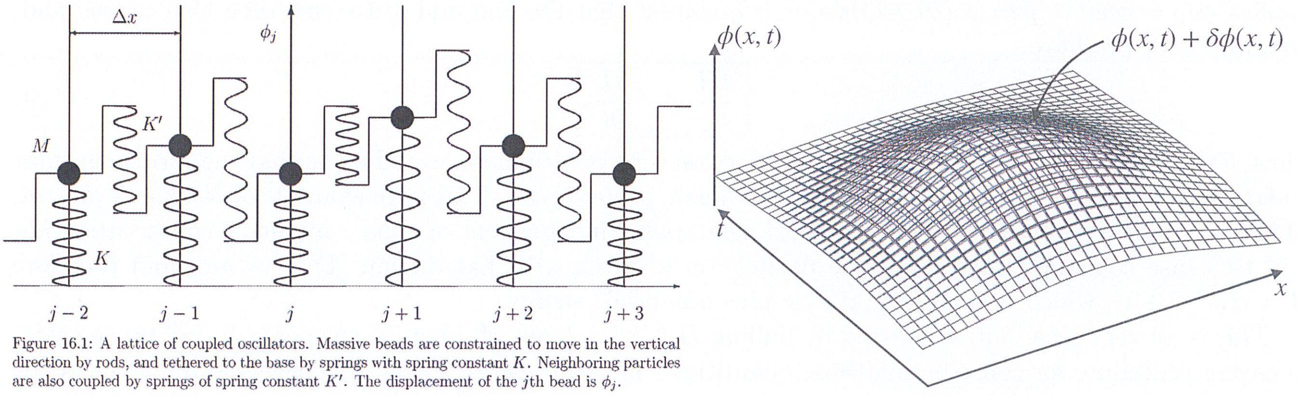


Figure 2: The 1D scalar elasticity lattice model for a scalar field. The discrete version is shown on the left. Taking the continuum limit, we get a field $\phi(x,t)$ whose equations of motion are obtained by considering a variation $\delta\phi(x,t)$ of the field, as illustrated on the right, and requiring that the action S – which becomes a 1+1 dimensional integral of a Lagrangian density – be stationary with respect to this variation.

2.2.2 The continuum limit

This discrete lattice system becomes a 1-D field theory if we take the *continuum limit*, and let the spacing dx between the rods become small and consider displacements that are assumed to vary smoothly on the, now microscopic, discreteness scale. In doing so, the discrete index i is replaced by the continuous variable $x = idx$ and $\phi_i(t) \Rightarrow \phi(x,t)$. We could then work out the form of the equations of motion and the energy etc. in the continuum limit, but it proves more fruitful to take the continuum limit of the Lagrangian (13), in which the relative displacements $\phi_{i+i} - \phi_i$ are replaced by dx times the spatial derivative of ϕ.

The Lagrangian can be written as the space integral of the *Lagrangian density*[5]

$$L = \int dx\, \mathcal{L} \tag{14}$$

where

$$\mathcal{L}(\dot{\phi}, \phi', \phi) = (A\dot{\phi}^2 - B\phi'^2 - C\phi^2)/2, \tag{15}$$

in which $\dot{\phi} \equiv \partial_t\phi$ and $\phi' \equiv \partial_x\phi$ (where ∂_t and ∂_x denote the partial derivatives at fixed x and t respectively) and where A, B and C are positive constants. The first, A, is determined by the masses M of the particles (and their spacing), while B is determined by the spring constant K' of the connecting springs, and C depends on the strength K of the base springs.

[4]Note that, in closing the ring, we have introduced another symmetry, this time a discrete one; the system is invariant under step-wise rotations of the ring.

[5]We could define a kinetic energy density $\mathcal{K} = A\dot{\phi}^2/2$, in terms of which $K = \int dx\, \mathcal{K}$, and a potential energy density $\mathcal{V} = (B\phi'^2 + C\phi^2)/2$, in terms of which $V = \int dx\, \mathcal{V}$, and write $\mathcal{L} = \mathcal{K} - \mathcal{V}$, i.e. as the sum of a 'kinetic' minus 'potential' energy densities. However, for reasons that are lost to obscurity, it is conventional to call $A\dot{\phi}^2/2 - B\phi'^2/2$ the 'kinetic term'.

The action $S = \int dtL$ is then a 1+1 dimensional space-time integral

$$S[\phi(x,t)] = \iint dt\, dx\, \mathcal{L}(\dot{\phi}, \phi', \phi). \tag{16}$$

To get the equations of motion implied by $\delta S = 0$ we imagine the 2-D *surface* $\phi(x,t)$ lying above the $x - t$ plane, and a vertically displaced surface $\phi(x,t) + \delta\phi(x,t)$ as illustrated in figure 2 to obtain[6]

$$\delta S = \iint dt\, dx \left(\frac{\partial \mathcal{L}}{\partial \dot{\phi}} \delta\dot{\phi} + \frac{\partial \mathcal{L}}{\partial \phi'} \delta\phi' + \frac{\partial \mathcal{L}}{\partial \phi} \delta\phi \right). \tag{17}$$

What we would now like to do is somehow convert this to an integral $\delta S = \iint dt\, dx\, \delta\phi[\ldots]$ involving only the variation $\delta\phi$ (and not $\delta\dot{\phi}$ and $\delta\phi'$). To do this, we write the first term and second terms in the above integral as

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial \dot{\phi}} \delta\dot{\phi} &= \partial_t \left(\frac{\partial \mathcal{L}}{\partial \dot{\phi}} \delta\phi \right) - \partial_t \left(\frac{\partial \mathcal{L}}{\partial \dot{\phi}} \right) \delta\phi \\ \frac{\partial \mathcal{L}}{\partial \phi'} \delta\phi' &= \partial_x \left(\frac{\partial \mathcal{L}}{\partial \phi'} \delta\phi \right) - \partial_x \left(\frac{\partial \mathcal{L}}{\partial \phi'} \right) \delta\phi \end{aligned} \tag{18}$$

so we are 'hiding' the unwanted $\delta\dot{\phi} = \partial_t \delta\phi$ and $\delta\phi' = \partial_x \delta\phi$ in the 'total' derivatives. The variation of the action then becomes

$$\delta S = \int dx \left[\frac{\partial \mathcal{L}}{\partial \dot{\phi}} \delta\phi \right]_{t_1}^{t_2} + \int dt \left[\frac{\partial \mathcal{L}}{\partial \phi'} \delta\phi \right]_{x_1}^{x_2} - \iint dt\, dx\, \delta\phi \left[\partial_t \frac{\partial \mathcal{L}}{\partial \dot{\phi}} + \partial_x \frac{\partial \mathcal{L}}{\partial \phi_{,x}} - \frac{\partial \mathcal{L}}{\partial \phi} \right] \tag{19}$$

So if we demand that $\delta\phi(x,t)$ vanish at $t = t_1$ and $t = t_2$ and similarly on the boundaries $x = x_1$ and $x = x_2$ (or one might impose *periodic boundary conditions* in x) the first two terms vanish and requiring that δS vanish for otherwise arbitrary $\delta\phi(x,t)$ gives the the *Euler-Lagrange equation*:

$$\boxed{\partial_t \frac{\partial \mathcal{L}}{\partial \dot{\phi}} + \partial_x \frac{\partial \mathcal{L}}{\partial \phi'} - \frac{\partial \mathcal{L}}{\partial \phi} = 0.} \tag{20}$$

So we see that in generalising to a field, the equation of motion has gained the extra term $\partial_x(\partial\mathcal{L}/\partial\phi')$. Note that time and space appear in the equation of motion – as indeed they do in the Lagrangian density – on an equal footing.

That is general. For the specific model $\mathcal{L} = (A\dot{\phi}^2 - B\phi'^2 - C\phi^2)/2$, the partial derivatives are $\partial\mathcal{L}/\partial\dot{\phi} = A\dot{\phi}$ and $\partial\mathcal{L}/\partial\phi' = -B\phi'$, so the equation of motion is

$$\boxed{A\ddot{\phi} - B\phi'' + C\phi = 0.} \tag{21}$$

This is a *linear, but dispersive, wave equation* which allows *travelling wave solutions* like

$$\phi(x,t) = \phi_0 \cos(kx - \omega t) \tag{22}$$

which, in (21) gives the *dispersion relation*

$$\boxed{A\omega^2 = Bk^2 + C} \tag{23}$$

so the temporal frequency is determined by the spatial frequency (radians per metre) k.

This has the same form as the relativistic energy-momentum relation

$$E^2 = c^2 p^2 + m^2 c^4. \tag{24}$$

This is equivalent to (23) if we replace E and p by $E \Rightarrow \hbar\omega$ and $p \Rightarrow \hbar k$, set $c = \sqrt{B/A}$, this being the phase velocity of waves in the limit of high spatial frequency $k \gg \sqrt{C/B}$, and choose the mass m so that

[6] Note that there are no derivatives with respect to time or space here. This is not because $\mathcal{L}$ is necessarily independent of x and/or t. If the spring constants or masses *were* varying with time, or position, we would still use the above expression. That's because it's inside the integral – where what we need is the change in the Lagrangian density at a fixed t and x.

$\sqrt{C/A}$ – the frequency of waves in the long-wavelength limit $k \ll \sqrt{C/B}$ – be $\sqrt{C/A} = mc^2/\hbar$, which is the *Compton frequency* for a particle of mass m.

It is also identical in form to the dispersion relation for electromagnetic waves in a cold plasma:

$$\omega^2 = c^2 k^2 + \omega_{\mathrm{p}}^2 \tag{25}$$

where ω_{p} is the plasma frequency, below which EM waves cannot propagate. This proves to be a very useful analogy for scalar waves as the DM, which, as we will see, are trapped in the potential wells of galaxies much as radio waves are trapped below the ionosphere.

2.2.3 Time translational invariance

We derived the conservation equation for energy for the 1-variable system $L(\dot{\phi}, \phi)$ by considering the *total time derivative* dL/dt considering L to be, for a particular solution $\phi = \phi(t)$, with associated $\dot{\phi}(t) = d\phi/dt$, a function of time $L(t)$.

Here we can do something very similar: we partially differentiate the Lagrangian density

$$\mathcal{L}(x,t) \equiv \mathcal{L}(\dot{\phi}(x,t), \phi'(x,t), \phi(x,t)), \tag{26}$$

considered now as a function of x and t for a particular solution $\phi = \phi(x,t)$, with respect to time. Applying the chain rule to the right hand side gives

$$\begin{aligned} \partial_t \mathcal{L}(x,t) &= \frac{\partial \mathcal{L}}{\partial \dot{\phi}} \partial_t \dot{\phi} + \frac{\partial \mathcal{L}}{\partial \phi'} \overbrace{\partial_t \phi'}^{\partial_x \dot{\phi}} + \overbrace{\frac{\partial \mathcal{L}}{\partial \phi}}^{\partial_t \frac{\partial \mathcal{L}}{\partial \dot{\phi}} + \partial_x \frac{\partial \mathcal{L}}{\partial \phi'}} \overbrace{\partial_t \phi}^{\dot{\phi}} \\ &= \underbrace{\frac{\partial \mathcal{L}}{\partial \dot{\phi}} \partial_t \dot{\phi} + \dot{\phi} \partial_t \frac{\partial \mathcal{L}}{\partial \dot{\phi}}}_{\partial_t \left(\dot{\phi} \frac{\partial \mathcal{L}}{\partial \dot{\phi}} \right)} + \underbrace{\frac{\partial \mathcal{L}}{\partial \phi'} \partial_x \dot{\phi} + \dot{\phi} \partial_x \frac{\partial \mathcal{L}}{\partial \phi'}}_{\partial_x \left(\dot{\phi} \frac{\partial \mathcal{L}}{\partial \phi'} \right)} \\ &= \partial_t \left(\dot{\phi} \frac{\partial \mathcal{L}}{\partial \dot{\phi}} \right) + \partial_x \left(\dot{\phi} \frac{\partial \mathcal{L}}{\partial \phi'} \right) \end{aligned} \tag{27}$$

where, as indicated, we are using the commutativity of partial derivatives and invoking the equation of motion.

Collecting together the two time derivative terms, what we have here is a *continuity equation*

$$\boxed{\partial_t \mathcal{E} + \partial_x \mathcal{F} = 0} \tag{28}$$

as illustrated in figure 3, and where we have defined

$$\mathcal{E} \equiv \dot{\phi} \frac{\partial \mathcal{L}}{\partial \dot{\phi}} - \mathcal{L}, \tag{29}$$

which looks a lot like the Hamiltonian for a simple 1 d.o.f. system, and

$$\mathcal{F} \equiv \dot{\phi} \frac{\partial \mathcal{L}}{\partial \phi'}. \tag{30}$$

That (28) expresses continuity of energy is seen if we specialise to the model

$$\mathcal{L} = \tfrac{1}{2}(A\dot{\phi}^2 - B\phi'^2 - C\phi^2) = \mathcal{K} - \mathcal{V} \tag{31}$$

from which we have $\partial \mathcal{L}/\partial \dot{\phi} = A\dot{\phi}$ and therefore

$$\mathcal{E} = \tfrac{1}{2}(A\dot{\phi}^2 + B\phi'^2 + C\phi^2) = \mathcal{K} + \mathcal{V} \tag{32}$$

which is evidently the energy density or what one might call the *Hamiltonian density*.

For the travelling wave solution (22), the energy density is

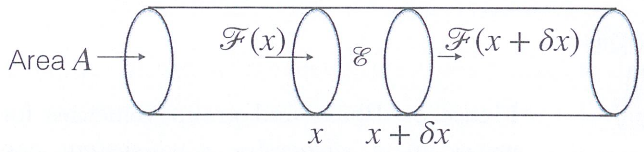
$$\begin{aligned} \mathcal{E} &= \tfrac{1}{2}\phi_0^2[(A\omega^2 + Bk^2)\sin^2(kx - \omega t) + C\cos^2(kx - \omega t)] \\ &= \tfrac{1}{2}\phi_0^2[2Bk^2 \sin^2(kx - \omega t) + C] \end{aligned} \tag{33}$$

as illustrated in figure 4.

If we integrate (28) over position x and assume either periodic boundary conditions or that $\mathcal{F} \to 0$ at $x \to \pm\infty$ we evidently have a globally conserved total energy $E = \int dx \mathcal{E}$, since

$$\frac{dE}{dt} = \frac{d}{dt} \int dx\, \mathcal{E} = \int dx\, \partial_t \mathcal{E} = -\int dx\, \partial_x \mathcal{F} = [\mathcal{F}]_{-\infty}^{+\infty} = 0. \tag{34}$$

The energy continuity (or conservation) law



- The energy in the volume element $\delta V = A\delta x$ is $\delta E = \mathscr{E}\delta V$
- the change in δE in time Δt is energy in minus energy out, or
  - $\Delta\delta E = A\Delta t(\mathscr{F}(x) - \mathscr{F}(x+\delta x))$ or $\Delta\delta E = -A\Delta t \times \delta\mathscr{F}$
- where $\mathscr{F}$ is the energy flux density = energy per area per time
- but $\Delta\delta E = \delta V\Delta t\partial\mathscr{E}/\partial t = A\Delta t \times \delta x\partial\mathscr{E}/\partial t$
  - $\partial\mathscr{E}/\partial t$ being the rate of change of energy density at that x
- so $\delta x\partial\mathscr{E}/\partial t = -\delta\mathscr{F}$ or, taking the limit, $\partial\mathscr{E}/\partial t = -\partial\mathscr{F}/\partial x$

Figure 3: Energy conservation in a 1-dimensional field theory is like a 1-dimensional flow of compressible fluid in a pipe. At any position in the pipe there is a 1-dimensional energy-density $\mathcal{E}$ (mass per unit length in the fluid analogy) and there is a flux-density $\mathcal{F}$. In 3-D the units of flux density are energy per area per time. In 1D it is just mass per unit time. If $\mathcal{F}$ is constant in x then there is no build up or decrease of density, so $\dot{\mathcal{E}}$ – note this is a *partial* derivative at fixed position – vanishes. In general $\dot{\mathcal{E}} = -\partial\mathcal{F}/\partial x$ (in 3D the right hand side becomes minus the *divergence* $-\nabla \cdot \mathcal{F}$).

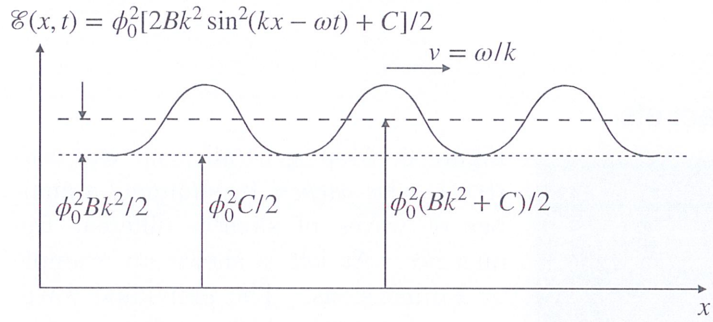


Figure 4: The energy density for a travelling wave $\phi(x,t) = \phi_0\cos(kx-\omega t)$ has a spatially constant component $\mathcal{E} = \phi_0^2 C/2$ plus a moving ripple, contributing a mean energy density $\langle\mathcal{E}\rangle = \phi_0^2 Bk^2/2$. The ripple travels at the phase velocity $v_\mathrm{p} = \omega/k$. The effective velocity at which energy is being transported is smaller than v_p by a factor $Bk^2/(C+Bk^2)$, and is equal to the *group velocity* $v_\mathrm{g} = d\omega/dk$.

2.2.4 The energy flux density

What about $\mathcal{F}$? This must represent the energy flux density. Is this reasonable?

From (15) we have $\partial\mathcal{L}/\partial\phi' = -B\phi'$, so

$$\mathcal{F} \equiv \dot{\phi}\frac{\partial\mathcal{L}}{\partial\phi'} = -B\dot{\phi}\phi'. \tag{35}$$

Does this make sense? Consider a wave $\phi = \phi_0\cos(kx-\omega t)$. This is a wave propagating in the $+x$ direction since $kx - \omega t =$ constant implies $kdx = \omega dt$ so $v = dx/dt = \omega/k$. The mean square value of ϕ^2 in the energy density (32) is $\langle\phi^2\rangle = \phi_0^2\langle\cos^2(kx-\omega t)\rangle = \phi_0^2/2$ where $\langle\ldots\rangle$ denotes average over time or space (or both). Similarly, $\langle\dot{\phi}^2\rangle = \omega^2\phi_0^2/2$ and $\langle\phi'^2\rangle = k^2\phi_0^2/2$ so

$$\langle\mathcal{E}\rangle = \tfrac{1}{2}(A\langle\dot{\phi}^2\rangle + B\langle\phi'^2\rangle + C\langle\phi^2\rangle) = \tfrac{1}{4}\phi_0^2(A\omega^2 + Bk^2 + C) = \tfrac{1}{2}A\omega^2\phi_0^2 \tag{36}$$

where, in the last step we have used the dispersion relation (23). In the energy flux density, $\dot{\phi} = \omega\phi_0\sin(kx-\omega t)$ and $\phi' = -k\phi_0\sin(kx-\omega t)$ so

$$\langle\mathcal{F}\rangle = -B\langle\dot{\phi}\phi'\rangle = \tfrac{1}{2}B\omega k\phi_0^2 \tag{37}$$

and it is reassuring that this is positive, as one would expect for a wave propagating in the $+x$ direction.

However, the speed v at which the energy is propagating must be the mean energy per unit time crossing a position (e.g. $x = 0$), which is $\langle\mathcal{F}\rangle$ divided by the mean energy per unit length, which is $\langle\mathcal{E}\rangle$, so, using (37) and (36), this is

$$v = \frac{\langle\mathcal{F}\rangle}{\langle\mathcal{E}\rangle} = \frac{B}{A}\frac{k}{\omega}. \tag{38}$$

This seems at odds with the velocity the wave is propagating: $v = \omega/k$. But the difference is simply that the latter is the *phase velocity* while that above is the *group velocity*, which is not ω/k but $d\omega/dk$ or

$$v_\mathrm{g} = \frac{d\omega}{dk} = \frac{k}{\omega}\frac{d\omega^2}{dk^2} = \frac{k}{\omega}\frac{d(Bk^2+C)/A}{dk^2} = \frac{B}{A}\frac{k}{\omega} = \frac{\langle\mathcal{F}\rangle}{\langle\mathcal{E}\rangle}. \tag{39}$$

as illustrated in figure 5.

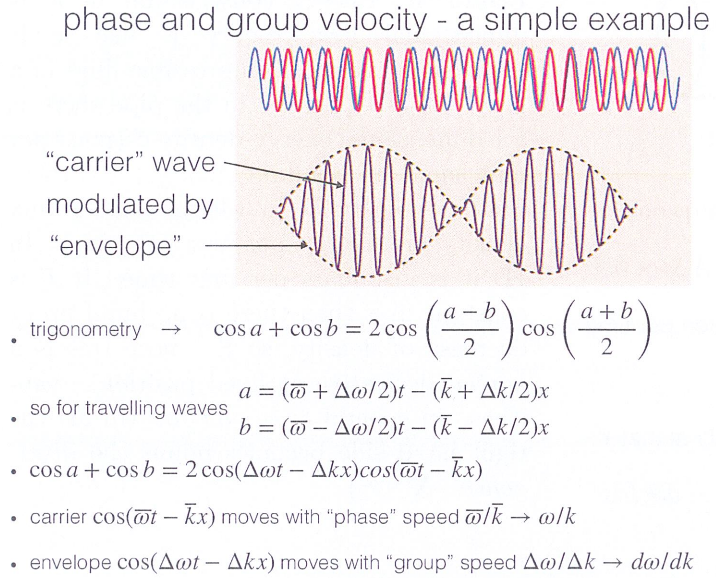


Figure 5: Phase and group velocities for waves in a *dispersive wave-system* can be understood in a model where, in 1-dimension for simplicity, we add two waves with slightly different spatial frequencies. Simple trigonometry shows that this gives a *carrier wave*, the wave-crests of which propagate at the *phase-velocity* $\overline{\omega}/\overline{k}$, modulated by an *envelope* that propagates at speed $\Delta\omega/\Delta k$, which, for small Δk, will be approximately $d\omega/dk$. A *non-dispersive wave-system* is one for which $\omega \propto k$, for which the phase- and group-velocities are the same.

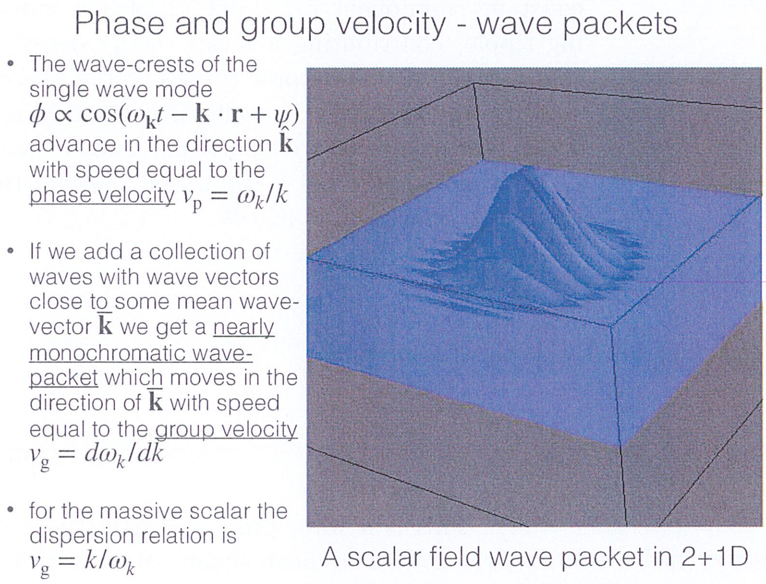


Figure 6: More generally, one can construct *wave-packets* by summing a number of waves of slightly different frequencies. At left is shown an example in 2-dimensions. The individual wave-crests move through the packet; appearing at the trailing edge and disappearing at the leading edge. The number of waves N is on the order of the inverse of the fractional width in spatial frequency $\Delta k/k$. Wave packets tend to spread as they propagate, but nearly monochromatic wave packets – which have large N – travel about N times their size before spreading appreciably.

The conservation law (28) is a consequence of the symmetry that the Lagrangian density does not depend explicitly on time t (i.e. the mass- and spring-coefficients A, B and C are constant) and is another instance of Noether's theorem.

2.2.5 Spatial translational invariance and wave momentum

We have assumed that all of the masses, all of the potentials and all of the connecting springs are identical, so, in the continuum limit, A, B and C, and hence also $\mathcal{L}$, are independent of position x. This invariance of $\mathcal{L}$ with respect to spatial displacements implies that there is another conserved quantity that we will call *wave-momentum*.

It is easy to obtain its continuity equation. If we swap $x \Leftrightarrow t$ and $\phi' \Leftrightarrow \dot{\phi}$ in (27) we obtain

$$\partial_x \mathcal{L} = \partial_x \left(\phi' \frac{\partial \mathcal{L}}{\partial \phi'}\right) + \partial_t \left(\phi' \frac{\partial \mathcal{L}}{\partial \dot{\phi}}\right) \tag{40}$$

which, combining the spatial derivative terms together, gives

$$\boxed{\partial_t \mathcal{P} + \partial_x \mathcal{S} = 0} \tag{41}$$

where we have defined the *wave-momentum density*

$$\mathcal{P} \equiv -\phi' \frac{\partial \mathcal{L}}{\partial \dot{\phi}} \tag{42}$$

and the *wave-momentum flux density* (or *wave-stress*, if you prefer)

$$\mathcal{S} \equiv \mathcal{L} - \phi' \frac{\partial \mathcal{L}}{\partial \phi'}. \tag{43}$$

If we integrate (41) over position x and assume either periodic boundary conditions or that $\mathcal{S} \to 0$ at $x \to \pm\infty$ we evidently have another globally conserved quantity, the *total wave-momentum* $P \equiv \int dx \mathcal{P}$, since

$$\frac{dP}{dt} = \frac{d}{dt} \int dx \mathcal{P} = \int dx\, \partial_t \mathcal{P} = -\int dx\, \partial_x \mathcal{S} = 0. \tag{44}$$

The above expressions are general, and can be used for self-interacting fields (lattices for which the base springs have non-quadratic potential energy). For the free-field model $\mathcal{L} = \frac{1}{2}(A\dot{\phi}^2 - B\phi'^2 - C\phi^2)$, the wave-momentum density is

$$\mathcal{P} = -A\phi'\dot{\phi} \tag{45}$$

so this is just A/B times the energy flux density $\mathcal{F}$. So, like $\mathcal{F}$, it is positive for a wave – or wave-packet – propagating toward positive x (i.e. with positive k).

On the other hand, it is not difficult to see that the total wave-momentum is *not* the same as the normal momentum (which is just the sum of the mass times velocity). For one thing, it is quadratic in the displacements while the normal momentum is linear[7]. Both normal momentum and wave-momentum are conserved. But their conservation arises from different symmetries. The normal momentum is conserved because of the homogeneity of *space*. The Lagrangian is the same no matter where in the universe the system is located. The wave momentum is conserved as a result of the properties of the lattice being independent of location on the lattice. One could imagine a lattice in which e.g. C were to vary with position. This would then *not* conserve wave momentum. But the normal momentum would still be conserved.

While the wave-momentum should not be confused with normal momentum, it does at least have the right units. The units of $\mathcal{L}$ are energy density (linear density that is) or [MLT^{-2}], and the units of $\mathcal{S}$ are evidently the same, so from (41) the units of $\mathcal{P}$ are [MT^{-1}] and so those of P are [MLT^{-1}] which are those of momentum. Similarly, the wave-momentum flux-density $\mathcal{S}$ or *stress* has units of stress in 1D (in 3D stress is momentum per area per time, here it is just momentum per time or [MLT^{-2}]) but should not be confused with the stress in the springs. There is stress in the springs, but it is linear in the displacement ϕ. In the ABC-model

$$\mathcal{S} = \tfrac{1}{2}(A\dot{\phi}^2 + B\phi'^2 - C\phi^2) \tag{46}$$

so just like $\mathcal{L}$ but with the sign of the ϕ'^2 term flipped. It is, like $\mathcal{P}$, quadratic. For a wave, or a wave-packet, or a collection of waves or wave packets, the average $A\langle\dot{\phi}^2\rangle = B\langle\phi'^2\rangle + C\langle\phi^2\rangle$ so the time and/or space average of the stress is

$$\langle \mathcal{S} \rangle = B\langle\phi'^2\rangle \tag{47}$$

and is positive regardless of the direction of the wave. A wave with positive (negative) k carries positive (negative) wave-momentum in the positive (negative) x-direction. Thus a wave packet composed of waves with negative k, in moving from $+x$ to $-x$, is transporting momentum in the positive direction. This is like kinetic pressure in a gas, where both positive and negative moving particles constitute a positive flux density of momentum.

The continuity equation (41) is, in essence, Newton's law that rate of change of momentum is equal to the force. Here we have that the rate of change of wave-momentum *density* is the *force density*, being (minus) the 1-dimensional divergence of the *momentum flux density* or, equivalently, the *pressure gradient*.

[7] If we have a system of N particles interacting via a potential $V(\mathbf{r}_1, \mathbf{r}_2, \ldots)$ then the change in V under a displacement of the system $d\mathbf{r}$ is $dV = d\mathbf{r} \cdot \sum_i \partial V/\partial \mathbf{r}_i = -d\mathbf{r} \cdot \sum_i \dot{\mathbf{p}}_i$. So if $V(\mathbf{r}_1, \mathbf{r}_2, \ldots)$ is invariant under displacements the total momentum $\mathbf{P} = \sum_i \mathbf{p}_i$ is conserved.

2.2.6 Some questions concerning wave momentum

1. Consider the sum of two waves with similar k and calculate the speed with which the 'beats' move. Convince yourself that a *nearly monochromatic wave-packet* – i.e. one composed of many waves with a small range of k moves at the same speed as the beats for the simple two-wave model.

2. Sketch the phase- and group-speeds for the 1-dimensional ABC-model as a function of k.

3. For a wave packet of finite extent show that if we integrate the rate of change of the energy- or momentum-density over all space the spatial derivative term vanishes. Obtain thereby a relation between total energy and momentum for wave-packet. What does this remind you of?

4. Show that for long wavelength fluctuations – such that $B\phi_{,xx} \ll C\phi$ or $\lambda^2 \gg B/C$ – one can make a change of variables $\phi(x,t) = (\psi(x,t)e^{imt} + \text{c.c})/2$ where c.c. denotes complex conjugation and $m \equiv \sqrt{C/A}$ and where $\psi(x,t)$ is slowly varying with time-scale for variation $\tau \sim (\sqrt{AC}/B)\lambda^2$ and in terms of which the Euler-Lagrange equations become (to leading order in $B/(C\lambda^2) \ll 1$) $i\partial\psi/\partial t = \frac{B}{2\sqrt{AC}}\partial^2\psi/\partial x^2$ and the momentum is $p = i\sqrt{CA}\int dx\ (\psi^*\partial\psi/\partial x + \text{c.c.})/4$ plus rapidly oscillating terms whose time average vanishes. What do these equations remind you of?

5. Regarding the previous question, does it seem strange that the original field ϕ has only one real degree of freedom while the complex field ψ has two? What gives? (hint: you changed from an equation that was second order in time to one that was first order).

6. Generalise the theory to allow the coefficient C to vary smoothly with position. How does that change the continuity equation for momentum? (Hint: be careful – what we called $\mathcal{L}_{,x}$ was the partial derivative of $\mathcal{L}$ considered as a function of x and t. That was fairly unambiguous since $\mathcal{L}$ did not have an explicit x dependence. Here you have $\mathcal{L}(\phi, \dot{\phi}, \phi', x)$. You may want denote what we called $\mathcal{L}_{,x}$ above as $\delta\mathcal{L}/\delta x$ or maybe $d_x\mathcal{L}$ to avoid confusion.) What does this modification imply for the rate of change of momentum of a wave-packet? Hint: You might want to draw the analogy with EM waves propagating in a plasma with varying plasma frequency (where, for instance, radio waves can be reflected from the ionosphere).

7. What if we had a *finite* lattice like this on a skate-board with a wave-packet propagating along it carrying energy and wave-momentum in the $+x$ direction say. When the packet reaches the end, it will reflect and the sign of the wave-momentum will flip. Would we see the skate-board start to move from the recoil?

8. Considering again the long wave-length limit, show that while the time-averaged wave momentum flux density is very small compared to the energy density, there is a very large fluctuating component for which $\langle \mathcal{S}^2 \rangle^{1/2} \simeq \langle \mathcal{E} \rangle$.

2.3 Transition to a relativistic massive scalar field

2.3.1 Lagrangian, action and equations of motion

The transition to a *relativistic real massive scalar field* is mathematically straightforward. First we make x a 3-vector $\mathbf{x}$. So we're now considering waves that will behave analogously to those on something like a springy mattress or crystal lattice. Such a lattice – if spatially homogeneous and non-time varying – will have, in the continuum limit, 4 conserved quantities; the energy and 3 components of the wave-momentum. Monochromatic waves $\phi = \phi_0 \cos(\mathbf{k} \cdot \mathbf{x} - \omega_{\mathbf{k}} t + \psi)$ are now defined by the amplitude, the phase, and by the 3-dimensional wave vector $\mathbf{k}$. As indicated, the temporal frequency will be a function of $\mathbf{k}$. We will determine this presently.

Second, we introduce $x^0 = ct$ as the 'time' coordinate, giving it the same units as $\mathbf{x}$. This is not essential; we will later work in coordinate systems where this is not the case, but it is convenient.

Third, we fix the ratio of the constants A and B so that the high-$|\mathbf{k}|$ asymptotic wave-speed (phase- or group-speed as these are identical in the high-k limit) is equal to the speed of light c.

Finally, we replace the displacement $\phi \rightarrow \phi^\star = \phi/\sqrt{B}$ and drop the star. The Lagrangian density is now

$$\boxed{\mathcal{L}(\phi_{,\alpha}, \phi) = -\tfrac{1}{2}(\phi_{,\alpha}\phi^{,\alpha} + \mu^2\phi^2)} \tag{48}$$

where μ^2 is a suitably re-scaled version of C and which, if ϕ is taken to be a Lorentz scalar (i.e. frame independent), is manifestly Lorentz invariant.

In the process of re-scaling the field, we changed its units. We will demand that the Lagrangian density have units of energy density: $[\mathrm{ML^{-1}T^{-2}}]$, so the field ϕ now has units $[\mathrm{M^{1/2}L^{1/2}T^{-1}}]$. A useful thing to remember is that ϕ^2 has units of force. The constant μ evidently has units of inverse length.[8]

Allowing for the possibility that the Lagrangian have an explicit dependence on space-time location $\vec{x}$ – as would be the case if the field were interacting somehow with other fields or particles – the action is

$$S = \frac{1}{c}\int d^4x\, \mathcal{L}(\phi_{,\alpha}, \phi, \vec{x}) \tag{49}$$

whose variation is

$$\delta S = \frac{1}{c}\int d^4x\, \delta\mathcal{L}(\phi_{,\alpha}, \phi, \vec{x}) \tag{50}$$

in which the variation of the Lagrangian density (at fixed $\vec{x}$, since $\mathcal{L}$ is inside the integral) is

$$\begin{aligned} \delta\mathcal{L}(\phi_{,\alpha}, \phi, \vec{x}) &= \overbrace{\frac{\partial\mathcal{L}}{\partial\phi_{,\alpha}}\delta\phi_{,\alpha}} + \frac{\partial\mathcal{L}}{\partial\phi}\delta\phi \\ &= \left(\frac{\partial\mathcal{L}}{\partial\phi_{,\alpha}}\delta\phi\right)_{,\alpha} - \delta\phi\left(\frac{\partial\mathcal{L}}{\partial\phi_{,\alpha}}\right)_{,\alpha} + \delta\phi\frac{\partial\mathcal{L}}{\partial\phi} \end{aligned} \tag{51}$$

where we have performed the usual trick of 'hiding' the variations of the derivatives $\delta\phi_{,\alpha}$ in the 'total' derivative term, so the variation of the action is

$$\delta S = \left[\int d^3x \frac{\partial\mathcal{L}}{\partial\phi_{,0}}\delta\phi\right]_{t_i}^{t_f} - \frac{1}{c}\int d^4x\, \delta\phi\left(\left(\frac{\partial\mathcal{L}}{\partial\phi_{,\alpha}}\right)_{,\alpha} - \frac{\partial\mathcal{L}}{\partial\phi}\right). \tag{52}$$

where we have imposed periodic spatial boundary conditions[9]. If we demand that $\delta\phi \to 0$ for $t_i \to -\infty$ and $t_f \to +\infty$ then the boundary term $[\ldots]$ vanishes, and the vanishing of δS for otherwise arbitrary $\delta\phi$ requires $(\ldots)$ in the integral vanishes, which gives us the *Euler-Lagrange equation*:

$$\boxed{\left(\frac{\partial\mathcal{L}}{\partial\phi_{,\alpha}}\right)_{,\alpha} = \frac{\partial\mathcal{L}}{\partial\phi}.} \tag{53}$$

This is valid for arbitrary $\mathcal{L}(\phi_{,\alpha}, \phi, \vec{x})$. From the specific Lagrangian density (48) – which has no explicit $\vec{x}$-dependence – we get the Euler-Lagrange equation for a real scalar field:

$$\boxed{\partial_\alpha\partial^\alpha\phi - \mu^2\phi = 0} \tag{54}$$

which is known as the *Klein-Gordon equation* for a real scalar field. This is a *free field*; it has no self-interaction, by virtue of the fact the the potential energy term is simply $\frac{1}{2}\mu^2\phi^2$, so we get a *linear wave equation*. This means that we can superpose solutions; waves, or wave-packets, can pass through one another and emerge undisturbed. It is for this reason that equation (48) is called the 'free-field' Lagrangian density.

Inserting a trial solution $\phi(\vec{x}) = \phi_0\cos(\mathbf{k}\cdot\mathbf{x} - \omega t)$, or equivalently, with $\vec{k} \longrightarrow (\omega/c, \mathbf{k})$ and $\vec{x} \longrightarrow (ct, \mathbf{x})$,

$$\phi(\vec{x}) = \phi_0\cos\left(\vec{k}\cdot\vec{x}\right) \tag{55}$$

gives the dispersion relation $-\vec{k}\cdot\vec{k} = \mu^2$ or

$$\omega^2 = c^2(k^2 + \mu^2). \tag{56}$$

[8]If we were working in *natural units* where the units of mass length and time are $M_{\mathrm{Pl}} = \sqrt{\hbar c/G} = E_{\mathrm{Pl}}/c^2$, $T_{\mathrm{Pl}} = \sqrt{\hbar G/c^5} = \hbar/E_{\mathrm{Pl}}$ and $L_{\mathrm{Pl}} = \sqrt{\hbar G/c^3} = \hbar c/E_{\mathrm{Pl}}$ (in which units, the values of the fundamental constants c, G and $\hbar$ are all numerically unity) then we could say that energy density has 'units' of E_{Pl}^4 and that both μ and ϕ have 'units' of $E_{\mathrm{Pl}} = \sqrt{\hbar c^5/G}$.

[9]There are two ways of thinking about periodic spatial boundary conditions. One is to say that, for all we know, our universe itself could be finite and periodic, but, provided it is large enough, this should have no effect on local physics. Another – and this is a much more satisfactory way of thinking about the situation in cosmological simulations that invoke periodic BCs – is that the universe is infinite, but that we choose to consider the evolution of fields that are periodic, with some long spatial period.

The KG equation (54) is actually often written with $mc/\hbar$ in place of μ. This is a source of considerable confusion, as the presence of $\hbar$ leads the uninitiated to think there is something quantum mechanical about this. But that is false; the field here is a *classical field*, just as the $\mathbf{E}$ and $\mathbf{B}$ fields in the classical Maxwell equation are classical fields. Writing the spatial frequency parameter μ in terms of $\hbar$ doesn't change this. Now of course, we *should* be treating the field using quantum mechanics. One way to do that, as discussed earlier, is to consider the Fourier modes of the ϕ-field as a set of independent simple harmonic oscillators. Each of these has a wave function – a function of the mode-amplitude $\phi_{\mathbf{k}}$ – and we can define creation and destruction operators and generate occupation number eigenstates etc. The particles thereby created, which are bosons, have Compton wave-number $k_C = mc/\hbar = \mu$. Transition amplitudes for scattering processes are then obtained by writing the *interaction Lagrangian density* in terms of these operators and then using the *non-relativistic Schrödinger equation* – not the equation above – to evolve the quantum state. In this framework, what (54) describes is the evolution of the *expectation value of the field* in such a quantum state, which, according to *Ehrenfest's theorem*, is described by the classical equations of motion.

As something of an aside, the boundary term $[\ldots]$ in (52) is a useful starting point for developing the Hamilton-Jacobi formalism as it tells us that the action, considered as a function of the field and time $S(\phi, t)$ for a family of fields that had the same value at some initial time t_i (but a range of time derivatives) has $\delta S/\delta\phi = c\partial\mathcal{L}/\partial\dot{\phi}$ and has $\partial S/\partial t$ given by minus the Hamiltonian: $\partial S/\partial t = \int d^3x\,(\mathcal{L} - \dot{\phi}\partial\mathcal{L}/\partial\dot{\phi}) = -\int d^3x\,T^{00}$. We could then write down an approximation to the wave function $\psi(\phi, t) \sim e^{iS/\hbar}$ *à la* Dirac and Feynman.

2.3.2 Stress-energy tensor for the real scalar field

For the 'free-field' (with no interaction to other fields or particles) the stress energy tensor is simply

$$T^{\mu\nu} = \begin{bmatrix} \mathcal{E} & \mathcal{P} \\ \mathcal{F} & \mathcal{S} \end{bmatrix} \tag{57}$$

where e.g. $\mathcal{F}$ is the generalisation to 3-dimensions of the 1-dimensional energy flux density used above and can be cast into a transparently Lorentz invariant form using the re-scaling described above. This is quite analogous to the stress-energy tensor for electromagnetic waves, with $\mathcal{F}$ being the analogue of the *Poynting flux*.

As we may want to include couplings between fields, or other external influences, we now show how the stress-energy tensor is obtained for the more general Lagrangian density $\mathcal{L}(\phi_{,\alpha}, \phi, \vec{x})$. The partial derivative of $\mathcal{L}(\vec{x}) = \mathcal{L}(\phi_{,\alpha}(\vec{x}), \phi(\vec{x}), \vec{x})$ is

$$\begin{aligned} \partial_\beta \mathcal{L}(\vec{x}) &= \frac{\partial\mathcal{L}}{\partial\phi_{,\alpha}}\partial_\beta\phi_{,\alpha} + \underbrace{\frac{\partial\mathcal{L}}{\partial\phi}\partial_\beta\phi} + \frac{\partial\mathcal{L}}{\partial x^\beta} \\ &= \underbrace{\frac{\partial\mathcal{L}}{\partial\phi_{,\alpha}}\partial_\alpha\phi_{,\beta} + \partial_\alpha\left(\frac{\partial\mathcal{L}}{\partial\phi_{,\alpha}}\right)\phi_{,\beta}} + \frac{\partial\mathcal{L}}{\partial x^\beta} \\ &= \partial_\alpha\left(\frac{\partial\mathcal{L}}{\partial\phi_{,\alpha}}\phi_{,\beta}\right) + \frac{\partial\mathcal{L}}{\partial x^\beta} \end{aligned} \tag{58}$$

where, in the first step we have used the commutativity of partial derivatives and have invoked the equation of motion (53). Note that here and below, if the arguments of $\mathcal{L}$ are not given explicitly $\mathcal{L}$ assumed to denote $\mathcal{L}(\phi_{,\alpha}, \phi, \vec{x})$. Thus, for example, final terms on the right hand side are the partial derivative of $\mathcal{L}(\phi_{,\alpha}, \phi, \vec{x})$ holding $\phi_{,\alpha}$ and ϕ fixed (whereas on the left hand side $\partial_\beta\mathcal{L}(\vec{x})$ means the derivative with respect to the β^{th} component of $\vec{x}$ holding the other components fixed).

Replacing $\partial_\beta\mathcal{L}(\vec{x})$ by $\delta^\alpha_\beta\partial_\alpha\mathcal{L}(\vec{x})$ gives

$$T^\alpha{}_{\beta,\alpha} = \mathcal{L}_{,\beta} \tag{59}$$

where the (mixed version) of the *stress-energy tensor* is

$$T^\alpha{}_\beta \equiv -\frac{\partial\mathcal{L}}{\partial\phi_{,\alpha}}\phi_{,\beta} + \delta^\alpha_\beta\mathcal{L} \tag{60}$$

or, raising the index β with the Minkowski metric,

$$\boxed{T^{\alpha\beta} \equiv -\frac{\partial\mathcal{L}}{\partial\phi_{,\alpha}}\phi^{,\beta} + \eta^{\alpha\beta}\mathcal{L}} \tag{61}$$

obeying the continuity equation(s)

$$\boxed{T^{\alpha\beta}{}_{,\alpha} = \mathcal{L}^{,\beta}} \tag{62}$$

where, at the risk of being repetitive, the right hand side means the (contravariant components of the) derivative of $\mathcal{L}(\phi_{,\alpha}, \phi, \vec{x})$ with the field and its derivatives held fixed.

For the free field with Lagrangian given by (48), the stress-energy tensor is

$$T^{\alpha\beta} = \phi^{,\alpha}\phi^{,\beta} + \eta^{\alpha\beta}(-\tfrac{1}{2}\phi_{,\gamma}\phi^{,\gamma} - \tfrac{1}{2}\mu^2\phi^2) \tag{63}$$

or, in 3+1 form,

$$T^{\alpha\beta} = \begin{bmatrix} \mathcal{E} & \boldsymbol{\mathcal{P}} \\ \boldsymbol{\mathcal{F}} & \boldsymbol{\mathcal{S}} \end{bmatrix} = \begin{bmatrix} \frac{1}{2}(\dot{\phi}^2/c^2 + |\boldsymbol{\nabla}\phi|^2 + \mu^2\phi^2) & -\dot{\phi}\boldsymbol{\nabla}\phi/c \\ -\dot{\phi}\boldsymbol{\nabla}\phi/c & \boldsymbol{\nabla}\phi \otimes \boldsymbol{\nabla}\phi + \frac{1}{2}(\dot{\phi}^2/c^2 - |\boldsymbol{\nabla}\phi|^2 - \mu^2\phi^2)\mathbf{I} \end{bmatrix} \tag{64}$$

where $\mathbf{I}$ is the 3×3 identity matrix and $\boldsymbol{\nabla}\phi \otimes \boldsymbol{\nabla}\phi$ is the (tensor) outer product. Note that this is symmetric. Note also that, for a purely time varying field, so $\boldsymbol{\nabla}\phi = 0$, as is assumed in inflation, the stress-energy tensor is diagonal: $T^{\alpha\beta} = \mathrm{diag}(\mathcal{E}, P, P, P)$ with $\mathcal{E} = \rho c^2 = \frac{1}{2}(\dot{\phi}^2/c^2 + \mu^2\phi^2)$ and pressure $P = \frac{1}{2}(\dot{\phi}^2/c^2 - \mu^2\phi^2)$, so, if $\dot{\phi}^2/c^2 \ll \mu^2\phi^2$ this gives $P = -\rho c^2$.

If there is no explicit dependence of $\mathcal{L}$ on $\vec{x}$, the right hand side of (62) vanishes, and we have $T^{\alpha\beta}{}_{,\alpha} = 0$. In that situation, there are 4 conserved quantities: $\int d^3r\, T^{0\beta}$, which are the space integrals of $\mathcal{E}$ and $\boldsymbol{\mathcal{P}}$, so the total 4-momentum is conserved.

It is interesting to compare (64) with the stress-energy tensor for electromagnetism. The (mixed) covariant form of this, which can be compared with (63), is

$$T^{\mu}{}_{\nu} = \mu_0^{-1}(F^{\mu\alpha}F_{\nu\alpha} - \tfrac{1}{4}\delta^{\mu}_{\nu}F^{\alpha\beta}F_{\alpha\beta}) \tag{65}$$

(where $F_{\alpha\beta} \equiv A_{\alpha,\beta} - A_{\beta,\alpha}$ is the Faraday tensor) and which can be derived,[10] as can Maxwell's equations, from the Lagrangian density

$$\mathcal{L}(A_{\mu,\nu}) = -\tfrac{1}{4\mu_0}F^{\alpha\beta}F_{\alpha\beta}. \tag{66}$$

The 3+1 form of this (as derived by Maxwell and Poynting) is

$$T^{\alpha\beta} = \begin{bmatrix} \mathcal{E} & \mathbf{P}/c \\ \mathbf{P}/c & -\boldsymbol{\sigma} \end{bmatrix} = \begin{bmatrix} \frac{1}{2}(\epsilon_0|\mathbf{E}|^2 + \mu_0^{-1}|\mathbf{B}|^2) & \mathbf{E}\times\mathbf{B}/\mu_0 c \\ \mathbf{E}\times\mathbf{B}/\mu_0 c & -\epsilon_0(\mathbf{E}\otimes\mathbf{E} - \frac{1}{2}|\mathbf{E}|^2\mathbf{I}) - \mu_0^{-1}(\mathbf{B}\otimes\mathbf{B} - \frac{1}{2}|\mathbf{B}|^2\mathbf{I}) \end{bmatrix} \tag{67}$$

in which we see the Poynting flux $\mathbf{P} = \mathbf{E}\times\mathbf{B}/\mu_0$ and the Maxwell 3-stress tensor[11] $\boldsymbol{\sigma}$. So we see that the stress-energy tensors for the scalar field and EM are quite similar. The latter is very similar to the former for a massless field ($\mu = 0$) and with, roughly speaking, $\mathbf{E}$ and $\mathbf{B}$ playing the role of the time and space derivatives of the field ϕ. The details of the 3-stress are somewhat different, however. In particular, while the EM field has tension along the field lines, the isotropic tension in inflation – probably the most important application of scalar fields in cosmology as it accounts for the creation of the universe – arises from the potential term (here $T_{ij} = -\frac{1}{2}\mu^2\phi^2\delta_{ij}$ but, more generally, $T_{ij} = -V(\phi)\delta_{ij}$).

2.3.3 More general relativistic field theories

It is possible to construct variants of the KG theory while maintaining the attractive properties of relativistic invariance.

One possibility is to replace the *harmonic potential energy* term $\mu^2\phi^2/2$ by a more general function of the field, usually denoted by $V(\phi)$. In the mechanical analogue, this would correspond to making the 'base springs' anharmonic. This changes things radically: the free-field theory described above has equations of motion that are linear in ϕ so we can add solutions and plane wave solutions, wave-packets etc., can propagate without interacting with each other. With a *non-harmonic potential* $V(\phi)$ the force $-dV/d\phi$ in the KG equation is no longer linear in ϕ. So there will be *wave-wave interactions* at the classical level and, correspondingly, scattering of particles in the quantised field.

[10] If you follow the procedure used here you will end up with a version of the stress-energy tensor which, like (65), has vanishing 4-divergence, but which is actually not symmetric. But, as described in Jackson, you can symmetrise this by adding a term that does not change the 4-divergence.

[11] Maxwell defined σ to be minus the momentum flux density

Another possibility is to have *multi-component scalar fields*. One can construct theories in which the Lagrangian is the sum of the individual free-field Lagrangians plus interaction term(s) involving the various fields. For example $\mathcal{L} = \mathcal{L}_\phi + \mathcal{L}_\psi + \mathcal{L}_{\rm int}(\phi, \psi)$ containing two free-field Lagrangians plus an interaction term. This again would allow scattering of waves also such phenomena as spontaneous symmetry breaking.

Another application of multi-component scalar field is the *complex scalar field* (a field with two components that can be represented as the real and imaginary parts of a complex field). This can represent an electrically charged field if coupled to the electromagnetic field by replacing the partial derivatives ∂_μ in the theory by the *gauge-covariant derivatives* $D_\mu \Rightarrow \partial_\mu - i(q/\hbar)A_\mu$. Another application of a complex scalar we will explore later – this time with a 'wine-bottle-bottom' potential, so it is self-interacting – can generate cosmic strings.

A great many variations on the above themes have been explored in cosmology. Much more radical – and questionable – are proposals to modify the Lorentz-scalar *kinetic term* $-\phi_{,\alpha}\phi^{,\alpha}$ in the Lagrangian density[12], by replacing it, for instance, by some function of this scalar.

2.3.4 What does it mean?

We started with a simple, and conceptually straightforward, model for a 'solid-state' lattice – a slightly modified version of what Ziman [?] calls the *scalar elasticity model* – where the field was simply the physical displacement of the masses and the energy was the kinetic and potential energies.

Waves and wave-packets on such a lattice, we noticed, had properties rather similar to that of relativistic particles, and the Lagrangian density has a symmetry very similar to that of relativistic systems under Lorentz boost and other transformations (with the high-k asymptotic sound speed in place of the speed of light).

The relativistic real massive scalar field is *mathematically* identical; the formal transition being essentially a matter of choosing appropriate constants for the three terms in the Lagrangian density. But you probably shouldn't take this literally and think of there being an underlying physical lattice on which the fields we observe are a physical displacement. Though it should be recognized that e.g. Maxwell *did* think of the EM fields as propagating through an *aether* that had some microphysical mechanism communicating disturbances. He talked of the *hidden underworld* in which *'the medium [...] may have rotatory as well as vibratory motion'*. He is also supposed to have said *'I didn't really get rid of action at a distance, I just replaced a big action by lots of little actions'*.

The field ϕ is probably best visualised as a displacement in some abstract space. In this view, the the elastic lattice model is seen as something that just happens to have exactly the same equations of motions, and demonstrates all of the continuity laws etc.. It is nonetheless perhaps helpful in giving a mathematically precise but conceptually unchallenging analogy that greatly helps visualise and concretise the mathematical concepts.

If ϕ 'lives' in an abstract space, we don't need to worry about the 'normal' momentum; we just have fields, and all there is is wave-momentum. It may be transferred between the different fields, but it obeys continuity by virtue of the symmetry of space-time with respect to translation and its space integral – if defined – is conserved.

The theory thus developed is that of a *classical* field; the equation of motion is, it turns out, identical to the relativistic equation for a single-particle wave function originally proposed – but then discarded – by Schrödinger, but the interpretation is very different. The free field can, as discussed above, be decomposed into independent harmonic oscillators which can be quantised in the usual way (exactly as we would for phonons) and this leads to bosonic particles, and that allows one to construct, for example, coherent states that are analogous to the coherent light from a laser. These are states which are a carefully organised superposition of occupation number eigenstates whose wave-functions – functions of the mode amplitude $\phi_{\mathbf{k}}$ that is – add up to give a a well defined *expectation value* for $\phi_{\mathbf{k}}$, about which the quantum uncertainty is relatively small, and in which the expectation value obeys the classical field equations derived above.

2.3.5 Applications of the scalar field

Here we expand, a little, on the introductory comments. As noted by Zel'dovich, scalar fields are 'the genie that escaped from the bottle' with the invention of the Higgs field, and have been used extensively in

[12] As already mentioned, this terminology may seem rather odd since, in the elastic analogy it is only the time derivatives that one would consider to be kinetic energy; the spatial gradients of ϕ contributing to the potential energy.

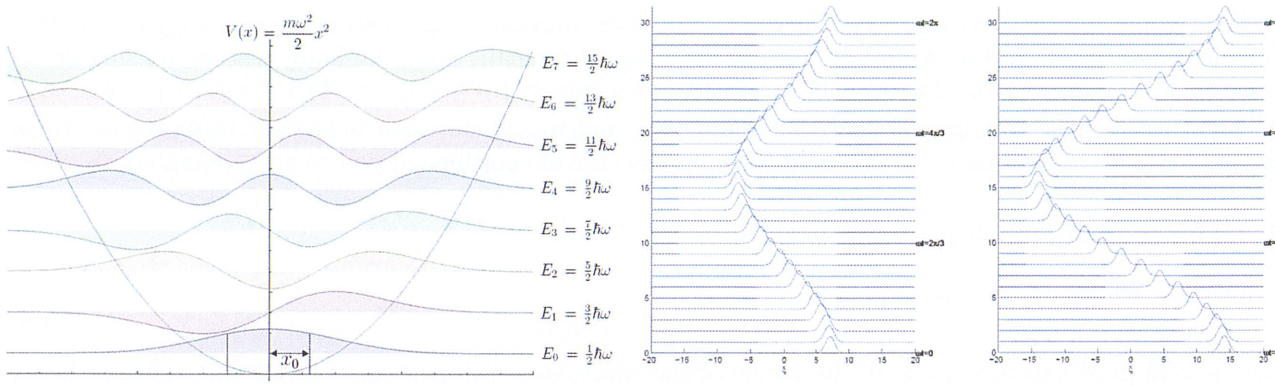


Figure 7: Energy eigenstates $|n\rangle$ for a quadratic potential are shown on the left. The expectation value of the displacement $\langle n|\, x(t)\, |n\rangle$ vanishes in each of these states. On the right are shown coherent states $|\alpha\rangle = e^{-|\alpha|^2/2} \sum\limits_n \frac{\alpha^n}{\sqrt{n!}} |n\rangle$ (discovered by Roy Glauber) for $|\alpha|^2 = 5$ and 10. These states have a well defined expectation value $\overline{x}(t) = \langle\alpha|\, x(t)\, |\alpha\rangle$ which oscillates between the classical turning points and which obeys the classical equation of motion $\ddot{x} = -\omega^2 x$. The assumption usually invoked in cosmology is that each **k**-mode of the field is in a coherent state, and this results in a well defined expectation value for the total field synthesised from these modes.

cosmology; most notably in inflation but there are many other applications ranging from the sublime to the ridiculous.

At the sublime end of the spectrum is the *Peccei-Quinn axion* which is a fairly well motivated candidate for the *dark matter* (DM). More speculatively, *ultra-light scalar fields* are another popular candidate for the DM. These are applications where the field is effectively free.

Scalar fields are also invoked to explain accelerated expansion in two different contexts: One is in the early universe, when, in what is called *inflation*, a phase preceding what is usually called the *big-bang*, the universe underwent accelerated expansion, driven by a hypotherical *inflaton field*. The other is in the late universe, where the universe seems to be entering a period of *late-time inflation* and hypothetical field invoked to drive this is often called *quintessence* (though it should be noted that *Einstein's cosmological constant* with a suitably small value is a viable alternative).

These latter applications typically, but not necessarily, invoke non-harmonic potentials $V(\phi)$. The essential idea is that if, within the region of the universe of interest, the scalar field is homogeneous, one can ignore the $(\nabla\phi)^2$ term in the stress-tensor and the energy density and pressure are then $\mathcal{E} = \rho c^2 \simeq \dot{\phi}^2/2 + V(\phi)$ and $P \simeq \dot{\phi}^2/2 - V(\phi)$. If, moreover, the time derivative is sufficiently small, this allows the pressure to be *negative*, and, if $P < -\rho c^2/3$ this drives accelerated expansion.

Yet another, extremely rich, application of scalar fields is in attempts to modify Einsteinian gravity. This goes back to the Brans-Dicke scalar-tensor theory for gravity.

There are many beautiful similarities, some of which will become apparent later, between scalar waves, and scalar field wave-packets in particular, and particles, both in how they behave and in the form of their stress-energy tensor, which appears as the source driving Einstein's equations. We will show, for example, that if one has a random sea of KG waves then the stress energy tensor is $\mathbf{T} = \int d^3k P_\phi(\mathbf{k}) \vec{k} \otimes \vec{k}$, which is identical to that for particles obtained above, $\mathbf{T} = \int d^3p/p^0 f(\mathbf{p}) \vec{p} \otimes \vec{p}$, with the phase-space density $f(\mathbf{p})/p^0(\mathbf{k})$ replaced by the power-spectrum of the waves: $P_\phi(\mathbf{k} = \mathbf{p}/k)$.

3 The classical scalar field in a general coordinate system

We allowed for, but have not yet much used, the possibility that the Lagrangian density may have an explicit dependence on location in space-time. Space-time dependence of the Lagrangian density comes about in at least two ways. First, if we have interactions between fields, for example two fields ϕ and ψ with a total Lagrangian density $\mathcal{L} = \mathcal{L}_\phi + \mathcal{L}_\psi + \mathcal{L}_{\rm int}(\phi, \psi)$ containing two free-field terms plus an interaction term, then we can study the evolution of one of the fields, say $\phi(\vec{x})$, considering the effect of $\psi(\vec{x})$ as an external influence. This changes the force term in the KG equation for ϕ and it gives a non-vanishing 4-divergence for the energy and momentum of the ϕ-field, as it can exchange energy and momentum with the other field.

Second, and this is the aspect we explore here, it allows one to obtain the equations of motion and continuity equations in alternative coordinate systems. One important application in cosmology is to obtain an efficient description of the evolution of waves that are statistically spatially homogeneous on surfaces of constant cosmic time. To do this, we will use FLRW coordinates (in Lecture 6: §2.1). Another is to model how scalar waves behave in inhomogeneous cosmologies; and we explore their behaviour using the metric of weak-field gravity in Lecture 6: §3. We also consider the case of coordinates tied to an accelerated observer.

3.1 The transformation of the action

Consider an element of the action

$$dS = d^4x\,\mathcal{L}(\phi_{,\alpha}, \phi, \vec{x}) \tag{68}$$

which is valid in Minkowski space, or, by virtue of the equivalence principle, in a locally inertial coordinate system. What does this look like in a general coordinate system?

Consider the free-field Lagrangian density was $\mathcal{L} = -\frac{1}{2}\phi_{,\alpha}\phi^{,\alpha} - \mu^2\phi^2$. The kinetic term contains $\phi_{,\alpha}\phi^{,\alpha} = \eta^{\alpha\beta}\phi_{,\alpha}\phi_{,\beta}$. This is the same, in inertial coordinates, as $g^{\alpha\beta}\phi_{;\alpha}\phi_{;\beta}$ since, in such coordinates, the components of (the inverse of $\mathbf{g}$) are $\eta^{\alpha\beta}$ and the Christoffels vanish. But that is a scalar, so it is valid in arbitrary coordinates. Also, the covariant derivavative of a scalar field is just the ordinary derivative: $\phi_{;\alpha} = \phi_{,\alpha}$, so we have, in general, $\phi_{,\alpha}\phi^{,\alpha} \Rightarrow g^{\alpha\beta}\phi_{,\alpha}\phi_{,\beta}$.

Alternatively, going back to basics, we may simply write the new (primed) coordinate system as

$$x^{\alpha'} = x^{\alpha'}(x^{\alpha}) \tag{69}$$

for which coordinate differentials transform as

$$dx^{\alpha'} = \Lambda^{\alpha'}{}_{\alpha}dx^{\alpha} \tag{70}$$

with transformation matrix $\Lambda^{\alpha'}{}_{\alpha} = \partial x^{\alpha'}/\partial x^{\alpha}$. The partial derivatives are then $\phi_{,\alpha} = \Lambda^{\alpha'}{}_{\alpha}\phi_{,\alpha'}$, so the kinetic term is[13] $\Lambda^{\alpha'}{}_{\alpha}\Lambda^{\beta'}{}_{\beta}\eta^{\alpha\beta}\phi_{,\alpha'}\phi_{,\beta'} = g^{\alpha'\beta'}\phi_{,\alpha'}\phi_{,\beta'}$.

Either way, we simply need to make the replacement

$$\phi_{,\alpha}\phi^{,\alpha} \Rightarrow g^{\alpha'\beta'}\phi_{,\alpha'}\phi_{,\beta'} \tag{71}$$

in the Lagrangian density. The mass term is invariant as ϕ^2 is a scalar, so for the free scalar field, the Lagrangian density becomes

$$\boxed{\mathcal{L}(\phi_{,\alpha'}, \phi) = -\tfrac{1}{2}g^{\alpha'\beta'}\phi_{,\alpha'}\phi_{,\beta'} - \tfrac{1}{2}\mu^2\phi^2} \tag{72}$$

and self-interacting fields would be the same, but with $\frac{1}{2}\mu^2\phi^2 \Rightarrow V(\phi)$

The other thing we need to do is replace d^4x by $|\partial x^{\alpha}/\partial x^{\alpha'}|d^4x'$ where we see the *Jacobian of the transformation*, which is the determinant of the transformation matrix. But we would rather express this in terms of the metric. In many of the cases we will consider the primed-frame metric is diagonal, and the 4-volume element is simply $d^4x = \prod_{\alpha'}\sqrt{|g_{\alpha'\alpha'}|(dx^{\alpha'})^2} = \sqrt{-|\mathbf{g}|}d^4x'$. For example, for the spatially flat FLRW metric $ds^2 = -c^2d\tau^2 + a(\tau)^2|d\mathbf{x}|^2$ this is straightforward and we have, in $x^{\alpha'} = (\tau, x, y, z)$ coordinates, $d^4x \Rightarrow ca^3d^4x'$. As another example, in Rindler rocket $x^{\alpha'} = (t, x, y, z)$ coordinates, $d^4x \Rightarrow c(1 + ax/c^2)d^4x'$. In general, a symmetric matrix $\mathbf{M}$ can be diagonalised: $\mathbf{M}' = \mathbf{R}\cdot\mathbf{M}\cdot\mathbf{R}^{-1} = \mathrm{diag}(\lambda_0, \ldots, \lambda_3)$ where the λ_i are the eigenvalues of $\mathbf{M}$ and $\mathbf{R}$ is a 'rotation' matrix (it has six degrees of freedom in 4-dimensions). But the determinant is invariant under this rotation, so, in general, we need to replace, in the element of the action above,

$$\boxed{d^4x \Rightarrow \sqrt{g}d^4x'} \tag{73}$$

where we are defining $\sqrt{g} \equiv \sqrt{-|\mathbf{g}|}$.

With these substitutions, and dropping the primes, the action is

$$\begin{aligned} S &= \frac{1}{c}\int d^4x\sqrt{g}\;\overbrace{\left(-\tfrac{1}{2}g^{\alpha\beta}(\vec{x})\phi_{,\alpha}\phi_{,\beta} - \tfrac{1}{2}\mu^2\phi^2\right)}^{\mathcal{L}(\phi_{,\alpha}, \phi, \vec{x})} \end{aligned} \tag{74}$$

[13] We use here the fact that the squared interval $ds^2 = \eta_{\alpha\beta}dx^{\alpha}dx^{\beta}$ is an invariant, so, using the transformation of differentials, this is $ds^2 = (\Lambda^{\alpha}{}_{\alpha'}\Lambda^{\beta}{}_{\beta'}\eta_{\alpha\beta})dx^{\alpha'}dx^{\beta'}$ which gives us the transformed metric $g_{\alpha'\beta'} = \Lambda^{\alpha}{}_{\alpha'}\Lambda^{\beta}{}_{\beta'}\eta_{\alpha\beta}$. The fact that its inverse transforms as $g^{\alpha'\beta'} = \Lambda^{\alpha'}{}_{\alpha}\Lambda^{\beta'}{}_{\beta}\eta^{\alpha\beta}$ is readily verified using the fact that $\Lambda^{\alpha'}{}_{\gamma}\Lambda^{\gamma}{}_{\beta'} = \delta^{\alpha'}_{\beta'}$.

where the first line is general and the second line is specific for the free massive scalar field, though self-interactions can be incorporated with $V(\phi)$ here in place of $\frac{1}{2}\mu^2\phi^2$. As indicated, the Lagrangian density in the new coordinates has acquired an explicit dependence on the coordinates coming through the metric in the kinetic term[14].

3.2 Scalar field equations in general coordinates

There are two ways to obtain the field equations; one is to vary the action (74), the other is simply to write the flat space-time KG equation in a generally covariant manner. These two routes yield expressions for the kinetic term that appear to be different, but are, in fact, equivalent. Both, as we shall see, are useful.

3.2.1 Field equations from variation of the action

Varying the action (1st line in (74)) gives

$$\begin{aligned}\delta S &= \frac{1}{c}\int d^4x \left(\sqrt{g}\frac{\partial\mathcal{L}}{\partial\phi_{,\alpha}}\delta\phi_{,\alpha} + \sqrt{g}\frac{\partial\mathcal{L}}{\partial\phi}\delta\phi \right) \\ &= \frac{1}{c}\int d^4x \left(\overbrace{\underbrace{\partial_\alpha\left(\sqrt{g}\frac{\partial\mathcal{L}}{\partial\phi_{,\alpha}}\delta\phi\right)}\underbrace{-\partial_\alpha\left(\sqrt{g}\frac{\partial\mathcal{L}}{\partial\phi_{,\alpha}}\right)\delta\phi + \sqrt{g}\frac{\partial\mathcal{L}}{\partial\phi}\delta\phi}} \right) \\ &= \frac{1}{c}\left[\sqrt{g}\frac{\partial\mathcal{L}}{\partial\phi_{,\alpha}}\delta\phi\right] + \frac{1}{c}\int d^4x\sqrt{g}\delta\phi\left[-\frac{1}{\sqrt{g}}\partial_\alpha\left(\sqrt{g}\frac{\partial\mathcal{L}}{\partial\phi_{,\alpha}}\right) + \frac{\partial\mathcal{L}}{\partial\phi}\right]\end{aligned} \tag{75}$$

Requiring that $\delta\phi$ vanish on the boundaries – which, if we have periodic spatial boundary conditions, means on the hypersurfaces $t = \pm\infty$ – and that δS vanish for arbitrary $\delta\phi$, gives the general form of *the scalar field equations of motion in an arbitrary coordinate system*:

$$\boxed{\frac{1}{\sqrt{g}}\partial_\alpha\left(\sqrt{g}\frac{\partial\mathcal{L}}{\partial\phi_{,\alpha}}\right) = \frac{\partial\mathcal{L}}{\partial\phi}.} \tag{76}$$

For the free massive scalar field Lagrangian density, for which $\partial\mathcal{L}/\partial\phi_{,\alpha} = -g^{\alpha\beta}\phi_{,\beta}$ and $V(\phi) = \frac{1}{2}\mu^2\phi^2$ so $\partial\mathcal{L}/\partial\phi = -dV/d\phi = -\mu^2\phi$, the *KG equation in an arbitrary coordinate system* is

$$\boxed{\frac{1}{\sqrt{g}}\partial_\alpha(\sqrt{g}g^{\alpha\beta}\partial_\beta\phi) - \mu^2\phi = 0.} \tag{77}$$

We see here that the d'Alembertian (or wave) operator $\Box$ that we have in the KG-equation in Minkowski coordinates has become

$$\Box = -\partial_\alpha\partial^\alpha \Rightarrow -\sqrt{g}^{-1}\partial_\alpha(\sqrt{g}g^{\alpha\beta}\partial_\beta). \tag{78}$$

We will use this below to study expanding scalar waves using FLRW coordinates.

3.2.2 Field equations from generalised covariance

The KG equation in locally inertial coordinates can be written in a generally covariant manner as

$$g^{\mu\nu}\phi_{;\mu;\nu} - \mu^2\phi = 0 \tag{79}$$

since then $g^{\mu\nu} = \eta^{\mu\nu}$ and $\phi_{;\mu;\nu} = \phi_{,\mu,\nu}$.

But this is a tensor (actually a scalar) equation, so should be valid in an arbitrary coordinate system. The field ϕ, being a scalar, has $\phi_{;\mu} = \phi_{,\mu}$ so

$$\phi_{;\mu;\nu} = \phi_{,\mu;\nu} = \phi_{,\mu\nu} - \Gamma^\alpha{}_{\mu\nu}\phi_{,\alpha}. \tag{80}$$

So the KG equation in general coordinates is

$$\boxed{g^{\mu\nu}(\phi_{,\mu\nu} - \Gamma^\alpha{}_{\mu\nu}\phi_{,\alpha}) - \mu^2\phi = 0.} \tag{81}$$

We will use this when we consider how scalar waves behave in gravitating structures.

[14] An alternative would be to define the Lagrangian density to be the entire integrand in the first line above; i.e. include the factor $\sqrt{g}$ in $\mathcal{L}$.

3.2.3 Equivalence of the two versions of the kinetic term

Comparing (81) with (77), it must be the case that

$$\begin{aligned}\frac{1}{\sqrt{g}}\partial_\mu(\sqrt{g}g^{\mu\nu}\partial_\nu\phi) &= g^{\mu\nu}\phi_{,\mu\nu} + \left(g^{\mu\nu}{}_{,\mu} + g^{\mu\nu}\sqrt{g}_{,\mu}/\sqrt{g}\right)\phi_{,\nu} \\ &= g^{\mu\nu}\phi_{,\mu\nu} + \left(g^{\mu\alpha}{}_{,\mu} + g^{\gamma\alpha}\sqrt{g}_{,\gamma}/\sqrt{g}\right)\phi_{,\alpha}\end{aligned} \tag{82}$$

is the same as

$$\begin{aligned}g^{\mu\nu}(\phi_{,\mu\nu} - \Gamma^\alpha{}_{\mu\nu}\phi_{,\alpha}) &= g^{\mu\nu}\phi_{,\mu\nu} - \tfrac{1}{2}g^{\mu\nu}g^{\alpha\gamma}(g_{\gamma\mu,\nu} + g_{\gamma\nu,\mu} - g_{\mu\nu,\gamma})\phi_{,\alpha} \\ &= g^{\mu\nu}\phi_{,\mu\nu} - g^{\mu\nu}g^{\alpha\gamma}(g_{\gamma\mu,\nu} - \tfrac{1}{2}g_{\mu\nu,\gamma})\phi_{,\alpha}.\end{aligned} \tag{83}$$

This can be verified using the fact that $(g^{\alpha\gamma}g_{\gamma\beta})_{,\nu} = \partial_\nu\delta^\alpha_\beta = 0$ so $g_{\gamma\beta}g^{\alpha\gamma}{}_{,\nu} = -g^{\alpha\gamma}g_{\gamma\beta,\nu}$ implying $g^{\mu\nu}{}_{,\gamma} = -g^{\mu\alpha}g^{\nu\beta}g_{\alpha\beta,\gamma}$ and an identity $\sqrt{g}_{,\gamma}/\sqrt{g} = \frac{1}{2}g^{\mu\nu}g_{\mu\nu,\gamma}$ from matrix algebra (see appendix A).

Alternatively, if you find the proof of the latter identity a headache, you may, quite legitimately, consider the above equivalence to be a proof of this.

3.3 Generalised energy and momentum continuity

There are two useful forms for the equations of continuity of energy and momentum. The first, and most general, as set out in §3.3.1, is to exploit the symmetry of space-time; simply differentiating the Lagrangian density provides the desired equations. The second, is to apply the comma $\Rightarrow$ semi-colon rule $T^\alpha{}_{\beta,\alpha} = 0$, assuming, that is, we are dealing with a field with no external influences. This is described, and shown to be equivalent to the first approach in §3.3.2.

3.3.1 Energy and momentum continuity from the symmetry of space-time

Much as before, to obtain the continuity equations we consider the differential of the integrand in the action – now $\sqrt{g}\mathcal{L}$ – considered as a function of $\vec{x}$:

$$\begin{aligned}\partial_\beta(\sqrt{g}(\vec{x})\mathcal{L}(\vec{x})) &= \underbrace{\sqrt{g}\frac{\partial\mathcal{L}}{\partial\phi_{,\alpha}}\phi_{,\alpha\beta} + \sqrt{g}\overbrace{\frac{\partial\mathcal{L}}{\partial\phi}}^{\sqrt{g}^{-1}\partial_\alpha(\sqrt{g}\partial\mathcal{L}/\partial\phi_{,\alpha})}\phi_{,\beta}} + \partial_\beta(\sqrt{g}\mathcal{L}(\phi_{,\alpha},\phi,\vec{x})) \\ &= \partial_\alpha\left(\sqrt{g}\frac{\partial\mathcal{L}}{\partial\phi_{,\alpha}}\phi_{,\beta}\right) + \partial_\beta(\sqrt{g}\mathcal{L})\end{aligned} \tag{84}$$

where, as indicated with the over-brace, we have invoked the equations of motion. Using, on the left side, $\partial_\beta = \delta^\alpha_\beta\partial_\alpha$, implies the continuity equation

$$\boxed{\partial_\alpha(\sqrt{g}T^\alpha{}_\beta) = \partial_\beta(\sqrt{g}\mathcal{L})} \tag{85}$$

where the stress-energy tensor is defined, just as before, by

$$\boxed{\begin{aligned}T^\alpha{}_\beta &\equiv -\frac{\partial\mathcal{L}}{\partial\phi_{,\alpha}}\phi_{,\beta} + \delta^\alpha_\beta\mathcal{L} \\ &= g^{\alpha\gamma}\phi_{,\gamma}\phi_{,\beta} + \delta^\alpha_\beta\mathcal{L}.\end{aligned}} \tag{86}$$

Note that, on the left hand side of (85) $T^\alpha{}_\beta$ is to be considered a function of $\vec{x}$; that is to say the partial derivative on the left is with respect to x^α holding the other components fixed. The right hand side, in contrast, contains the derivative of $\mathcal{L}$ holding the field (and its derivatives) constant.

Equation (85) is conceptually the cleanest way to express the continuity equations. An alternative expression is

$$\boxed{T^\alpha{}_{\beta,\alpha} = \sqrt{g}^{-1}(\partial_\beta(\sqrt{g}\mathcal{L}) - \sqrt{g}_{,\alpha}T^\alpha{}_\beta).} \tag{87}$$

3.3.2 Energy and momentum continuity from generalised covariance

An simple alternative route, particularly if we are interested only in the effect of the coordinate transformations, on the form of the continuity equations is to realise that $T^\alpha{}_\beta$ as defined in (86) is a tensor[15]. If

[15] The first term is $\phi^{;\alpha}\phi_{;\beta}$, as we are assuming the usual kinetic term, while $\mathcal{L}$ is a Lorentz scalar (we will assume) and δ^α_β is a tensor.

the field is not interacting with anything else (other than being affected by gravity) then, in a local inertial frame, continuity is simply $T^{\alpha}{}_{\beta;\alpha} = 0$, since commas and semi-colons are then equivalent. But this is a tensor equation, so

$$\begin{aligned} 0 = T^{\alpha}{}_{\beta;\alpha} = T^{\alpha}{}_{\beta,\alpha} &+ \overbrace{\tfrac{1}{2}g^{\alpha\sigma}(\cancel{g_{\sigma\gamma,\alpha}} + g_{\sigma\alpha,\gamma} - \cancel{g_{\gamma\alpha,\sigma}})T^{\gamma}{}_{\beta}}^{\Gamma^{\alpha}{}_{\gamma\alpha}T^{\gamma}{}_{\beta}} - \overbrace{\tfrac{1}{2}g^{\gamma\sigma}(\cancel{g_{\sigma\beta,\alpha}} + g_{\sigma\alpha,\beta} - \cancel{g_{\beta\alpha,\sigma}})T^{\alpha}{}_{\gamma}}^{\Gamma^{\gamma}{}_{\beta\alpha}T^{\alpha}{}_{\gamma}} \end{aligned} \tag{88}$$

where we see that many of the connection terms cancel, or

$$\boxed{T^{\alpha}{}_{\beta,\alpha} = \tfrac{1}{2}g^{\mu\sigma}(g_{\sigma\tau,\beta}T^{\tau}{}_{\mu} - g_{\mu\sigma,\tau}T^{\tau}{}_{\beta}).} \tag{89}$$

An important application of this is to provide the equation of hydrostatic equilibrium, either in a gravitational field or in an accelerated frame, which comes from the spatial components of this equation.

This looks a little different to (87) but is equivalent. To show this, we may note, first of all, using the identity $\sqrt{g}_{,\tau}/\sqrt{g} = \frac{1}{2}g^{\mu\sigma}g_{\mu\sigma,\tau}$, that the final terms in (87) and (89) are identical. Secondly, recalling that, in the first term on the RHS of (87), we are to differentiate $\mathcal{L} = -g^{\mu\nu}\phi_{,\mu}\phi_{,\nu} - V(\phi)$ holding ϕ and $\phi_{,\mu}$ fixed, we have $\partial_\beta\mathcal{L} = -\frac{1}{2}\phi_{,\mu}\phi_{,\nu}g^{\mu\nu}{}_{,\beta}$ so the first term is $\sqrt{g}^{-1}\partial_\beta(\sqrt{g}\mathcal{L}) = \mathcal{L}\sqrt{g}_{,\beta}/\sqrt{g} - \frac{1}{2}\phi_{,\mu}\phi_{,\nu}g^{\mu\nu}{}_{,\beta}$. Eliminating $\partial_\beta\sqrt{g}/\sqrt{g}$ using the above identity once more, then gives

$$\begin{aligned} \sqrt{g}^{-1}\partial_\beta(\sqrt{g}\mathcal{L}) &= \tfrac{1}{2}g_{\sigma\tau,\beta}\,(g^{\sigma\tau}\mathcal{L} + g^{\mu\sigma}g^{\nu\tau}\phi_{,\mu}\phi_{,\nu}) \\ &= \tfrac{1}{2}g_{\sigma\tau,\beta}(g^{\sigma\tau}\mathcal{L} - g^{\mu\sigma}\underbrace{\phi_{,\mu}\frac{\partial\mathcal{L}}{\partial\phi_{,\tau}}}_{-T^{\tau}{}_{\mu}+\mathcal{L}\delta^{\tau}_{\mu}}) \\ &= \tfrac{1}{2}g_{\sigma\tau,\beta}g^{\mu\sigma}T^{\tau}{}_{\mu} \end{aligned} \tag{90}$$

showing that the first terms on the RHS of (87) and (89) are also identical.

4 Inflation in the early universe

4.1 Motivation

The hot big bang model, in which the universe is described by a spatially homogeneous FLRW model, is successful in many ways. While the 'Copernical principle' (that the world looks the same to all fundamental observers) was initially a simplifying assumption, observations have show that it appears to hold to a high degree when the universe is observed on a very large scale. And the success of big-bang nucleosynthesis suggests that this model holds back to when the universe was only a second or so old.

But, as we have already discussed, there are some problematic features of these models. One of these is the *horizon problem* which says that the remarkable uniformity we see on large scales today must be imposed acausally. The other is the *flatness problem* which is that the observational fact that (Ω is currently not very different from unity) requires it to have been astonishingly close to unity in the distant past. Both of these problems render the models unattractive since the basic properties of flatness and homogeneity are not really explained by the theory, rather they must be imposed as finely tuned initial conditions.

To these problems we can add the *monopole problem*. In *grand unified theories* (GUTs) massive magnetic monopoles are predicted to exist. In the hot big bang model, at the time of GUT symmetry breaking (as the Universe cools through the GUT temperature of around 10^{16}GeV) these monopoles appear as topological defects, with a number density on the order of one per horizon size. These objects are a definite prediction of GUTs, yet their existence in anything like this abundance would be a disaster for cosmology, as they would have a density today hugely in excess of that observed.

There are also a number of additional unsettling features of the hot big bang model. One might ask what happened before the initial singularity? What are the seeds of the structure that we see in the Universe? What explains the *baryon asymmetry* of the Universe? There are now $\sim 10^8$ photons per baryon, which seems to imply that there was initially a slight asymmetry between baryons and anti-baryons at the one part in 10^8 level. Why has the Universal expansion started to accelerate? And why did it start so recently? What sets the masses of the neutrinos? Which appear to be such that they have only recently become non-relativistic.

tension 张力；拉紧.

没有解决 neutrino, baryon/antibaryon.

4.2 The Inflationary Scenario

In the *inflationary scenario* – which emerged in the late 70s – several of these problems appear to be solved or at least ameliorated. The essence of inflation is to assume that at early times the Universe passed through a phase with a strongly negative pressure (i.e. positive tension).

Let us start with the horizon problem. As we have already discussed, this can be traced directly to the deceleration of the expansion Universe; if $\ddot{a} < 0$ then the velocity difference between any two observers, which is proportional to $\dot{a}$, decreases with time. Therefore, going back in time, the relative velocity inexorably increases and at some finite time in the past reaches the speed of light c, and before that the two observers cannot exchange information or causal influences.

The only way to avoid this is for the Universe to have undergone an *accelerating phase* with $\ddot{a} > 0$ in its early history. From the acceleration equation $\ddot{a} = -(4\pi/3)G(\rho + 3P/c^2)a$ this requires $\rho + 3P/c^2 < 0$, or a strong negative pressure $P < -\rho c^2/3$. This is the strange, and somewhat counterintuitive, feature of the general relativistic expansion law; just as a positive pressure augments the gravitational deceleration, a sufficiently strong negative pressure can cause the expansion to accelerate.

At first sight it is hard to see how a negative pressure can arise. For a gas of particles interacting through localized collisions, the pressure cannot be negative. As shown in Weinberg's book, for instance, a relativistic ideal gas must have pressure in the range $0 \leq P \leq \rho c^2/3$. Also, pressure is the flux of momentum. A particle moving in the positive x direction carries a positive x-component of momentum, and therefore the flux of x-momentum passing in the positive x-direction through a surface must be positive.

However, if we consider *fields*, rather than particles, then the possibility of negative pressure is not at all unreasonable. After all, the most commonplace field that we can feel macroscopically is the magnetic field. Anyone who has played with a pair of bar-magnets or pulled magnets off a fridge knows that such fields have strong tension. However, such fields do not have *isotropic* tension; there is tension along the field lines — you have to do work to stretch the field out and create more of it — but in the transverse directions the opposite is true; as we see from images of the field produced with iron filings the field between a pair of magnets clearly wants to burst out sideways. And high current solenoids need to be physically constrained from flying apart. This transverse pressure follows directly from energetic considerations, along with flux conservation. Imagine you try to confine the field to pass through a smaller area. Flux conservation means that the field strength must increase inversely with the area, but the energy density scales as the square of the field strength, so the total energy is larger the smaller the cross-sectional area. A static magnetic field then has negative pressure along the field lines but positive pressure in the transverse directions.

This anisotropy of the pressure for a macroscopic quasi-static magnetic field is associated with the fact that electromagnetism is a *vector field*. As we have seen, however, a *scalar field* with Lagrangian density $\mathcal{L} = -\frac{1}{2}\phi^{,\mu}\phi_{,\mu} - V(\phi)$ contains a term in the pressure tensor $P_{ij} = -\delta_{ij}V(\phi)$ which is isotropic and which, for positive potential, is negative.

The only scalar field known to exist is the Higgs field which has a 'w'-shaped potential with two minima, which allows the field to undergo 'spontaneous symmetry breaking', and which led the universe to transition from a state in which the weak and EM forces were indistinguishable to what we see today. The Higgs field is not itself thought to be responsible for inflation, but it provided Alan Guth – who was working on 'grand-unification'; postulated to unify the strong nuclear force as well – and others with the inspiration that if there was another scalar field it may have provided the negative pressure required to make the universal expansion accelerate.

The field equation for a classical scalar field with potential $V(\phi)$ – which may be simply a mass term $V = \frac{1}{2}m^2\phi^2$ – in FLRW coordinates is

$$\ddot{\phi}/c^2 + 3H\dot{\phi} - \nabla^2\phi/a^2 + dV/d\phi = 0 \tag{91}$$

(as discussed more fully below). A possible solution is that the field be spatially homogeneous: so $\boldsymbol{\nabla}\phi = 0$, in which case the stress energy tensor, whose components are, in a locally inertial coordinate system tied to a fundamental observer, are $T^{\mu}{}_{\nu} = -\phi^{,\mu}\phi_{,\nu} + \delta^{\mu}_{\nu}\mathcal{L}$, becomes $T^{\mu}{}_{\nu} = \mathrm{diag}(-\mathcal{E}, P, P, P)$, with energy density $\mathcal{E}$ and pressure P given by

$$\mathcal{E} = \tfrac{1}{2}\dot{\phi}^2/c^2 + V(\phi) \tag{92}$$

$$P = \tfrac{1}{2}\dot{\phi}^2/c^2 - V(\phi). \tag{93}$$

If this field dominates any other contributions to $T^{\mu}{}_{\nu}$, the Friedmann acceleration equation is then

$$\frac{\ddot{a}}{a} = \frac{4\pi G}{3c^2}(\mathcal{E} + 3P) = \frac{8\pi G}{3c^2}(\dot{\phi}^2/c^2 - V(\phi)) \tag{94}$$

so provided $\dot{\phi}^2/c^2 < V(\phi)$ (and provided V is positive), the universal expansion will accelerate.

Why would the field satisfy this condition? If we have either a mass term for the potential $V = \frac{1}{2}m^2\phi^2$, or some other positive-index power-law, it is possible that the field started out with some large value but with $\dot{\phi} = 0$, in which case the pressure would, initially, be very strongly negative $P = -\mathcal{E}$, so the equation of state parameter would be, initially, $w = -1$. But ϕ would then start to 'roll down' the potential, and as it picks up speed and V decreases w would increase. A critical ingredient of the field equation here is the 'Hubble-dampling' or frictional term $3H\dot{\phi}$ in the field equation. Depending on the form of the potential (and the initial conditions) it is possible that the field will reach 'terminal velocity' $\dot{\phi} = -(dV/d\phi)/3H$ (which is independent of the initial conditions) and, if $\dot{\phi}^2 \ll c^2 V$ the equation of state will be $w \simeq -1$, leading to exponential expansion, and if, moreover, the terminal velocity and field satisfy $|\dot{\phi}| \ll \phi$, the field will move relatively little in one expansion time, and the Universe could undergo a large number of e-foldings before eventually exiting the inflationary phase.

This then is the essential 'scenario' of inflation. There are clearly a number of conditions that the potential (and initial field value and velocity) need to satisfy for this to work successfully. There is also the big assumption that the field is spatially homogeneous. We will return to the question of how to 'design' a potential that will cause inflation. First we will look in a bit more detail at the solutions of the acceleration equation.

4.2.1 Solutions for $a(\tau)$ during inflation

Assuming $\dot{\phi}^2 \ll c^2 V$ we have $P = -\mathcal{E} = -\rho c^2$, and the continuity equation $\dot{\mathcal{E}} = -3H(\mathcal{E} + P)$ tells us that $\dot{\mathcal{E}} = 0$. The cosmological expansion does work against the tension of the field at just the rate required to keep the energy density constant; for this reason the inflationary universe has been dubbed the *ultimate free lunch*.

The acceleration equation is

$$\frac{\ddot{a}}{a} = -\frac{4}{3}\pi G(\rho + 3P/c^2) = \frac{8}{3}\pi G\rho, \tag{95}$$

with $\rho = \mathcal{E}/c^2 = \text{constant}$. The general solution of this second order equation is

$$a(t) = a_+ e^{+Ht} + a_- e^{-Ht} \tag{96}$$

with constants $a_\pm$ and

$$H = \sqrt{8\pi G\rho/3} = \text{constant}. \tag{97}$$

For generic initial conditions, a potential dominated universe, will tend towards an exponentially expanding solution $a \propto e^{Ht}$.

4.2.2 How inflation solves the horizon problem

The comoving horizon size — defined here as the comoving distance that a photon can travel per expansion time — is $r_{\rm H} \sim cH^{-1}/a$. During inflation, H is constant so $r_{\rm H}$ decreases exponentially as $r_{\rm H} \propto e^{-Ht}$. At early times during inflation photons can travel great comoving distances but this decreases as time goes on. In a viable inflationary model, inflation cannot continue forever, but must end, with conversion of the energy density — all stored in the scalar field — into 'ordinary' matter with $P = \rho c^2/3$, i.e. we must make a transition from a scalar field dominated universe to a radiation dominated hot-big bang model. Side-stepping, for the moment, the issue of exactly how this so-called 're-heating' occurs, the overall behavior of the comoving horizon scale (as we have defined it above) is shown as the solid line in figure 8. This allows the possibility that the entire Universe was initially in causal contact.

Let's look at this from the point of view of a pair of comoving observers. These have a constant comoving separation, as indicated by the horizontal dashed line say. During inflation, the velocity difference between these observers increases as they accelerate apart, and a pair of observers with initial recession velocity $v < c$ will at some time lose causal contact with each other once their relative velocity[16] reaches the speed

[16] When we say 'relative velocity' here we mean the rate at which their physical separation is changing with time.

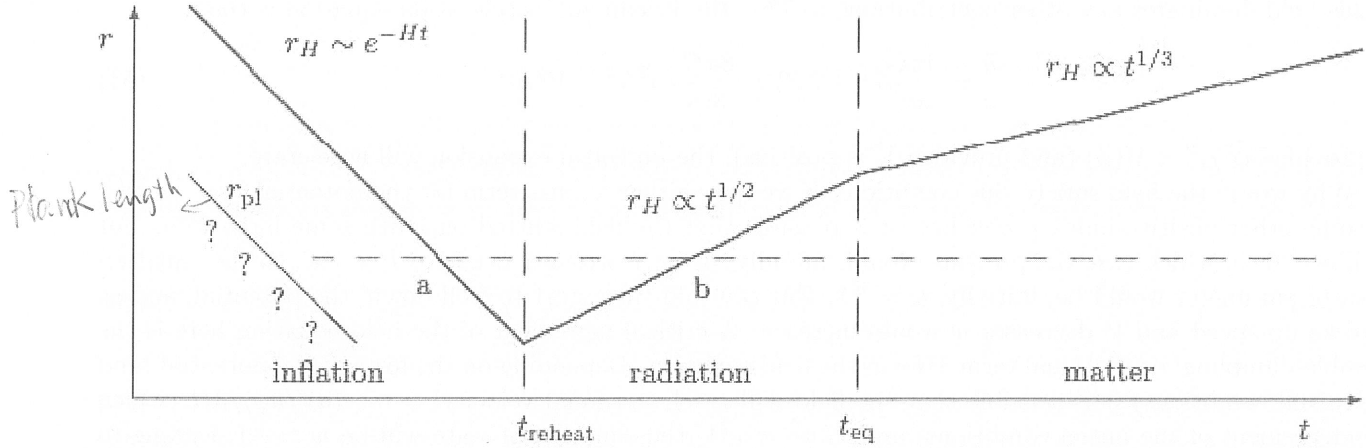


Figure 8: The evolution of the comoving horizon scale (heavy solid line) in a Universe which passes through three phases; an inflationary stage followed by a radiation dominated and then a matter dominated era. The r-axis is logarithmic, while the time axis abscissa is linear in the inflationary era and logarithmic thereafter. The diagonal line labeled $r_{\rm pl}$ indicates the Planck length. The horizontal dashed line indicates the comoving separation of a pair of comoving observers, who start separated by the Planck scale at some time during the inflationary era. The observers first accelerate away from one another. At the point 'a' their physical separation is increasing with time as $dr_{\rm phys}/dt \sim c$, at which point they lose contact with one another. Much later, inflation ends, the universe starts to decelerate, and $dr_{\rm phys}/dt$ decreases. At point 'b', when $dr_{\rm phys}/dt \sim c$ once more, the observers can communicate once more. We say their separation exited the horizon at 'a' and re-entered at 'b'.

of light. If the universe later becomes radiation dominated, the relative velocity will subsequently fall and these observers can regain causal contact.

For those who feel uneasy with the somewhat hand-waving definition of the horizon size as the distance light can travel in an expansion time, consider instead the rigorous definition of the comoving distance to a distant source as a function of the 'look-back time' $\tau = t_0 - t$

$$r(\tau) = c \int_0^\tau \frac{d\tau}{a(t_0 - \tau)}. \tag{98}$$

In the matter dominated era this increases with decreasing τ at first, but tends towards a limiting asymptote. Back in the inflationary era, however, this integral grows exponentially and becomes arbitrarily large.

Coming back to our pair of comoving observers, to the left of point 'a' their separation is such that they can exchange light signals, and, if they did, they would perceive an increasing redshift. Their fixed comoving separation becomes equal to the horizon scale at 'a'. At that time their relative redshift becomes infinite. Subsequently they are unable to exchange signals. At the reheating epoch the Universe starts to decelerate, and at point 'b' the recession velocity falls below the speed of light. The observers then re-appear on each other's horizon; they can exchange signals which are received with steadily decreasing redshift.

The separation chosen here is such that it re-enters the horizon during the radiation dominated era. Larger separations enter the horizon at later times. To set the scale of this plot, The current horizon scale is $ct_0 \sim c/H_0 \simeq 4000$Mpc. Since the comoving horizon scale is proportional to $t^{1/3}$ in the matter era, the horizon scale at $t_{\rm eq}$ (where the slope of the comoving horizon size changes) is smaller than the current horizon by a factor ~ 100, or about ~ 40Mpc, roughly the scale of large super-clusters. Originally, inflation was conceived to happen (or, more precisely, the temperature to which the Universe reheats is) around the GUT scale, or around 10^{16}GeV. If so, this plot is highly distorted, as matter-radiation equality occurs when the temperature is a few eV, the comoving horizon increased by about 25 orders of magnitude during the radiation era.

4.2.3 How inflation solves the flatness problem

What about the flatness problem? Recall that departure from flatness is an indication of an imbalance between the kinetic and potential energy terms in the energy equation

$$\dot{a}^2 = \frac{8}{3}\pi G \rho a^2 - kc^2. \qquad (99)$$

For an exactly spatially flat universe $k = 0$ and these two terms are exactly equal. Now consider a universe which is initially open say, with $k = -1$. During an inflationary phase, the expansion accelerates, $\dot{a}$ increases, as must the potential energy term on the right hand side. Inflation acts to increase, exponentially, the kinetic and potential terms in the energy equation. Thus even if there is a non-zero initial energy constant, it will tend to become exponentially small at the end of inflation. Inflation therefore drives the universe towards flatness; the $\Omega = 1$ state becomes an attractor rather than an unstable state. Another way to look on this is to realize that in FLRW models — and the inflationary universe is an FLRW model, just one with a weird equation of state — the curvature scale is a comoving scale. Relative to the horizon scale, the curvature scale is stretched exponentially. Thus it might be that our universe is open or closed, but that the curvature scale has been stretched to be enormously larger than the currently observable region of the universe.

What about the monopole problem? These are topological defects of a field. Inflation allows this field to be coherent over very large scales; up to the initial comoving horizon scale. Provided the universe re-heats to a temperature less than the GUT scale, monopoles — which have a mass around the GUT energy scale — will not be effectively created.

4.2.4 The required number of e-foldings

It is interesting to ask, how many e-foldings of inflationary expansion are required in order to establish causality over the region of the universe (size $l \sim c/H_0$) that we can currently observe? The answer depends on the temperature at which reheating occurs. If this reheating temperature is around the energy scale of *grand unification*, or $T \sim T_{\rm GUT} \sim 10^{16}$GeV, then the temperature falls by about a factor 10^{25} before the Universe becomes matter dominated at a temperature[17] of about a few eV. During that period the comoving horizon grows as $r_{\rm H} \propto t^{1/2} \propto a \propto 1/T$, or by about 25 orders of magnitude. Once the universe becomes matter dominated the horizon grows as $r_h \propto t^{1/3} \propto a^{1/2}$ or by about another factor of 100. The current horizon is therefore about $10^{27} \simeq e^{62}$ times larger now than at the reheating time, so we need at least about 60 e-foldings of inflation.

4.2.5 The size of the observable universe during inflation

If the reheating temperature is $T \sim T_{\rm GUT} \sim 10^{16}$GeV (or about 3 orders of magnitude less than the Planck energy[18]) that would say (since $H^2 \propto T^4$) the expansion time during inflation is about 6 orders of magnitude greater than the Planck time, putting the horizon scale at about 10^{-29}m. We estimated that the universe must have undergone about 62 e-foldings ($e^{62} \sim 10^{27}$) between the time the currently observable universe left the horizon and the end of inflation, that would say that, at the end of inflation, the current horizon volume had a physical size on the order of a cm. The result is highly dependent on the reheating energy scale; an early model gave this size to be that of a grapefruit.

4.3 Chaotic Inflation

Originally, it was imagined that the field driving inflation, the *inflaton field*, had a w-shaped potential of the kind involved in spontaneous symmetry breaking with the Higgs field. For reasons we shall not go into here, such models have fallen out of favor. Instead, most attention is currently focused on so-called *chaotic*

[17] The conversion from eV to Kelvin is that eV is the same as $k_{\rm B}T$ for $T \simeq 10^4$K.

[18] The Planck mass $m_{\rm P}$ is obtained by asking what is the mass m such that the gravitational radius $r \sim Gm/c^2$ is also equal to the Compton length $\hbar/mc$. The result is $m_{\rm P} = \sqrt{\hbar c/G} \simeq 2 \times 10^{-5}$gm. Multiplying by c^2 gives the Planck energy $E_{\rm P} = \sqrt{\hbar c^5/G} \simeq 10^{19}$GeV. The Planck time is $t_{\rm P} = \hbar/E_{\rm P} = \sqrt{G\hbar/c^5} \simeq 5 \times 10^{-44}$s. And the Planck length is $l_{\rm P} = ct_{\rm P} = \sqrt{G\hbar/c^3} \simeq 1.6 \times 10^{-35}$m. This defines the 'natural units' of mass, length and time in which units c, G and $\hbar$ are all numerically unity. It is convenient to express these all in terms of the Planck energy. Thus $m_{\rm P} = E_{\rm P}/c^2$, so masses can be expressed as the equivalent energy. Similarly, $t_{\rm P} = \hbar/E_{\rm P}$ and $l_{\rm P} = c\hbar/E_{\rm P}$ allow us to express times and lengths as inverse energies. The Lagrangian density has units of energy density [M/LT^2] (or like $E_{\rm P}^4$) and contains terms like $\mu^2\phi^2$ where μ is an inverse length. So the units of ϕ are [$\sqrt{\rm ML/T^2}$] (or like $E_{\rm P}$).

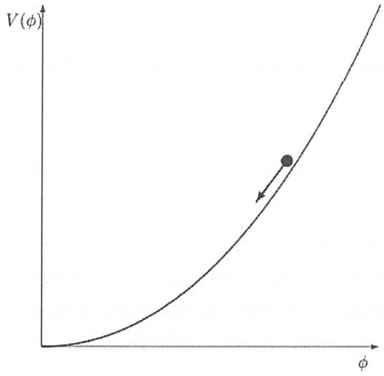


Figure 9: In the *chaotic inflation* scenario, the potential function $V(\phi)$ is assumed to be a monotonically increasing function with $V(0) = 0$. The potential could simply be a mass term $V \propto \phi^2$ or perhaps $V \propto \phi^4$. Here we will consider, as an illustrative example, $V(\phi) = \lambda\phi^4$ where λ is a 'coupling constant' describing the strength of the self-interactions of this field. The idea is that the field started at some initial value away from the origin, as indicated, and then rolled down, reaching a terminal velocity that is sufficiently small that the fractional change in ϕ in one expansion time is no larger than about $\epsilon = 1/62$, in order that there can be sufficiently many e-foldings so that the current observable size c/H_0 can exit the horizon during inflation.

inflation models in which the field has a potential function as sketched in figure 9. It is assumed that the field starts out at some point far from the origin, and then evolves to smaller values much as a ball rolling down a hill. In this section we shall explore what is required in order to obtain a viable inflationary scenario, i.e. one in which there are sufficiently many e-foldings.

For concreteness, we will consider a field with Lagrangian density

$$\mathcal{L} = -\tfrac{1}{2}\phi^{,\mu}\phi_{,\mu} - \lambda\phi^4. \tag{100}$$

This is a massless field with a self-interaction term parameterised by the constant λ. The units of λ are $[\mathrm{T}^2/\mathrm{L}^3\mathrm{M}]$, but if we work in natural units this is dimensionless, something one can also see from the fact that the units of the Lagrangian density are those of energy density (or like E_P^4) while that of the field is like E_P.

Assuming the field to be spatially uniform, the equation of motion is

$$\ddot{\phi} + 3H\dot{\phi} + 4\lambda c^2\phi^3 = 0, \tag{101}$$

where the last term is the potential gradient and the second term is the damping due to the cosmological expansion. The energy density and pressure are

$$\mathcal{E} = \frac{1}{2c^2}\dot{\phi}^2 + \lambda\phi^4 \tag{102}$$

$$P = \frac{1}{2c^2}\dot{\phi}^2 - \lambda\phi^4 \tag{103}$$

and the expansion rate is given by

$$H^2 = \frac{8\pi G}{3c^2}\mathcal{E} = \frac{8\pi G}{3c^2}\left(\frac{1}{2c^2}\dot{\phi}^2 + \lambda\phi^4\right). \tag{104}$$

Equation (101) is like that of a unit mass ball rolling down a hill with a frictional force, the coefficient of friction $3H$ being dependent on the field and the field velocity through (104). For such a system there are two limiting types of behavior, depending on the value of the field. In one, the friction is negligible and the field is in free-fall with $\ddot{\phi}$ equal to the potential gradient. In the other, the friction is important, the first term in (101) is negligible compared to the other terms and the field moves at a 'terminal velocity' such that the friction force just balances the potential gradient.

Let's assume, for the moment, that the former is the case. The effective equation of motion is then

$$\ddot{\phi} + 4\lambda c^2\phi^3 = 0. \tag{105}$$

The time-scale for changes in the field velocity is

$$t_\mathrm{accel} \sim \sqrt{\phi/\ddot{\phi}} \sim 1/\sqrt{\lambda c^2\phi^2}. \tag{106}$$

After one acceleration time-scale the field will acquire a velocity

$$\dot{\phi} \sim \ddot{\phi}t_\mathrm{accel} \sim \sqrt{\lambda}c\phi^2. \tag{107}$$

Squaring this we get, to order of magnitude, the the kinetic energy term in the energy density:

$$\dot{\phi}^2/c^2 \sim \lambda\phi^4 \tag{108}$$

so the potential and kinetic terms in the energy density are comparable.

Now the condition that the friction should still be negligible is $3H\dot{\phi} \ll \lambda c^2\phi^3$. Using (104), this inequality becomes

$$3H\dot{\phi} \simeq 3\sqrt{\frac{8\pi G\lambda\phi^4}{3c^2}}\sqrt{\lambda}c\phi^2 \ll \lambda c^2\phi^3. \tag{109}$$

The dimensionless interaction strength factors out of this inequality, so the condition that the friction be negligible is simply that the field be sufficiently weak ($\phi \ll \sqrt{c^4/G}$). Or equivalently that ϕ be strongly sub-Planckian.

Conversely, the condition that the friction should dominate — called *slow-roll condition* — which, of course, is what we desire, is

$$\phi \gg \sqrt{\frac{c^4}{G}}. \tag{110}$$

or that the field strength correspond to a strongly super-Planckian energy.

If so, the friction will dominate and the field will be unable to roll freely down the potential, rather it will roll slowly down the hill at the terminal velocity

$$\dot{\phi} = -4\lambda c^2\phi^3/3H. \tag{111}$$

Assuming that the inequality (110) holds, what is the equation of state, or equivalently how large is the positive kinetic energy term $\sim \dot{\phi}^2/c^2$ in the pressure as compared to the potential term $V = \lambda\phi^4/\hbar c$? Squaring the terminal velocity (111) and using the inequality $H^2 \geq 8\pi GV/3c^2$ yields

$$\frac{1}{2c^2}\dot{\phi}^2 = \frac{8\lambda^2c^2\phi^6}{9H^2} \leq \frac{8\lambda^2c^2\phi^6}{9}\left(\frac{8\pi G\lambda\phi^4}{3c^2}\right)^{-1} = \left(\frac{c^4}{G\phi^2}\right)\lambda\phi^4. \tag{112}$$

Which tells us that if the slow-roll condition (110) is obeyed, we must have

$$\frac{1}{2c^2}\dot{\phi}^2 \ll V(\phi). \tag{113}$$

The kinetic energy terms in the pressure and density are therefore much less than the potential terms and we therefore have $P \simeq -\rho c^2$ (or $w \simeq -1$) as required for inflation to proceed.

As already mentioned, in a viable model, inflation must be sustained for many e-foldings in order to solve the flatness, horizon problems. In one e-folding, the field will move a distance $\Delta\phi \sim \dot{\phi}/H$. For GUT scale inflation, where we need ~ 60 e-foldings, we need

$$\frac{\Delta\phi}{\phi} = \frac{\dot{\phi}}{H\phi} \lesssim \frac{1}{60} \equiv \epsilon. \tag{114}$$

Using (111) and $H^2 \sim GV/c^2 \sim G\lambda\phi^4/c^2$ this becomes

$$\phi \geq \sqrt{\frac{c^4}{\epsilon G}}. \tag{115}$$

Thus, the field needs to exceed the Planck value $\sqrt{c^4/G}$ by at least a factor $\epsilon^{-1/2} \sim 8$ in order to achieve sufficiently many e-foldings of inflation.

In this model, the field rolls slowly — very slowly at first — down the potential and the universe inflates. The expansion rate H does not remain precisely constant, but decreases slowly with time. Eventually the field reaches the value $\phi \simeq \sqrt{c^4/G}$, at which point the friction term $H\dot{\phi}$ in the equation of motion is no longer effective and the field starts to oscillate about the potential minimum.

What happens then is that the pressure oscillates, and is no longer strongly negative, so the energy density that has created itself in the scalar field will start to decrease. This is not what we want, which is a transition to a radiation dominated cosmology. So it is necessary that there be coupling of the 'inflaton' to other fields, so that the coherent oscillations of the ϕ-field can become incoherent and, eventually, one presumes thermalise.

The details of this so-called 're-heating' process depends on the details of the interactions between the field. In figure (8) we have assumed that re-heating happens promptly once the inflaton starts to oscillate.

We have considered a rather specific model above, with a $\lambda\phi^4$ potential. The main results are not specific to this choice. Had we instead assumed $V = \frac{1}{2}(m^2c^2/\hbar^2)\phi^2$ — i.e. a non-interacting, but massive, field, then we again find that the 'slow-roll' condition is simply that $\phi \gg \sqrt{c^4/G}$.

4.4 Discussion

The 'inflationary scenario' described above purports to be the answer to the question "what came before the hot big bang?". It provides a possible mechanism for the creation of all of the matter in the universe essentially out of nothing. It has the advantage that, starting from fairly generic initial conditions – that there be some region where the field is has small spatial field variation – the details of the initial state will get largely erased and the universe will be prepared in the state needed – nearly exactly spatial flat on the scales available to observations – as the, otherwise seemingly finely tuned, initial conditions for the following radiation phase.

All we require is that there be a field with some potential that starts off at a sufficiently high value; the initial value of the field velocity $\dot{\phi}$ is largely irrelevant, since the cosmic drag term rapidly reduces $\dot{\phi}$ to the terminal velocity.

As we shall discuss later, the inflationary scenario also creates density fluctuations which can seed the structures we see in the distribution of galaxies and in the cosmic microwave background. The amplitude of these fluctuations is strongly model dependent, but the prediction is for fluctuations with dependence on wavelength very much like that which seem to be required.

These results make the inflationary model highly attractive. On the down side, one has to invoke a new field, the inflaton, precisely to obtain these desirable results. Initially, the development of this field of research was strongly linked to developments in fundamental particle physics — spontaneous symmetry breaking etc. — but the subject has now taken on a life of its own. While we have used GUT-scale inflation in order to derive e.g. the number of e-foldings, there is really no need to assume this (though reheating to super-GUT temperatures would be problematic). Indeed, studies of the expansion rate using supernovae have suggested that the universal expansion is now accelerating; it would seem that we are entering another inflationary phase. The ideas described above can readily be re-cycled to describe late-time inflation by choosing appropriate parameters (specifically, this requires that the fields be very light). The inflaton field must be coupled to other fields in order to allow re-heating, and in principle this allows empirical tests of the theory. However, the requirements on the form and strength of the interaction are not very specific, and the energies required to make GUT-scale inflatons is beyond the reach of terrestrial particle accelerators. Aside from the 'predictions' of flatness, homogeneity and density fluctuations — all of which were observed before inflation was invented — it is hard to find testable predictions. One hope is that the inflaton field and its potential will emerge as the low-energy some more fundamental theory which unifies all of the forces, including gravity. This is an area of much activity at present, but hopefully will explain why there is an inflaton; why it has the potential it needs; why the minimum of the potential is at zero energy density and so on.

There is another rather unsettling aspect of the inflationary scenario, which is that we had to assume that the field is highly homogeneous. Many discussions of the subject simply argue that any inhomogeneity will be stretched to super-horizon scale in order to justify this assumption. Others invoke 'anthropic' arguments; the idea being that even if it is very unlikely to have a region which is sufficiently smooth to inflate, it will end up becoming very large, so it is not unreasonable that we find ourselves in a region which inflated (particularly if it is necessary for the existence of life for the Universe to have lived long enough to make stars etc.). This seems to me to be overly complacent. Recall that in figure (8) the boundary of the domain which we can describe without a theory of quantum gravity is not a fixed time $t = t_{\rm pl}$, rather the time at which a region starts to be describable classically depends on the size of the region. Each time the universe doubles in size, each Planck-scale region gets replaced by eight new Planck-scale volumes. Predicting the 'initial' state of such regions requires a quantum theory of gravity, but it is commonly imagined that the classical universe emerges from some chaotic space-time foam. Now even if this process were to generate quite small occupation numbers for these Planck-scale modes, this would give a positive contribution to the pressure which would stop inflation taking place. If we want to invoke inflation then we must assume that this quantum-gravitational process produce an almost perfect vacuum.

There is one other peculiar feature of a potential dominated medium that merits being mentioned. We have developed the theory here simply as we did for the FRW models, with the sole modification being the adoption of the equation of state $P = -\rho c^2$. There is, however, an important distinction to be drawn between such a medium and a fluid with $P = \rho c^3/3$ or $P = 0$ say. In the latter cases there are a preferred set of observers — the 'comoving observers' — for whom the stress energy tensor takes the symmetric form $T^{\mu\nu} = \mathrm{diag}(\rho c^2, P, P, P)$. At each point in space, this zero momentum density condition picks out a unique velocity, and this gives us a unique 'congruence' of comoving observers who are, in our Universe, expanding away from one another. In the inflationary phase, in contrast, when $P = -\rho c^2$ to high accuracy, there is no such unique congruence of comoving observers, since with $P = -\rho c^2$ the stress energy tensor has the same form in all inertial frames. One can construct a set of test particle world-lines which are exponentially expanding, as we have done here, and these observers would say that mass-energy is being created spontaneously by the universal expansion. However, one can also find a set of test particles whose world-lines are initially converging (the acceleration equation only tells us that $\ddot{a} > 0$, and one can have test particles with $\dot{a} < 0$ initially). Such observers would not agree that mass-energy is being created. One can also construct a congruence of world-lines which are tilted (i.e. in a state of motion) with respect to our comoving observers, and they would also see vanishing momentum density for the scalar field. The usual response to this is to argue that the pressure is not precisely $P = -\rho c^2$, rather there will be a small correction, either due to the field velocity $\dot{\phi}$ or due to the presence of other matter fields, which will break the exact invariance of $T^{\mu\nu}$ under Lorentz boosts. The other thing that breaks this symmetry is the field itself. It is like a bit like a clock in that it rolls down the potential and when it reaches the value $\phi \simeq \sqrt{c^4/G}$, the equation of state changes. And that determines the hypersurface on which inflation ends.

Finally, coming back to the question of initial conditions, we noted that with the inflationary equation of state, the general solution for the expansion factor is the sum of exponentially growing and decaying terms (96), and we said that generically the former will come to dominate the behaviour of $a(t)$ at late times. But would that not suggest that the decaying term would be expected to dominate at early times, and that the Universe would be expected to have undergone a 'bounce'?

In the context of the models developed here, however, that would be a misconception. There is a quantitative difference in the behaviour of a scalar field in a contracting universe as compared the the expanding model considered above. In the latter case, H is positive and the term $3H\dot{\phi}$ in the equation of motion is a friction, and the evolution of the field will relax towards the slowly rolling terminal velocity solution. In a collapsing phase H is negative, so we have negative friction. In this case the slowly rolling solution — while possible, since the system as a whole is time symmetric — is an unstable one. For generic initial conditions going into a 'big-crunch' we do not expect the field to become potential dominated, and so the inflationary equation of state will not arise. This is discussed very nicely in Zel'dovich's *My universe* monograph.

5 Cosmological structure from scalar fields

We have previously studied the evolution of density perturbations from some given initial state we now explore how the initial seeds for structure may have arisen. We first consider the 'spontaneous' generation of fluctuations from the effect of non-gravitational forces in the hot big bang model, and show that it is very difficult to generate large-scale structure in this way. We then consider the generation of density fluctuations from quantum fluctuations in the scalar field during inflation and finally we consider topological defects.

Before embarking on these calculations, it is worth describing what seems to be required observationally. In fact, long before inflation and when cosmological structure formation was still a relatively immature subject, Edward Harrison and Jacob Zel'dovich pointed out that if the initial spectrum of fluctuations had a power-law spectrum[19] $P_\rho(k) \propto k^n$ extending over a very wide range of scales then it should have index $n = 1$. The argument is that for a power-law, the fluctuations in the potential, and therefore in the curvature, also have a power law spectrum. For most spectral indices, the curvature fluctuations will either diverge at small scales or at large scales. This would result in small black-holes if n is too large, or would lead to the universe

[19] A field $f(\mathbf{r})$, assumed to be periodic within a very large (fictitious) scale L, can be synthesised as $f(\mathbf{r}) = \sum_{\mathbf{k}} f_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{r}}$ where the modes have spacing $\Delta k = 2\pi/L$. The mean squared field is $\langle f^2 \rangle \equiv L^{-3} \int d^3r f^2(\mathbf{r}) = \sum_{\mathbf{k}} f_{\mathbf{k}} f^\star_{\mathbf{k}}$. replacing $\sum_{\mathbf{k}} \ldots \Rightarrow (\Delta k)^{-3} \int d^3k \ldots$ gives $\langle f^2 \rangle = (2\pi)^{-1/3} \int d^3k \, P_f(k)$ where the power-spectrum - for an isotropic random field – is $P_f(k) \equiv L^3 \langle f_{\mathbf{k}} f^\star_{\mathbf{k}} \rangle$. The 'smoothed' field $\overline{f}_R(\mathbf{r}) \equiv \int d^3r' W(\mathbf{r}') f(\mathbf{r} - \mathbf{r}')$, where $W(\mathbf{r}) = W(r)$ is an 'averaging window' of size $\sim R$, has $\overline{f}_{R\mathbf{k}} = W_{\mathbf{k}} f_{\mathbf{k}}$, so its mean square is $\langle \overline{f}^2_R \rangle = (2\pi)^{-3} \int d^3k |W_{\mathbf{k}}|^2 P_f(\mathbf{k}) \sim (k^3 P_f(k))_{k \sim 1/R}$. For a flat power spectrum $P_f(k) \propto k^n$ with $n = 0$, $\langle \overline{f}^2_R \rangle^{1/2} \propto 1/R^{3/2} \propto 1/\sqrt{N}$. So the spectral index $n = 0$ is what one would get for the density of particles scattered at random; a so-called 'Poissonian' or 'white-noise' process.

being highly inhomogeneous on large scales if n is too small. The 'happy medium' (in which the curvature fluctuations diverge at *both* small and large scales, but only logarithmically fast) is that for which the root mean square density fluctuations scale as $\delta\rho/\rho \propto 1/\lambda^2$, so the Newtonian gravitational potential fluctuations $\delta\varphi \sim (H\lambda)^2 \delta\rho/\rho$ are independent of λ. For a power-law power spectrum, the variance per octave of wave-number is $\langle(\delta\rho/\rho)^2\rangle_k \sim k^3 P_\rho(k) \propto k^{3+n} \propto \lambda^{-(3+n)}$. Thus, for $n = 1$, the potential fluctuations are scale invariant. This is known as the *Harrison-Zel'dovich spectrum*. While somewhat philosophically motivated, this kind of spectral index has much to commend it. Richard Gott and Martin Rees had argued that the structure we see on scales of galaxies, clusters and super-clusters seemed to require an spectral index for the perturbations emerging after $z_{\rm eq}$ of $n \sim -1$. This is not the Harrison-Zel'dovich index, but allowing for the suppression of the growth of small scale perturbations during the physical processes described above during the era around $z_{\rm eq}$, which we will discuss later, these are consistent. The real clincher for the $n = 1$ spectrum came with the detection by COBE of roughly scale invariant ripples in the large-angle anisotropy of the CMB. Normalizing the spectrum to cluster or super-cluster scale structures, these fit very nicely to an extrapolation to larger scales using the Harrison-Zel'dovich spectrum.

5.1 Spontaneous generation of density fluctuations

Consider an initially homogeneous universe and let the pressure spontaneously become inhomogeneous (this might happen during a phase transition in the early universe, or at much later times when stars form and explode). There will then be non-gravitational forces will generate density perturbations. We saw how density perturbations grow, by a process called 'gravitational instability'; could it be that the structure we see in the universe originated this way? The answer is no; structure 'seeded' in this way produces too little power on large scales.

For a spontaneous cosmological phase transition the pressure fluctuations should be uncorrelated on scales larger than the horizon scale at that time. Similarly, a natural model for the pressure perturbation from randomly exploding stars has a flat power spectrum and the fluctuations in the pressure, when averaged over a large volume containing mass M, falls off as $M^{-1/2}$ or as $1/\sqrt{N}$ where N is the number of perturbing 'cells'. What is the amplitude and spectrum of mass fluctuations on large scales generated by such a process? Naively, one might imagine that there might be root-N perturbations, with N the number of independent fluctuation regions, giving $\delta\rho/\rho \propto M^{-1/2}$. Alternatively, one might imagine there would be 'surface fluctuations' giving $\delta\rho/\rho \propto M^{-5/6}$. Now it is true that if we measure the density within a sharp-edged top-hat sphere, then there will be fluctuations in the mass of this order, but, it turns out, the fluctuations in growing modes will be much smaller than this; the amplitude of the growing mode is in fact $\delta\rho/\rho \propto M^{-7/6}$.

Let's first obtain this result from a Newtonian analysis. What we shall do is compute the perturbation to the large-scale gravitational potential $\delta\varphi$ — since this is associated with the growing mode density perturbations — from which we can obtain $\delta\rho/\rho$. Consider first a homogeneous expanding dust-filled cosmology containing an agent who can re-arrange the surrounding matter, but can only influence material at distances $r < R$ (see figure 10). What is the perturbation to the Newtonian potential at large scales? The potential is

$$\delta\varphi(\mathbf{r}) = -G \int d^3r' \, \frac{\delta\rho(\mathbf{r}')}{|\mathbf{r}' - \mathbf{r}|} \tag{116}$$

where the integrand vanishes for $r' \gtrsim R$. At large distances $r \gg R$, we can expand the factor $1/|\mathbf{r}' - \mathbf{r}|$ as

$$\frac{1}{|\mathbf{r}' - \mathbf{r}|} = (\mathbf{r}\cdot\mathbf{r} - (2\mathbf{r}'\cdot\mathbf{r} - \mathbf{r}'\cdot\mathbf{r}'))^{-1/2} = \frac{1}{r}\left(1 - \frac{1}{2}\frac{2\mathbf{r}'\cdot\mathbf{r} - \mathbf{r}'\cdot\mathbf{r}'}{r^2}\right)^{-1/2} \tag{117}$$

Making a Taylor expansion gives

$$\frac{1}{|\mathbf{r}' - \mathbf{r}|} = \frac{1}{r}\left(1 + \frac{\mathbf{r}\cdot\mathbf{r}'}{r^2} + \frac{1}{2}\left(\frac{\mathbf{r}'\cdot\mathbf{r}'}{r^2} - 3\frac{(\mathbf{r}'\cdot\mathbf{r})^2}{r^4}\right) + \cdots\right) \tag{118}$$

Using this in (116) gives an expansion in powers of $1/r$. The coefficient of the leading order term (for which $\delta\varphi \sim 1/r$) is $\int d^3r' \, \delta\rho(\mathbf{r}')$. This is the *monopole moment* of the mass distribution, but this vanishes by virtue of conservation of mass. The next term has $\delta\varphi \propto 1/r^2$, and has coefficient proportional to $\hat{\mathbf{r}} \cdot \int d^3r' \, \delta\rho(\mathbf{r}')\mathbf{r}'$, which is the *dipole moment*. This vanishes by virtue of momentum conservation. The next term has $\delta\varphi \propto 1/r^3$ with coefficient proportional to the *quadrupole moment*. The does not, in general vanish; the

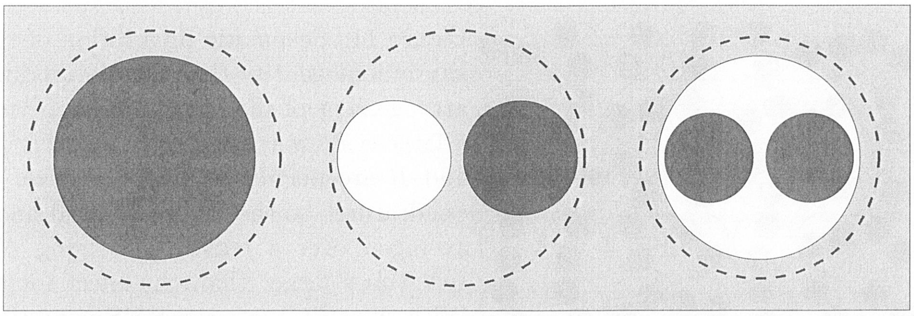


Figure 10: Illustration of the type of perturbation that can be generated by a physical process that operates locally (within region delimited by the dashed circle). On the left is a monopole perturbation. This has a net excess mass and would generate a potential perturbation at large scales $\delta\varphi \propto 1/r$. Such a perturbation is not allowed, since it requires importing mass from large distances; if one is constrained only to re-arrange the mass within the dashed circle, then for a symmetric mass configuration the net mass excess must vanish. In the center is shown a dipole perturbation with an over-dense region on the right and an under-dense region on the left. Such a perturbation would generate a large-scale gravitational potential $\delta\varphi \propto 1/r^2$. The net mass excess inside the dashed circle is now zero, but such perturbations are still now allowed as, in order to generate such a perturbation, one would need to impart a net momentum to the matter. On the right is a quadrupole perturbation. Such a perturbation can be generated by a local physical process while still conserving mass and momentum. A quadrupole source generates a large-scale potential perturbation $\delta\varphi \propto 1/r^3$; this falls off much faster than for an 'un-shielded' monopole perturbation.

agent, can, for example, rearrange the matter into a 'dumb-bell' shaped configuration without exchanging any mass or momentum with the exterior (see figure 10). If the mass contained within the perturbation region is ΔM, the large-scale gravitational potential is $\delta\varphi(r) \sim G\Delta M R^2/r^3$ where R is the scale of the fluctuation region and this is smaller than the un-shielded monopole term $G\Delta M/r$ by two powers of R/r.

Now consider a multitude of such agents, with separation $\sim R$, each of whom re-arranges the surrounding matter in accordance with mass and momentum conservation, but otherwise in a random manner, such that different fluctuation regions are uncorrelated with each other (see figure 11). The mean square large scale potential —- averaged over a region containing mass M, or size $r \sim (M/\rho)^{1/3}$ — is then the sum of $N \sim (r/R)^3 \sim M/\Delta M$ quadrupole sources adding in quadrature, so the root mean square potential perturbation is

$$\delta\varphi_M = \langle(\delta\varphi)^2\rangle_M^{1/2} \sim \sqrt{N} \times \frac{G\Delta M R^2}{r^3} \propto M^{-1/2} \tag{119}$$

since both N and r^3 are proportional to M.

Thus it is the fluctuations in the *potential* $\delta\varphi$ that form a *white-noise* process. Now, for the growing mode, the potential and mass – or density – fluctuations are related by $\delta\varphi \sim G\delta M/r$ where $\delta M \sim M \times \delta\rho/\rho$ is the fluctuation in the mass. So we can write $\delta\varphi \sim (GM/r) \times \delta\rho/\rho$. But $M \sim \rho r^3$, so $GM/r \sim G\rho r^2 \sim H^2 r^2$, so $\delta\varphi \sim (Hr)^2\delta\rho/\rho$ where Hr is the Hubble velocity at separation r. The key point is that, since $r^2 \propto M^{2/3}$, the root mean squared growing mode density perturbations induced by this kind of small-scale local rearrangement of mass has rms

$$\delta\rho/\rho \propto M^{-7/6}. \tag{120}$$

Put another way, this spectral index of the density fluctuations is[20] $n = 4$ as, with $P_\rho(k) \propto k^4$, the mean square fluctuations as $\langle(\delta\rho)^2\rangle_r \sim (k^3 P_\rho(k))_{k\sim 1/r} \propto r^{-7} \propto M^{-7/3}$. The mass distribution on large-scales at late times is much smoother than the 'root-N' mass fluctuations, and smoother even than the 'surface fluctuations'.

The argument given above is Newtonian and assumes conservation of mass. Do these conclusions still hold with fluctuations of the relativistic plasma? For example, consider a universe in which the process of baryogenesis is spatially inhomogeneous. If the photon-to-baryon ration — the specific entropy that is — is an incoherent random function of position, this will generate an initially isocurvature perturbation such that the number density of baryons is a white-noise process, but with the initial density fluctuation

[20] An alternative, and simpler, way to reach this conclusion is to note that Poisson's equation $\nabla^2\delta\varphi = 4\pi G\delta\rho$ becomes, in Fourier space, the algebraic equation $-k^2\varphi_{\mathbf{k}} = 4\pi G\rho_{\mathbf{k}}$, so the power spectra (defined as $P_X(k) \equiv L^3\langle X_{\mathbf{k}} X^\star_{\mathbf{k}}\rangle$) are related by $P_\rho(k) = (4\pi G)^{-2}k^4 P_\varphi$. Thus if $P_\varphi(k) \propto k^0$, $P_\rho(k) \propto k^4$.

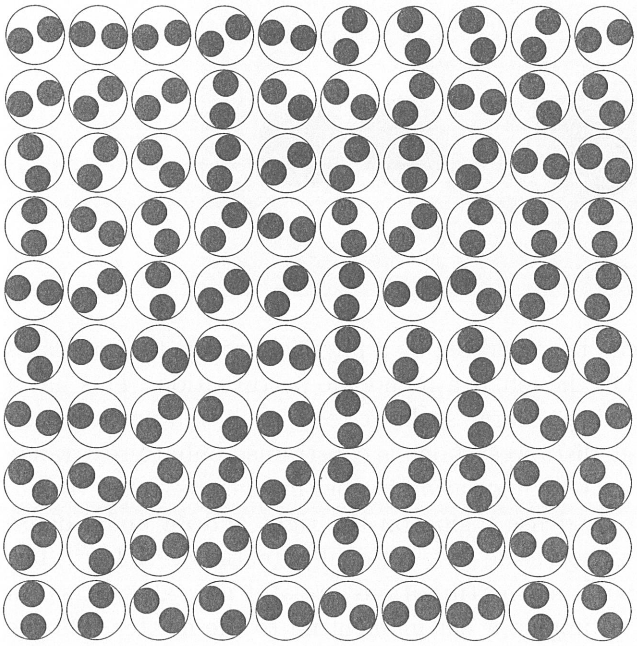


Figure 11: Schematic illustration of the type of density inhomogeneity than can be produced by local rearrangement of the matter. Each perturber generates a large-scale potential $\delta\varphi \sim G\Delta M R^2/r^3$, where ΔM and R are perturber mass and size. These add 'in quadrature', so the RMS potential fluctuations, when averaged over a region containing N perturbers is $\sqrt{N}$ times larger than the effect for a single region. With $N \propto M \propto r^3$, the RMS potential fluctuation is $\delta\varphi \propto \sqrt{M}/r^3 \propto r^{-3/2}$. This is a 'white-noise process', with a power spectrum $P_\varphi(k) \propto |\delta\varphi_{\mathbf{k}}|^2 =$ constant, for which the mean square potential fluctuations are $\langle(\delta\varphi)^2\rangle_r \sim (k^3 P_\varphi(k))_{k\sim 1/r} \propto 1/r^3$. Poisson's equation $\nabla^2\delta\varphi = 4\pi G\delta\rho$ becomes, in Fourier space, algebraic, so the growing mode density perturbations are $\delta\rho_{\mathbf{k}} \sim G^{-1}k^2\delta\varphi_{\mathbf{k}}$ and the power-spectrum of the mass-fluctuations is therefore $P_\rho(k) \sim \langle|\delta\rho_{\mathbf{k}}|^2\rangle \propto k^4$, with spectral index is $n = 4$, and the RMS mass fluctuations $\delta\rho/\rho \propto M^{-7/6}$.

in the baryons being compensated by the radiation density. Now as the universe expands the radiation will redshift away and will eventually become negligible. At late times then there will be fluctuations in the net proper mass contained within any comoving region with rms amplitude scaling inversely as the square root of the number of fluctuation regions, or $\delta\rho/\rho \propto M^{-1/2}$. Does this not conflict with the $M^{-7/6}$ rule? Not necessarily, since we do not know what fraction of these perturbations is in the growing mode. To resolve this, recall the behavior of spherical perturbations. To generate a decaying mode we delay the 'bang-time' keeping the energy constant, and the proper mass contained in the perturbation is fixed. In the growing mode we perturb the binding energy φ. The *gravitational* mass of the perturbation must equal the unperturbed mass, but as the binding energy is negative, we must actually have a slight enhancement of the net proper mass $\delta M \sim -M\delta\varphi$ within the perturbation. Thus, the fluctuations in proper mass within comoving regions (which scale as $M^{-1/2}$ in this incoherent isocurvature model) measure $\delta\varphi \sim (H^2\lambda^2)\delta_{\rm growing}$ and we recover the $\delta\rho/\rho \propto M^{-7/6}$ behavior for the growing modes.

The large-scale growing perturbations produced by small-scale rearrangement of mass are therefore very small and this effectively excludes the possibility that the large-scale structure results from curdling of the universe during a phase transition at early times because the horizon size is small then. It would also seem quite difficult to produce the largest scale structures seen from hydrodynamical effect of supernovae explosions (though the fact that a simple estimate of the net energy released based on the abundance of the results of nuclear burning in stars does not fall very far short of what is desired is tantalizing). In any such scenario, accounting for the large-angle fluctuations in the CMB is very difficult indeed, since the prediction is for temperature fluctuations falling off as $\delta T/T \sim \delta\varphi \propto \theta^{-3/2}$.

5.2 Density fluctuations from inflation

5.2.1 Introduction

A much more promising way to generate density fluctuations is from zero-point quantum fluctuations of the scalar 'inflaton' field driving inflation. In the inflationary scenario, the field is assumed to be spatially uniform, and it is necessary that it have $|\nabla\phi|^2/a^2 \ll V(\phi)$ in order to create accelerated expansion. But the field cannot be perfectly smooth; there must, at the very least, be zero-point quantum fluctuations of the co-moving Fourier modes $\phi_{\mathbf{k}}$.

The wavelength $\lambda = 2\pi/|\mathbf{k}|$ of such a mode – being a comoving distance – is a horizontal line in figure 8. It 'appears' when $a\lambda$ is of order $l_{\rm P}$ – this being the boundary of our ignorance – at some time during inflation, when it is sub-horizon scale, and the physical wavelength $a\lambda$ increases until it exits the horizon. In models like chaotic inflation, as we shall see, these modes are effectively massless, so they oscillate with frequency $\omega_{\mathbf{k}} = |\mathbf{k}|/a$ and with, initially $\omega_{\mathbf{k}} \gg H$, so they evolve adiabatically. If they are assumed initially to be in the ground state $|n = 0\rangle$ they stay that way, but only until horizon crossing, at which time the frequency becomes of order H, and adiabaticity no longer holds, and the occupation number will change. Nonetheless, to order of magnitude, the amplitude of the fluctuations can be estimated as being roughly that for zero-point fluctuations.

A key feature of inflation is that it is very nearly 'scale-invariant', so different modes, corresponding to structures of different scales, must evolve essentially identically. Thus a rather generic prediction of inflation is that there be fluctuations in ϕ at horizon crossing that are scale invariant.

The picture that is employed is that different regions – say the positive and negative halves of a wave – will then evolve independently – as they are then super-horizon scale – but starting from slightly different points on the potential curve (figure 9). The field rolls down the potential until $\phi \sim \sqrt{c^4/H}$ at which time the universe reheats and the energy of the scalar field is converted to that of the thermal plasma of the hot big bang. But a region where $\delta\phi$ was positive at horizon crossing will have had slightly more e-foldings of expansion. So it will occupy a larger proper volume than the region where $\delta\phi$ was negative. And more matter will have been created there. And, if we consider a perturbation of size that re-enters the horizon in the matter dominated era, there will be more or less proper mass depending on whether $\delta\phi$ was positive or negative. The way that the – originally homogeneous – space-time accommodates this is by 'herniating' slightly; a region with a proper mass excess does not appear as a 'monopole' source for an external gravitational field. Rather the space there has slightly positive curvature – or a slightly negative Newtonian gravitational potential $\delta\varphi$ – and, to the outside world, its gravitating mass is unchanged. The upshot of this is ripples of the dimensionless potential $\Phi_{\mathbf{k}} = \varphi_{\mathbf{k}}/c^2$ – which are equal to the density perturbation $\rho_{\mathbf{k}}/\rho$ when the perturbation region re-enters the horizon – that are roughly equal to $\phi_{\mathbf{k}}/\phi$ when the scale λ left the horizon.

This was first worked out – in the few years following the invention of the inflationary scenario – by several groups and individuals, who showed that inflation predicts density fluctuations re-entering the horizon with amplitude

$$\boxed{\frac{\delta\rho}{\rho} \sim \sqrt{\frac{\hbar}{c}}\frac{H^2}{\dot{\phi}}} \tag{121}$$

where H and $\dot{\phi}$ are evaluated as the perturbations leave the horizon during the inflationary era. Since, H and $\dot{\phi}$ are slowly varying during inflation, this naturally predicts seeds for structure formation close to the preferred Harrison-Zel'dovich form.

Below we will flesh out some of the details in the context of the chaotic inflation model with a $V(\phi) = \lambda\phi^4$ potential considered earlier.

5.2.2 Fluctuogenesis in chaotic inflation

Since we are dealing with small amplitude fluctuations, the natural approach is to decompose the field into spatial Fourier modes, and compute the evolution of these separately. As shown in figure 8, such a mode, being fixed in comoving wavelength, first appears, or rather becomes describable without a quantum theory of gravity, when the physical wavelength is on the order of the Planck length. As already discussed, for inflation to take place we require that the field fluctuation at that time be in the vacuum state to very high accuracy. This then sets the initial conditions; the initial occupation number for inflatons of this scale must vanish. A detailed calculation of the evolution is extremely technical, involving such tricky issues as the nature of the vacuum in curved space-time, as well as requiring a full general-relativistic treatment for the modes while they are outside the horizon. Here we shall only give a rather hand-waving sketch of the important processes and thereby physically justify the form of the key result (121). We will show that the requirement that the final density fluctuation amplitude agree with that required observationally puts a strong constraint on the strength of the interaction term (or mass term) in the inflaton potential. We will also discuss how inflation predicts, in addition to density fluctuations, fluctuations in all fields, and, in particular, predicts a stochastic background of gravitational waves. This provides, potentially, a powerful test of the theory.

First, we need to establish the nature of the fluctuations about the large-scale average inflaton field during inflation. We will denote the 'background' field by ϕ_0, and the fluctuations, which, as we shall see, are relatively small, by ϕ_1. The general equation of motion for the inflaton field is

$$\ddot{\phi} + 3H\dot{\phi} - \frac{c^2}{a^2}\nabla^2\phi + 4c^2\lambda\phi^3 = 0, \tag{122}$$

where ∇ denotes the derivative with respect to comoving coordinates. If we decompose the field as $\phi = \phi_0 + \phi_1$, where the 'background' field ϕ_0 is assumed to have $\nabla\phi_0 = 0$, and make a Taylor expansion of the interaction term assuming that the fluctuations about the background are relatively small (i.e $\phi_1 \ll \phi_0$) then the equation of motion for the perturbation, which we will not assume to have vanishing spatial gradient, is

$$\ddot{\phi}_1 + 3H\dot{\phi}_1 - \frac{c^2}{a^2}\nabla^2\phi_1 + 12\lambda\phi_0^2 c^2\phi_1 = 0. \tag{123}$$

Comparing this with the equation of motion for a free massive scalar field

$$\ddot{\phi} + 3H\dot{\phi} - \frac{c^2}{a^2}\nabla^2\phi + \frac{m^2c^4}{\hbar^2}\phi = 0. \tag{124}$$

we see that the fluctuations about the background field behave like a free field with mass

$$m = \sqrt{12\lambda\hbar^2\phi_0^2/c^2}. \tag{125}$$

The Compton wavelength for the inflaton fluctuations is

$$\lambda_{\rm C} = h/mc \tag{126}$$

which we can compare to the horizon scale c/H. With $H \sim \sqrt{G\rho} = \sqrt{G\mathcal{E}/c^2} \sim \sqrt{G\lambda\phi_0^4/c^2}$, the ratio of these is

$$\frac{\lambda_{\rm C}}{c/H} \sim \sqrt{\frac{G\phi_0^2}{c^4}}. \tag{127}$$

Therefore, if the field is large enough to allow inflation ($\phi_0 \gg \sqrt{c^4/G}$) then $\lambda_{\rm C} \gg c/H$; the Compton wavelength is much larger than the horizon. Thus, to a very good approximation, the classical equation governing fluctuations ϕ_1 is that of a free, massless field. This result is not specific to the $V \propto \phi^4$ form for the inflaton potential; the same is true for a $V \propto \phi^2$ theory or for other polynomial potentials.

Next we consider ϕ_1 to be a superposition of comoving Fourier modes: $\phi_1(\mathbf{x}, \tau) = \sum_{\mathbf{k}} \phi_{\mathbf{k}}(\tau)e^{i\mathbf{k}\cdot\mathbf{x}}$. We don't need to put a subscript '1' on $\phi_{\mathbf{k}}$ since the 'background field' ϕ_0 has vanishing spatial fluctuations, so if we refer to $\phi_{\mathbf{k}}$ it is obvious we are talking about ϕ_1. We are assuming as usual here that ϕ_1 is periodic within some large (fictitious) comoving volume L^3, so the modes live on a lattice of spacing $\Delta k = 2\pi/L$. The reality of ϕ_1 means that the complex mode amplitudes must be symmetric: $\phi_{-\mathbf{k}} = \phi_{\mathbf{k}}^\star$.

These mode amplitudes obey, classically, a damped oscillator equation $\ddot{\phi}_{\mathbf{k}} + 3H\dot{\phi}_{\mathbf{k}} + (c^2|\mathbf{k}|^2/a^2)\phi_{\mathbf{k}} = 0$. With the transformation $\phi_{\mathbf{k}} = \varphi_{\mathbf{k}}/a^{3/2}$, we find (with $a \propto e^{Ht}$) that $\ddot{\varphi}_{\mathbf{k}} = -\Omega_{\mathbf{k}}^2(\tau)\varphi_{\mathbf{k}}$, with $\Omega_{\mathbf{k}}(\tau) = \sqrt{c^2|\mathbf{k}|^2/a^2 - 3H^2/4}$. When the mode is within the horizon (i.e. $\lambda = 2\pi a/|\mathbf{k}| \ll c/H$) this is an undamped oscillator with a real frequency that is slowly time varying in the sense that $\dot{\Omega}_{\mathbf{k}} \ll \Omega_{\mathbf{k}}^2$. As is well known, such an oscillator evolves such that the 'energy' $\dot{\varphi}_{\mathbf{k}}^2$ is proportional to $\Omega_{\mathbf{k}}$, from which we find that the 'envelope' of the oscillations $\overline{\varphi}_{\mathbf{k}} \propto \sqrt{a}$ so $\phi_{\mathbf{k}} \propto 1/a$ and the true energy density varies as $\mathcal{E} \sim \omega_{\mathbf{k}}^2\phi_{\mathbf{k}}^2 \propto 1/a^4$; i.e. just like a gas of massless – or highly relativistic – particles whose total number is conserved. After leaving the horizon this suggests an exponential decay of $\varphi_{\mathbf{k}}$ with time (power-law in a), but that is a little misleading, as the small neglected "mass' term needs to be considered. In the regime $a/|\mathbf{k}| \gg c/H$ it is better to realise that the solution of the full equation tells us that starting at some $\phi + \delta\phi$ gives a solution which is a time translated version of what we get starting at ϕ.

Quantum mechanically, the essential assumption is that, initially (shortly after appearing at the Planck scale, perhaps), the harmonic oscillator of which $\phi_{\mathbf{k}}$ is the classical displacement is in its ground state. I.e. in the energy eigenstate with energy[21] $E = \hbar\omega_{\mathbf{k}}/2$. And as long as $a/|\mathbf{k}| \ll c/H$, adiabaticity ensures that the oscillator will remain in the ground state. One can then readily calculate the expectation $\langle 0_{\mathbf{k}}|\, |\phi_{\mathbf{k}}|^2 \,|0_{\mathbf{k}}\rangle$. For the oscillator with classical Hamiltonian $H(p, q) = \frac{1}{2}(p^2/2m + m\omega^2q^2)$ we have $\langle 0|\, q^2 \,|0\rangle = \frac{1}{4}\hbar/m\omega$. Here we have a classical Lagrangian $L_{\mathbf{k}}(\phi_{\mathbf{k}}, \dot{\phi}_{\mathbf{k}}) = L^3a^3\mathcal{L} = \frac{1}{2}L^3a^3(|\dot{\phi}_{\mathbf{k}}|^2/c^2 - (|\mathbf{k}|^2/a^2)|\phi_{\mathbf{k}}|^2)$, so the canonical momentum is $p_{\mathbf{k}} = \partial L/\partial\dot{\phi}_{\mathbf{k}}^\star = L^3a^3\dot{\phi}_{\mathbf{k}}/c^2$ and hence the Hamiltonian is

$$H_{\mathbf{k}}(p_{\mathbf{k}}, \phi_{\mathbf{k}}) = p_{\mathbf{k}}^\star\dot{\phi}_{\mathbf{k}} - L_{\mathbf{k}} = \tfrac{1}{2}(c^2|p_{\mathbf{k}}|^2/L^3a^3 + L^3a|\mathbf{k}|^2|\phi_{\mathbf{k}}|^2) \tag{128}$$

[21] We can think of only the modes in one hemisphere of $\mathbf{k}$-space as being independent as the others are determined by the symmetry. But each of the independent modes has a complex amplitude, with two degrees of freedom, so the net result is the same as assigning $E = \hbar\omega_{\mathbf{k}}/2$ to all of the modes

so just like $H(p,q)$ with $m \Rightarrow L^3a^3/c^2$ and $m\omega^2 \Rightarrow L^3a|\mathbf{k}|^2$. With $\omega_\mathbf{k} = c|\mathbf{k}|/a$ that means $m\omega \Rightarrow L^3a^2|\mathbf{k}|/c$ and hence

$$\langle 0_\mathbf{k}|\,|\phi_\mathbf{k}|^2\,|0_\mathbf{k}\rangle = \tfrac{1}{4}\hbar/m\omega \Rightarrow \tfrac{1}{4}\hbar c/L^3a^2|\mathbf{k}| = \tfrac{1}{4}\hbar c^2/L^3a^3\omega_\mathbf{k}. \tag{129}$$

Adding up the variance from all the modes we get

$$\langle|\,\phi_1^2\,|\rangle = \sum_\mathbf{k} \langle 0_\mathbf{k}|\,|\phi_\mathbf{k}|^2\,|0_\mathbf{k}\rangle \Rightarrow \left(\frac{L}{2\pi}\right)^3 \int d^3k\, \langle 0_\mathbf{k}|\,|\phi_\mathbf{k}|^2\,|0_\mathbf{k}\rangle = \tfrac{1}{4}\left(\tfrac{1}{2\pi a}\right)^3 \int d^3k\, \tfrac{\hbar c^2}{\omega_\mathbf{k}} = \tfrac{\hbar}{8\pi^2 c}\int d\omega_\mathbf{k}\,\omega_\mathbf{k} \tag{130}$$

where we note that, as required, the fictitious periodicity scale L does not appear the in the final result[22].

This is nice. But one might reasonably object that it is a bit of a cheat. After all, with these energy eigenstates, the expectation of the mode amplitude $\phi_\mathbf{k}$ vanishes. So we can't really interpret the square root of this as giving the classical displacement ϕ_1 we should add to the background field ϕ_0. This is where the non-adiabaticity comes to our rescue. As the mode leaves the horizon this breaks down, and there will be a non-vanishing amplitude for a mode to be in $|1_\mathbf{k}\rangle$, $|2_\mathbf{k}\rangle$ etc.. Just as in the hydrogen atom, where the energy eigenstates have, individually, vanishing current, and so do not radiate classically, an atom with an electron in a superposition of two eigenstates $|n\rangle$ and $|n+1\rangle$ say has a non-vanishing expectation value for the current $\langle \mathbf{j}(t)\rangle = \int d^3r (i\hbar q/m)(\psi\boldsymbol{\nabla}\psi^\star + \text{c.c.})$ for the current, which, moreover, for large n oscillates at just the classical frequency of an electron orbiting a proton with angular momentum $n\hbar$. This, along with the assumption that the amplitude to be in the excited states is not small, is, I think, what justifies the claim that, at horizon crossing, there is a non-vanishing expectation value for the field, whose square can be estimated using the above, and by adding up the contributions to $\langle|\,\phi_1^2\,|\rangle$ from a logarithmic interval of frequencies (so using $\int d\omega_\mathbf{k}\,\omega_\mathbf{k} \sim \omega_\mathbf{k}^2$ above). With $\omega_\mathbf{k} \sim H$ at horizon crossing, this gives

$$\langle\phi_1^2\rangle \sim \frac{\hbar H^2}{c}. \tag{131}$$

The prediction then is for inflaton field fluctuations at horizon exit of amplitude $\delta\phi \sim \sqrt{\hbar H^2/c}$. To understand how these couple to density and curvature fluctuations at the end of inflation, and subsequently to density fluctuations at horizon re-entry, consider first a region where $\delta\phi$ happens to be zero. This region will inflate by a certain number $N \sim H\phi/\dot\phi$ of e-foldings, and will then re-heat to a density determined solely by the nature of the inflaton potential and its couplings to other fields. Now consider a region of the same initial size, but where the field fluctuation happens to be positive. The field in this region starts up 'higher up the hill', so this region inflates for slightly longer, and ends up occupying a slightly larger volume when it re-heats (to the same density as the unperturbed region). The extra expansion factor is $\exp(H\delta t)$, where δt is the time taken for the field to roll from $\phi = \phi_0 + \delta\phi$ to $\phi = \phi_0$. This is just $\delta t = \delta\phi/\dot\phi$. For small $\delta\phi$ we can expand the exponential as $\exp(H\delta t) \simeq 1 + H\delta t \simeq 1 + H\delta\phi/\dot\phi$. This is the excess of volume occupied by the perturbation region as compared to what it would have been had $\delta\phi$ been zero; clearly to replace a given volume in the background model by a slightly larger volume requires that the perturbed region have a slight positive spatial curvature, which, as we have seen, can be related to the Newtonian potential fluctuations. But this needs to be treated with caustion, as we are talking about the spatial metric on a surface of constant density (or constant H). A more robust argument, I believe, is to consider for a large-scale perturbation which enters the horizon after matter domination, for which there will be an excess of proper mass within the perturbed region $\delta M/M \sim H\delta t$, but the universe outside should, by causality, be unaffected, so the proper mass has to be in a slight potential well, as we argued above. This is admittedly a little hand-waving, but provides at least a justification for the essential result, which is that there will emerge, at late times, scale-invariant growing-mode density perturbations with horizon-crossing amplitude given by (121).

The late-time density fluctuation amplitude is therefore set by the values of H and $\dot\phi$ at horizon exit. Since the field will be rolling slowly at the terminal velocity $\dot\phi = -4\lambda\phi_0^3/3H$, it follows that $H^2/\dot\phi \propto H^{3/2}$. As this changes steadily, but rather slowly, with time, and therefore with comoving horizon-exiting wavelength, the prediction is for a spectrum with index n *close* to unity but with what is called a slight 'tilt'. Note that the modes we can probe observationally are, on a logarithmic scale, close to the current horizon scale. If the latter exited soon after the start of inflation, we might expect to see some impact of this on large-scale CMB anisotropies, but there is no strong indication of these. As further consequence is that since the initial 'zero-point' fluctuations are statistically independent, so also will be the complex amplitudes for the density

[22] Different people use different conventions for the definition of $\phi_\mathbf{k}$, but they will agree on the final mean squared value.

fluctuations; i.e. the prediction is that the density perturbations will take the form of a Gaussian random field.

Observations of the microwave background and/or large-scale structure tell us that the density fluctuation $\delta_{\rm H}$ at horizon re-entry is a few times 10^{-5}. Matching this requirement places a constrain on the interaction strength parameter λ (or its equivalent for other choices of the inflaton potential form). Using $H^2 \sim G\mathcal{E}/c^2 \sim G\lambda\phi_0^4/c^4$ and $\dot{\phi} \sim \lambda\phi_0^3/H$ we have

$$\delta_{\rm H} \sim \sqrt{\frac{\hbar}{c}\frac{H^2}{\dot{\phi}}} \sim \left(\frac{G\phi_0^2}{c^4}\right)^{3/2} \lambda^{1/2}. \tag{132}$$

Now the first factor must be greater than unity for inflation to take place. In fact, we found that we needed $\phi_0 \gtrsim \sqrt{c^4/\epsilon G}$ where ϵ is the inverse of the number of e-foldings required to solve the horizon problem. This means that the pre-factor on the right hand side of (132) is around 100 (though the precise value is dependent on the energy scale of re-heating), and therefore a viable model must have an interaction strength, expressed in natural units,

$$\lambda \sim 10^{-4}\delta_{\rm H}^2 \sim 10^{-13} \tag{133}$$

so one might worry about the naturalness of this, or how this small number gets explained.

Finally, while we have focused on the fluctuations in the inflaton field, since it is these which give rise to density fluctuations, the first part of the argument here can be used to predict the horizon-exit value of the amplitude of any fields which are effectively massless during inflation. In particular, the theory predicts that there should be fluctuations in the graviton field — gravitational waves that is — with amplitude on the order of the expansion rate in units of the Planck frequency. The prediction for these is much less model dependent; the amplitude just depends on the energy scale of inflation; the higher the energy the larger the amplitude. These waves are 'frozen-in' which the perturbation is outside the horizon and then start to oscillate on horizon re-entry. The announcement of a detection of the signature of these waves in the polarisation of the CMB in 2014 sparked considerable interest, as it seemed to indicate inflation at around the GUT scale, but it was later shown that the signal can be attributed to dust in the Milky Way. A high priority for future measurements of the microwave background anisotropies is to measure the strength of these waves.

5.3 Self-Ordering Fields

An alternative possibility is that the seeds of structure may be due to *self-ordering fields*. The idea here is to have some scalar field or such-like which is initially in a highly disordered thermal state, but which has potential function of the kind invoked in spontaneous symmetry breaking. As the universe expands, the field temperature decreases and eventually it becomes energetically favorable for the field to fall into the minimum of the potential function. Such fields try to 'comb themselves smooth', but are frustrated in this due to the formation of *topological defects*. The most common example of this phenomenon at low energies is a ferro-magnetic material which, if cooled from high temperature, will undergo a phase transition and will develop domains which are bounded by walls. In cosmology, as at low energies, the character and evolution of such systems depends critically on the dimensionality of the field involved. Here we shall consider first 1-dimensional fields, which give rise to *domain walls*, but, unfortunately do not seem to be consistent the observed state of the universe. We then consider 2-dimensional fields, which, as we shall see, give rise to *cosmic strings*, and which seemed at first to be a promising mechanism for seeding cosmological structure, but the simplest model has now been ruled out observationally.

5.3.1 Domain Walls

Consider a real scalar field with, as usual, the Lagrangian density

$$\mathcal{L}(\phi, \dot{\phi}, \nabla\phi) = \frac{1}{2c^2}\dot{\phi}^2 - \frac{1}{2}(\nabla\phi)^2 - V(\phi) \tag{134}$$

with potential

$$V(\phi) = \lambda(\phi^2 - \phi_0^2)^2 \tag{135}$$

as sketched in figure 12. This field has a self-interaction parameterized by the coupling constant λ (which, as before, is dimensionless if we measure the field in units of Planck energy). The potential has asymmetric

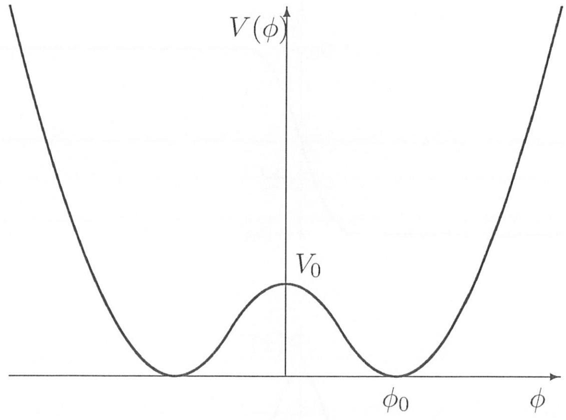


Figure 12: The potential function for a real scalar field involved in the generation of domain walls.

minima at field values $\phi = \pm\phi_0$, the solutions of $dV/d\phi = 0$, and its value at $\phi = 0$ – an unstable maximum – is $V(0) = \lambda\phi_0^4$.

In thermal equilibrium at temperature T, the state of the modes of the field are incoherent, with energy per mode $E_{\mathbf{k}} = (n_{\mathbf{k}} + 1/2)\hbar\omega_{\mathbf{k}}$ with $n_{\mathbf{k}} = (e^{\hbar\omega_{\mathbf{k}}/kT} - 1)^{-1}$. If the field is effectively free and massless, the dispersion relation is simply $\omega_{\mathbf{k}} = ck$. Ignoring the zero point energy, and taking the universe to be a periodic box of size L, the energy density is

$$\mathcal{E}(T) = L^{-3}\sum_{\mathbf{k}} n_{\mathbf{k}}\hbar\omega_{\mathbf{k}} = \int \frac{d^3k}{(2\pi)^3}\, n_{\mathbf{k}}\hbar\omega_{\mathbf{k}} \sim \frac{\hbar}{c^3}\omega_T^4 \sim \frac{(kT)^4}{(\hbar c)^3} \tag{136}$$

where $\omega_T = kT/\hbar$. This is just the Stefan-Boltzmann law. The energy density is also related to the mean square field fluctuations by the stress-energy tensor: $\mathcal{E} \sim \dot\phi^2/c^2 \sim \omega^2\phi^2/c^2$, so equating these two expressions for $\mathcal{E}$ gives the root mean square field fluctuation at temperature T:

$$\langle\phi^2\rangle_T^{1/2} \sim \sqrt{\frac{\hbar}{c}}\omega_T \sim \frac{kT}{\sqrt{\hbar c}}. \tag{137}$$

This says that the typical field value for a field in thermal equilibrium is proportional to the temperature (in natural units the root mean squared value of the field — which has dimensions of energy — is just equal to kT). If we use this to compute the potential energy density $\epsilon_{\rm int}$ due to the interaction term we find

$$\mathcal{E}_{\rm int} \sim \frac{\lambda}{\hbar c}\langle\phi^2\rangle^2 \sim \frac{\lambda(kT)^4}{(\hbar c)^2}. \tag{138}$$

Thus, provided the dimensionless interaction strength $\lambda\hbar c$ is much less than unity, as we shall assume, for a thermal state the interaction energy is a small perturbation to the total energy (136).

At high temperatures such that $kT \gg \sqrt{\hbar c}\phi_0$ the typical field values are much greater than ϕ_0 and the hill at the center of the potential is then relatively unimportant for the motion of the field. However, as the universe expands the temperature and the field amplitude decrease until the thermal field fluctuations become of order ϕ_0 and below this temperature the field will be trapped in one or other of the local minima. This *phase transition* occurs at a critical temperature

$$kT_c \sim \sqrt{\hbar c}\phi_0 \sim \frac{mc^2}{\sqrt{\lambda}}. \tag{139}$$

Now since the field configuration is initially highly spatially incoherent, different regions of space will want to settle into different minima. What happens is that the field will become locally smooth within domains with value $\phi = \pm\phi_0$, since this minimizes the $(\nabla\phi)^2$ contribution to the energy density, with domains separated by domain walls where there is a strong localized gradient of the field. One can estimate the thickness of these walls on energetic grounds to be

$$\Delta x \sim \frac{\phi_0}{\sqrt{V_0}} \tag{140}$$

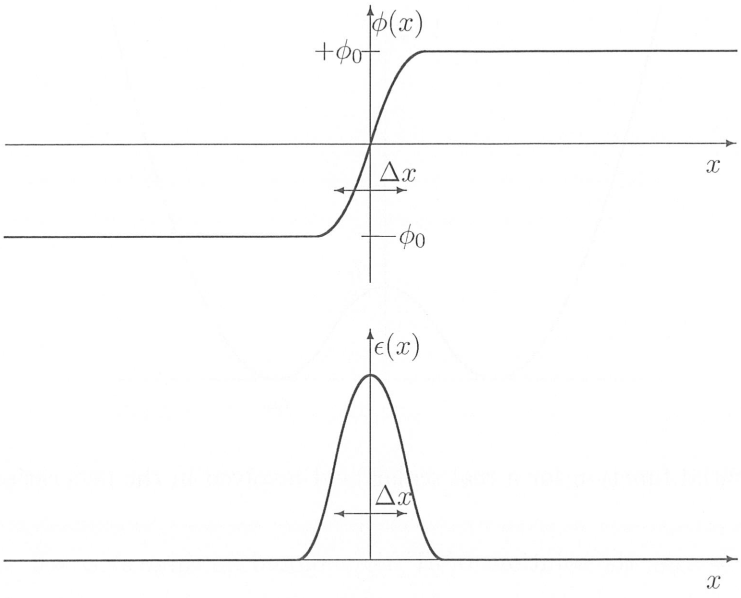


Figure 13: The upper panel shows the variation of the field passing through a domain wall of thickness Δx. We can estimate the width, and surface mass density of a domain wall, as follows. If the wall has width Δx then the typical field gradient within the wall is $\nabla\phi \sim \phi_0/\Delta x$. The energy density is then $\epsilon \sim (\phi_0/\Delta x)^2 + V_0$, so the density per unit area Σ is given by $c^2\Sigma \sim \epsilon\Delta x \sim \phi_0^2/\Delta x + V_0\Delta x$. If the wall is too thin, the gradient energy term becomes large while if the wall is too thick the potential term is increased. The total energy is minimized for $\Delta x \sim \phi_0/\sqrt{V_0}$. This is the width of a stable domain wall, for which the mass surface density is $\Sigma \sim \phi_0\sqrt{V_0}/c^2$.

(see figure 13 and its accompanying caption), and the mass-energy surface density in a stable wall is

$$\Sigma \sim \frac{\phi_0\sqrt{V_0}}{c^2}. \tag{141}$$

The single static planar wall is highly idealized. The initial walls configuration will be highly disordered. Again, energetic considerations tell us that the system will evolve to minimize the total energy in the walls. A simply connected region bounded by a wall will tend to shrink. In doing so, it will convert the potential energy into kinetic energy, so we expect walls to be moving at speeds on the order of the speed of light. Such a region will shrink to zero size on a time scale of order its size divided by c. The energy released will propagate away as waves, but these will damp adiabatically, so between the walls the field will remain relatively smooth. The expectation then is that any regions smaller than the horizon scale $\sim ct$ will disappear, but the field at separations bigger than the horizon scale will remain uncorrelated; the field dynamics will result in domains, at any time, on the order of the horizon size. In fact we expect a *scaling solution* where the field looks the same at any time save for scaling of the mean wall separation with the horizon scale.

This behavior is illustrated, for a field in 2-dimensions in figure 14. The equations for a scalar field in 2-dimensions, with a W-shaped potential and with a weak damping term were evolved numerically using a simple centered algorithm. The intial field was a Gaussian random field with a flat spectrum, aside from a smoothing with a small kernel to make the field smooth at the spatial sampling scale. The initial field amplitude was somewhat higher than ϕ_0, but the damping term cools the field, which starts to separate into domains where $\phi \simeq \pi\phi_0$. As the system evolves, enclosed regions can shrink to zero size and then disappear with a release of energy in a circular out-going wave. The scale of the walls gradually increases with time.

If we say there is on the order of one wall, of area $\sim (ct)^2$ per horizon volume $(ct)^3$, the mean mass-energy density in walls is

$$\rho_{\rm walls} \sim \frac{\Sigma}{ct}. \tag{142}$$

This is a serious problem, since the density of the matter, or radiation, in the universe is $\rho = 3H^2/8\pi G$, which scales as $1/t^2$. Thus the walls will rapidly come to dominate the universe; and one would have very large density inhomogeneity on the horizon scale. This is not what is observed.

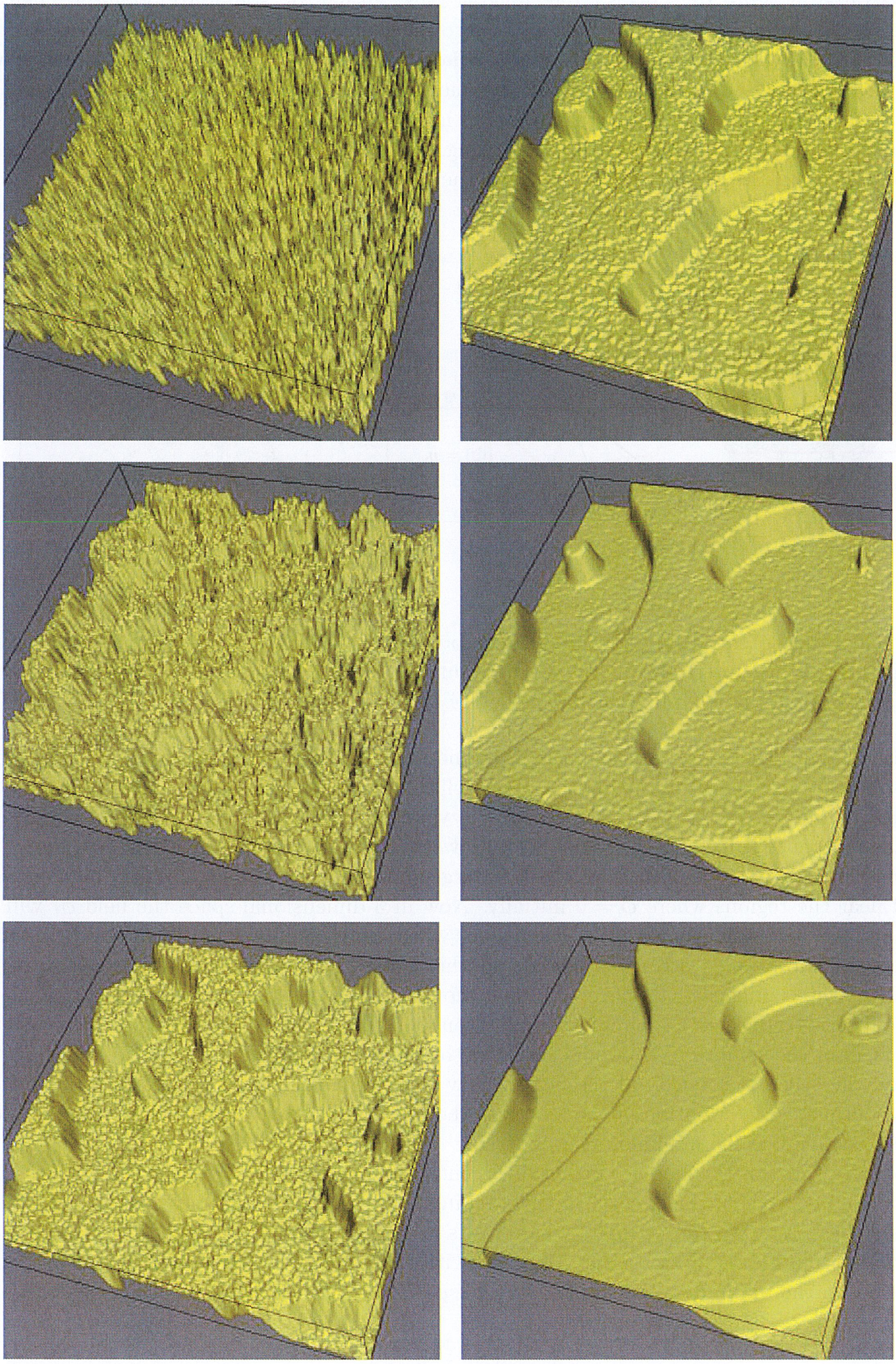


Figure 14: A set of snapshots from a computation of the spontaneous symmetry breaking of a field with potential $V = \lambda(\phi^2 - \phi_0^2)^2$ in an expanding universe. The initial field fluctuations were somewhat larger than ϕ_0 (the vertical axis here being compressed somewhat at early times).

5.3.2 Cosmic Strings

Now consider a two-component scalar field ϕ for which the potential $V(\phi)$ is the two dimensional analog of (135) as illustrated in figure 15. This is often called a 'Mexican-hat' or 'sombrero' potential. The minimum energy is on the circle $|\phi| = \phi_0$ and the field will try to relax towards this. There will be oscillations about the minimum, but the amplitude of these decreases adiabatically and the field will develop regions where the field lies in the minimum and varies slowly with position. While a completely uniform field is energetically favored, just as for domain walls, the assumed initial incoherence of the field limits the scale of coherence; the formation of a single infinitely large domain being frustrated by the formation of a network of *cosmic strings* — localized regions of energy density where the field sits at $\phi = 0$.

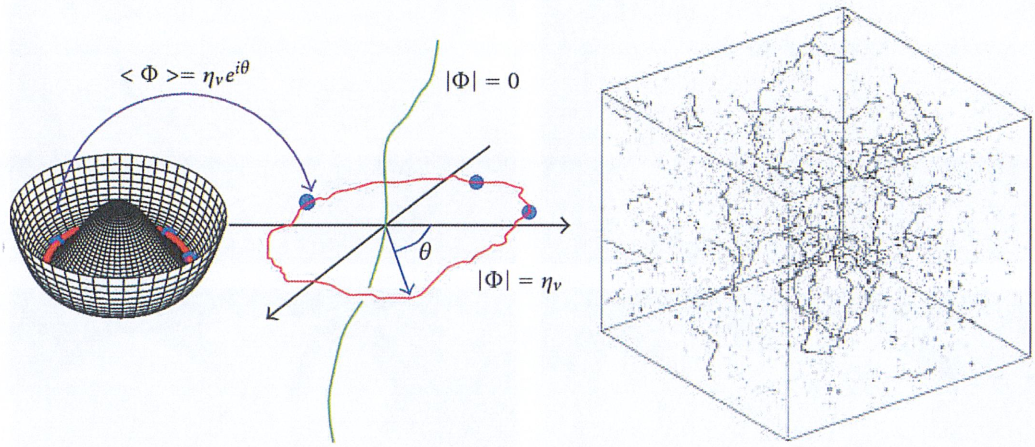


Figure 15: Left: the potential function for a 2 component scalar field involved in the generation cosmic strings. Right: result of numerical simulations (by Paul Shellard's group) of the evolving network of strings that develops from initially spatially incoherent field *à la* Kibble.

To get an idea of the topology of the initial string network, picture the initial field as filtered white noise with some coherence length set by the filter, and model the initial field evolution as simply rolling 'downhill' to the nearest minimum. In most places the field will vary quite smoothly with position, with $\nabla\phi \sim \phi_0/\lambda$, where λ is the coherence length for the initial field. However, at positions where both components of the field ϕ_1 and ϕ_2 were initially very small there will be very large gradients — and therefore very high energy density — localized near the regions where $\phi = 0$ initially. Now in 3-dimensional space the field ϕ_1 will generally vanish on a surface, and similarly for ϕ_2, so the regions where both components vanish are the intersection of these surfaces; i.e. on lines, or 'strings'. There is another way of looking at this; if we traverse an arbitrary loop in the real three dimensional space, the field moves along a closed trajectory in 2-dimensional field space. If the field is trapped in the circular trough then it is possible that the field trajectory will pass once around the brim of the sombrero; we would say that this loop has a *winding number* of one (or minus one, depending on the sense of rotation of the field). Now this winding number is a *topological invariant*; we can make a continuous deformation of the loop and the winding number cannot change, provided the field is everywhere confined to the minimum energy circle.

Now the energy for this toy model is in fact divergent. What is energetically more favorable is for the field to sit at $\phi = 0$ along the string axis, with the field falling to the potential minimum within some distance — the string thickness Δx. We can estimate what this is as follows. Consider a perfectly axi-symmetric field with unit winding number, and let's assume to start with that the field everywhere (except perhaps exactly on the axis) lies in the minimum. We can choose the spatial coordinate axes such that the field is

$$\begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix} = \frac{\phi_0}{r} \begin{bmatrix} x \\ y \end{bmatrix}. \tag{143}$$

The field gradient term in the energy density is

$$(\nabla\phi)^2 \equiv \frac{\partial \phi_i}{\partial r_j}\frac{\partial \phi_i}{\partial r_j} = \frac{\phi_0^2}{r^2}. \tag{144}$$

If we integrate this from $r_{\min}$ to $r_{\max}$ we find a contribution to the line density

$$\sigma c^2 = \phi_0^2 \int \frac{d^2 r}{r^2} = 2\pi\phi_0^2 \left[\log(r)\right]_{r_{\min}}^{r_{\max}} \tag{145}$$

so the line density diverges logarithmically if we let $r_{\min} \to 0$. Now consider a crude model in which the field lies in the zero potential for $r \gtrsim \Delta x$ but has $\phi \simeq 0$ for $r \lesssim \Delta x$ with some smooth transition between these. The gradient contribution to the line density will then contain a component $\sigma \sim \phi_0^2 \log(r_{\max}/\Delta x)/c^2$ and there will be a contribution from the potential $\sigma \sim V_0 \Delta x^2/c^2$, so the total line density will be

$$\sigma \simeq \frac{1}{c^2}(\alpha\sigma_0^2 \log(r_{\max}/\Delta x) + \beta V_0 \Delta x^2) \tag{146}$$

where α, β are dimensionless coefficients of order unity. Setting the derivative of this with respect to Δx to zero gives the string width for minimum line density (i..e. energy)

$$\Delta x \sim \frac{\phi_0}{\sqrt{V_0}} \tag{147}$$

just as above for domain walls. The linear mass-density is

$$\sigma \sim \frac{V_0 \Delta x^2}{c^2} \sim \frac{\phi_0^2}{c^2}. \tag{148}$$

The generation of this network of strings during a phase-transition involving a two-component field is known as the *Kibble mechanism*.

Just as for walls, the initial string network will be quite contorted. Calculating the stress-energy tensor (or more simply applying energetics arguments) again tells us that the string network will not be static but will develop transverse velocities $\sim c$. The character of the evolution of the string network is qualitatively different, however. Strings can reconnect when they intersect and so loops can be chopped off the network. Such a loop may further intersect itself, but there are stable loop configurations which sit there and oscillate. Such loops have large quadrupole moments and are moving relativistically, so they are quite efficient at radiating gravitational radiation. One can show that such loops will decay after $\sim c^2/G\sigma$ oscillations.

It is reasonable to expect that such a network will evolve towards a *scaling solution* with roughly one long string per horizon volume (that being the distance a string section will typically move). If we estimate the mean density in such strings we find

$$\rho_{\rm string} \sim \frac{\sigma}{(ct)^2}. \tag{149}$$

This is quite different from the case of walls where the density falls as $1/t$; here the string density evolves in the same manner as the mean density of matter or radiation, whichever happens to dominate. Thus we expect to have a constant fraction of the total energy density in string at any time. That the system should tend towards the scaling solution seems very reasonable — if there were too much string in some region then the interconnection would be more vigorous than on average and *vice-versa* — and early simulations of the evolution of the string network were performed and seemed to confirm this. This led to a simple picture of a continuously evolving network of long strings with a debris of oscillating loops lying around (whose mass spectrum could be crudely estimated from the dynamics of loop production) and it was supposed that the loops would act as point-like 'seeds' for structure formation. In this picture the density fluctuations would be highly non-Gaussian, in contrast to the fluctuations arising from inflation for instance. However, subsequent higher resolution simulations showed that this picture was somewhat flawed. The simple intuitive expectation (and low resolution simulations) did not incorporate an important feature; each time strings chop, discontinuities form and propagate along the string as traveling waves. As time proceeds the network develops more and more fine scale structure. It is still suspected that a scaling solution will result, but performing the needed simulations is quite a challenge. Analysis of the higher resolution simulations suggest that the simple one loop-one object picture for structure formation was overly simplified and that the myriad of rapidly moving loops produces something more akin to Gaussian fluctuations.

Perhaps the nicest feature of the string model is that the model has only one free parameter — the line-density of the strings σ. This sets the amplitude of density fluctuations at horizon crossing. We can estimate this as follows: The total density is $\rho_{\rm tot} \sim H^2/G$, so the ratio of string to total density is

$$\frac{\rho_{\rm string}}{\rho_{\rm tot}} \sim \frac{\sigma G}{c^2 H^2 t^2} \sim \frac{G\sigma}{c^2} \sim \frac{G\phi_0^2}{c^4} \sim \frac{G}{c^4}\frac{(kT_c)^2}{\hbar c} \simeq \left(\frac{kT_c}{E_{\rm pl}}\right)^2 \tag{150}$$

where we have used $\sigma \sim \phi_0^2/c^2$, $E_{\rm pl} \sim \sqrt{\hbar c^5/G}$ and $kT_c \sim \sqrt{\hbar c}\phi_0$. Now the energy density fluctuations in the strings are of order unity at the horizon scale — there being on the order of one string per horizon — and therefore the total density perturbation at horizon crossing is

$$\frac{\delta\rho}{\rho} \sim \frac{\rho_{\rm string}}{\rho_{\rm tot}} \sim \left(\frac{kT_c}{E_{\rm pl}}\right)^2. \tag{151}$$

The gravity associated with the string network drives motions of the rest of the matter and thus excites growing density perturbations which could plausibly account for the structure we see. This is very encouraging. First, the theory naturally generates perturbations with scale invariant amplitude at horizon crossing; the Harrison-Zel'dovich spectrum. Second, for strings formed at around the GUT scale of $kT_c \sim 10^{16}GeV \sim 10^{-3}E_{\rm pl}$, this predicts $\delta \sim 10^{-6}$, which is not far from that observed. Unfortunately, while the formation of strings at the GUT time is not mandatory, the formation of monopoles is, and these monopoles are a disaster. They can be gotten rid of by inflation — and one major motivation for inflation was the monopole problem — but then one would inflate away the strings as well.

An interesting feature of the negative tension is that the stress-energy tensor for an infinite static string is trace-free and consequently the string produces no tidal field. Outside of such a string spacetime is flat, but it is topologically different from ordinary Minkowski space in that there is a small deficit in azimuthal angle $\delta\varphi = 4\pi G\sigma/c^2$. A particularly distinctive features of the cosmic string model arises *via* gravitational lensing; lensing by long strings can produce a unique signature both in images of distant galaxies and in the microwave background. Observations of the latter, however, have ruled out cosmic strings as being the source of structure in our Universe. It may be that strings are present, but they cannot be the sole source of structure, so interest in these objects has diminished.

There are other defects which can form. We have already discussed formation of walls from 1-dimensional fields, and monopoles from three dimensional fields, both of which are pathological. A four-component field is more benign and gives rise to *texture*. A texture is not a topologically stable defect. Textures are most easily pictured in 1-D — where they result from having a 2-component field — and such a texture can shrink until $(\nabla\phi)^2 \sim V(0)$ at which point it will unwind.

A Useful formulae from matrix algebra

A.1 Derivative of the metric determinant

For a $N \times N$ square matrix $\mathbf{A}$ with components M_{ij}, the inverse can be computed as

$$\mathbf{A}^{-1} = \frac{1}{|\mathbf{A}|}\mathbf{C}^{\rm T} \tag{152}$$

where $|\mathbf{A}|$ denotes the *determinant* and where $\mathbf{C}^{\rm T}$ is the transpose of the *cofactor matrix* $\mathbf{C}$ whose components are $(-1)^{i+j}$ times those of the *minor* $\mathbf{M}$ defined such that M_{ij} is the determinant of the $(N-1)\times(N-1)$ matrix formed by deleting the $i^{\rm th}$ row and the $j^{\rm th}$ column of $\mathbf{A}$.

Multiplying by $\mathbf{A}$ and taking the trace gives an expression for the determinant

$$|\mathbf{A}| = N^{-1}\,{\rm Tr}\left(\mathbf{A}\mathbf{C}^{\rm T}\right) \tag{153}$$

though of course you would never use this as it all the diagonal elements of $\mathbf{A}\mathbf{C}^{\rm T}$ are the same and any one of them is equal to $|\mathbf{A}|$.

Applying this to the metric, $\mathbf{g} \to g_{\mu\nu}$, whose inverse has components we denote by $g^{\mu\nu}$, gives

$$|\mathbf{g}| = \tfrac{1}{4}\,{\rm Tr}(\mathbf{g}\mathbf{C}^{\rm T}) \tag{154}$$

which is come complicated function of the components of the metric $\mathbf{g}$. But the matrix representing $\mathbf{g}\mathbf{C}^{\rm T}$ is diagonal, and a particular component of $\mathbf{g}$, say $g_{\alpha\beta}$, will appear once in each diagonal component, where it will appear multiplied by $C^{\beta\alpha}$. But, by virtue of its definition, the cofactor matrix $C^{\alpha\beta}$ is independent of the particular component $g_{\alpha\beta}$. So, in the derivative of $\mathbf{g}$ with respect to x^γ, which, on applying the chain rule, is $|\mathbf{g}|_{,\gamma} = (\partial|\mathbf{g}|/\partial g_{\alpha\beta})g_{\alpha\beta,\gamma}$, the factor $\partial|\mathbf{g}|/\partial g_{\alpha\beta} = C^{\beta\alpha}$. Thus we have

$$|\mathbf{g}|_{,\gamma} = C^{\beta\alpha}g_{\alpha\beta,\gamma} = |\mathbf{g}|g^{\alpha\beta}g_{\alpha\beta,\gamma} \tag{155}$$

or

$$|\mathbf{g}|_{,\gamma}/|\mathbf{g}| = g^{\alpha\beta} g_{\alpha\beta,\gamma} \tag{156}$$

so the inverse of $\mathbf{g}$ (matrix-)multiplied by the derivative of $\mathbf{g}$ – which sounds like a logarithmic derivative of $\mathbf{g}$ – is just that: it is the logarithmic derivative of the determinant $\partial_\gamma \log(|\mathbf{g}|)$.

What appears with great frequency is not the determinant $|\mathbf{g}|$ – which is generally negative as space-time is locally Minkowskian (three of the eigenvalues of the metric being positive and one negative) – but $\sqrt{g} \equiv \sqrt{-|\mathbf{g}|}$, whose (logarithmic) derivative is

$$\boxed{\sqrt{g}_{,\gamma}/\sqrt{g} = \tfrac{1}{2} g^{\alpha\beta} g_{\alpha\beta,\gamma}} \tag{157}$$

A.2 Derivative of the inverse metric

To obtain the derivatives of the components of the inverse of the metric we can simply use

$$g^{\mu\nu} g_{\nu\sigma} = \delta^\mu_\sigma \tag{158}$$

whose space-time derivative vanishes, so

$$g_{\nu\sigma} g^{\mu\nu}{}_{,\gamma} = -g^{\mu\nu} g_{\nu\sigma,\gamma} \tag{159}$$

from which

$$g^{\alpha\beta}{}_{,\gamma} = -g^{\alpha\mu} g^{\beta\nu} g_{\mu\nu,\gamma} \tag{160}$$

which was invoked, along with (157) to demonstrate the equivalence of the two different forms of the d'Alembertian in §3.2.3. This can also be written as

$$g^{\alpha\beta}{}_{,\gamma} = -2 g^{\alpha\beta} \sqrt{g}_{,\gamma}/\sqrt{g}. \tag{161}$$

B Interacting fields

Another situation where an explicit $\vec{x}$-dependence of the Lagrangian density arises is with interacting fields. Imagine we have two scalar fields ϕ and ψ that are interacting through some extra term in the interaction to the total Lagrangian density is

$$\mathcal{L}(\phi_{,\alpha}, \phi, \psi_{,\alpha}, \psi) = \mathcal{L}_\phi(\phi_{,\alpha}, \phi) + \mathcal{L}_\psi(\psi_{,\alpha}, \psi) + \mathcal{L}_{\text{int}}(\phi, \psi) \tag{162}$$

where $\mathcal{L}_\phi(\phi_{,\alpha}, \phi)$ is the *free-field Lagrangian density* for the ϕ-field and similarly for $\mathcal{L}_\psi(\psi_{,\alpha}, \psi)$ and where, for the sake of concreteness, we'll assume the interaction is an addition to the potential energy density:

$$\mathcal{L}_{\text{int}}(\phi, \psi) = -\lambda \phi^2 \psi^2 \tag{163}$$

where λ is a *coupling constant*. This type of interaction could easily be 'cooked up' in the scalar elasticity model by having two lattices with some kind of extra springs coupling them. The interaction could, of course, depend on the derivatives of the fields also.

It is straightforward to obtain the Euler-Lagrange equations; there being two, one for ϕ and another for ψ. In the equation for ϕ, the *generalised force* is $\partial\mathcal{L}/\partial\phi = -\mu^2\phi - 2\lambda\psi^2\phi$, and similarly for the equation for ψ.

The total Lagrangian density above has no explicit dependence on $\vec{x}$, so it follows, from considering the derivative of $\mathcal{L}(\vec{x}) = \mathcal{L}(\phi_{,\alpha}(\vec{x}), \phi(\vec{x}), \psi_{,\alpha}(\vec{x}), \psi(\vec{x}))$ that there is a stress-energy tensor

$$T^\mu{}_\nu = -\phi_{,\nu} \frac{\partial\mathcal{L}}{\phi_{,\mu}} - \psi_{,\nu} \frac{\partial\mathcal{L}}{\psi_{,\mu}} + \delta^\mu_\nu \mathcal{L} \tag{164}$$

which we see, from the definition $\mathcal{L}$, has three components:

$$T^\mu{}_\nu = T^\mu_{\phi\ \nu} + T^\mu_{\psi\ \nu} + \delta^\mu_\nu \mathcal{L}_{\text{int}} \tag{165}$$

where $T^\mu_{\phi\ \nu} \equiv -\phi_{,\nu}(\partial\mathcal{L}_\phi/\partial\phi_{,\mu}) - \delta^\mu_\nu \mathcal{L}_\phi$ and similarly for $T^\mu_{\psi\ \nu}$, and which satisfies

$$T^\mu{}_{\nu,\mu} = 0 \tag{166}$$

and that there are four quantities $Q_\nu = \int d^3x T^0{}_\nu$ that are globally conserved since

$$\frac{dQ_\nu}{dt} = \frac{d}{dt}\int d^3x T^0{}_\nu = \frac{1}{c}\int d^3x T^0{}_{\nu,0} = -\frac{1}{c}\int d^3x T^i{}_{\nu,i} = 0. \tag{167}$$

But if we think of the ψ-field as a given 'external' influence on the ϕ-field the effective Lagrangian density for the the ϕ-field is

$$\tilde{\mathcal{L}}_\phi(\phi_{,\alpha}, \phi, \vec{x}) = \mathcal{L}_\phi(\phi_{,\alpha}, \phi) + \mathcal{L}_{\text{int}}(\phi, \psi(\vec{x})) \tag{168}$$

and has a $\vec{x}$-dependence. The stress-energy tensor obtained from this

$$\tilde{T}^\mu_{\phi\ \nu} = -\phi_{,\nu}\frac{\partial\tilde{\mathcal{L}}_\phi}{\phi_{,\mu}} + \delta^\mu_\nu\tilde{\mathcal{L}}_\phi = T^\mu_{\phi\ \nu} + \delta^\mu_\nu\tilde{\mathcal{L}}_{\text{int}} \tag{169}$$

obeys the continuity equation

$$\tilde{T}^\mu_{\phi\ \nu,\mu} = \tilde{\mathcal{L}}_{,\nu} = \frac{\partial\mathcal{L}_{\text{int}}}{\partial\psi}\psi_{,\nu} \tag{170}$$

which is generally non-vanishing. This is very reasonable, the $\nu = 0$ component of this equation has, on the right hand side, the generalised force on the ψ-field times its velocity – i.e. the rate at which work is being done on the ψ-field – whereas on the left we have the rate of change with time of $\tilde{T}^0_{\phi\ 0}$, which is (for a weak interaction) minus the energy density of the ϕ-field, along the 3-divergence of the energy flux density. So this would be the equivalent of Poynting's theorem.

Note that it is not the case that the sum of the stress-energy tensors for the two fields as defined here is conserved – that's because, in general, there is energy and momentum in the interaction term. But if it is weak – but acts for a long time, so as to have a significant effect in transferring energy and momentum between the two components – the sum $T^\mu_{\phi\ \nu} + T^\mu_{\psi\ \nu}$ will be approximately conserved. The space parts of this would express Newton's third law; that any momentum gained by one sub-system is equal to that lost by the other.

Alternatively, we could write the total Lagrangian density as the sum of two terms with one being, say, the free-field Lagrangian density for the ψ field; thus arbitrarily assigning the interaction energies to the ϕ-field. This is what happens if, for instance, one has a complex classical scalar field – for which the free-field Lagrangian is $-\frac{1}{2}\partial_\mu\phi^\star\partial^\mu\phi - \frac{1}{2}\mu^2\phi\phi^\star$ to which we add the free-field EM field $-F^{\mu\nu}F_{\mu\nu}/4\mu_0$ and then couple these by replacing $\partial_\mu \Rightarrow D_\mu = \partial_\mu + iqA_\mu$. In that case all of the interaction energies are in the modified scalar field term.

C The stress-energy tensor for EM plus charges

We will now apply the concepts developed above to the electromagnetic field. This may seem somewhat redundant as we have already developed the stress-energy tensor for EM in chapter ??. The exercise is an interesting one nonetheless, since the $T^{\mu\nu}$ that one obtains *à la* Noether from the symmetry of the system with respect to space-time translations is actually different to that we found before in that it is actually *asymmetric*.

C.1 The Lagrangian density for EM plus charges

The Lagrangian (??) is that of a free particle $L = -m/\gamma$ plus an interaction term $L_{\text{int}} = q\dot{x}^\mu A_\mu$. Generalising to a collection of particles, or a continuous distribution of charge, we have $L_{\text{int}} = \sum q\dot{x}^\mu A_\mu = \int d^3x A_\mu q \int d^3p f(\mathbf{x}, \mathbf{p})\dot{x}^\mu = \int d^3x A_\mu j^\mu$. So $L_{\text{int}} = \int d^3x \mathcal{L}_{\text{int}}$ with interaction Hamiltonian density

$$\mathcal{L}_{\text{int}} = j^\mu A_\mu. \tag{171}$$

To this we need to add the free-field Lagrangian density for radiation which, it turns out, is

$$\mathcal{L}(A_{\alpha,\beta}) = -\frac{1}{4\mu_0}F^{\mu\nu}F_{\mu\nu}. \tag{172}$$

which, as indicated, only depends on derivatives of the potential and not on the potential itself, and which, in terms of the EM fields is

$$\mathcal{L}(A_{\alpha,\beta}) = \tfrac{1}{2}(\epsilon_0|\mathbf{E}|^2 - \mu_0^{-1}|\mathbf{B}|^2) \tag{173}$$

The Lagrangian density for the electromagnetic field in the presence of a current $j^\mu(\vec{x})$

$$\mathcal{L}(A_\alpha, A_{\alpha,\beta}, x^\gamma) = -\tfrac{1}{4\mu_0} F^{\mu\nu} F_{\mu\nu} + j^\mu A_\mu. \tag{174}$$

with the dependence on x^γ because we are considering this to be the Lagrangian density for the EM field in the presence of some given current density $\vec{j}(\vec{x})$.

To justify (174) we may note that from this and the definition $F_{\mu\nu} \equiv A_{\mu,\nu} - A_{\nu,\mu}$ we find $\partial\mathcal{L}/\partial A_{\alpha,\beta} = F^{\beta\alpha}$ while $\partial\mathcal{L}/\partial A_\alpha = j^\alpha$ so the Euler-Lagrange equations

$$\frac{\partial}{\partial x^\beta}\left(\frac{\partial\mathcal{L}}{\partial A_{\alpha,\beta}}\right) = \frac{\partial\mathcal{L}}{\partial A_\alpha} \tag{175}$$

become simply $F^{\beta\alpha}{}_{,\beta} = \mu_0 j^\alpha$, which are the inhomogeneous Maxwell's equations.

C.2 The canonical stress-energy tensor for the radiation

To obtain a continuity equation for energy and momentum of the radiation à la Noether we take the partial derivative of $\mathcal{L}(x^\gamma) = \mathcal{L}(A^\alpha(x^\gamma), A^\alpha{}_{,\beta}(x^\gamma), x^\gamma)$ with respect to x^ν. Using the chain rule gives

$$\frac{\partial\mathcal{L}(\vec{x})}{\partial x^\nu} = \frac{\partial\mathcal{L}}{\partial A_\alpha} A_{\alpha,\nu} + \frac{\partial\mathcal{L}}{\partial A_{\alpha,\mu}} A_{\alpha,\mu\nu} + \frac{\partial\mathcal{L}}{\partial x^\nu} \tag{176}$$

where, just to be clear, the last term represents the derivative of $\mathcal{L}(A_\alpha, A_{\alpha,\beta}, x^\gamma)$ with respect to its final argument keeping A_α and $A_{\alpha,\beta}$ fixed, and is equal to $j^\alpha{}_{,\nu} A_\alpha$ since the only *explicit* functional dependence of $\mathcal{L}$ on $\vec{x}$ is through the current $j^\alpha(\vec{x})$.

Eliminating $\partial\mathcal{L}/\partial A_\alpha$ from this using (175), and using $A_{\alpha,\mu\nu} = \partial_\mu A_{\alpha,\nu}$, the first two terms on the right combine, as usual, to become the derivative of a single product, so

$$\partial_\nu \mathcal{L}(x^\gamma) = \partial_\mu\left(A_{\alpha,\nu}\frac{\partial\mathcal{L}}{\partial A_{\alpha,\mu}}\right) + \frac{\partial\mathcal{L}}{\partial x^\nu} = \mu_0^{-1}\partial_\mu\left(A_{\alpha,\nu}F^{\mu\alpha}\right) + j^\alpha{}_{,\nu}A_\alpha. \tag{177}$$

Finally, using $\partial_\nu \mathcal{L}\vec{x} = \delta^\mu_\nu \partial_\mu \mathcal{L}(\vec{x}) = -\delta^\mu_\nu \partial_\mu(\frac{1}{4\mu_0}F^{\alpha\beta}F_{\alpha\beta} - j^\alpha A_\alpha)$, we obtain the continuity equation

$$T_{\rm c}^\mu{}_{\nu,\mu} = -j^\mu A_{\mu,\nu} \tag{178}$$

where the *canonical stress tensor for radiation* is

$$\boxed{T_{\rm c}^\mu{}_\nu \equiv \mu_0^{-1}(F^{\alpha\mu}A_{\alpha,\nu} - \tfrac{1}{4}\delta^\mu_\nu F^{\alpha\beta}F_{\alpha\beta})} \tag{179}$$

C.3 The symmetric stress-energy tensor

The stress-energy tensor (179) is, like the canonical stress-energy tensor for particles, gauge-dependent. This might not seem surprising since we have obtained this from a Lagrangian density (174) containing a gauge-dependent interaction term. But even if we remove the interaction term, and start with the gauge invariant free-field electromagnetic Lagrangian density $\mathcal{L} = -\frac{1}{4\mu_0}F_{\mu\nu}F^{\mu\nu}$ alone, we still end up with the gauge dependent stress-energy tensor (179). Another unsatisfactory feature of (179) is that the source-term for its divergence in (178) is also gauge-dependent. It is in fact, however, minus the source term for the canonical stress tensor for the particles, so the sum of the canonical stress-energy tensors for the radiation and particles has a vanishing divergence; i.e. the total canonical 4-momentum is conserved. Also, it does not, at first sight, seem to agree with what one might expect from Poynting's theorem, but again that is perhaps not surprising as what appears in that theorem is $\mathbf{j} \cdot \mathbf{E}$ which is the rate at which the mechanical, rather than the canonical, energy of the particles is changing.

These unsatisfactory features are readily avoidable. This is because the continuity equation (178), while valid, does not uniquely specify the stress-energy tensor. It is possible to modify the radiation stress-energy tensor without affecting the continuity equation, and, in the process get rid of these problems.

Imagine we were to add to $T_{\rm c}^\mu{}_\nu$ an additional term of the form $\partial_\alpha(C^{\mu\alpha}{}_\nu)$. Then, when we take the divergence, we get an extra term $\partial_\mu\partial_\alpha(C^{\mu\alpha}{}_\nu)$. If $C^{\mu\alpha}{}_\nu$ is anti-symmetric under $\alpha \leftrightarrow \mu$ then, since $\partial_\mu\partial_\alpha$ is symmetric, the extra divergence will vanish. Looking at the (gauge-dependent) first term in (179) suggests that we might want to try something like $C^{\mu\alpha}{}_\nu = \frac{1}{\mu_0} F^{\mu\alpha} A_\nu$. This is indeed anti-symmetric under $\alpha \leftrightarrow \mu$ and would add to $T_c^\mu{}_\nu$ a divergence-free contribution

$$\partial_\alpha C^{\mu\alpha}{}_\nu = \frac{1}{\mu_0}(F^{\mu\alpha} A_{\nu,\alpha} + F^{\mu\alpha}{}_{,\alpha} A_\nu) = \frac{1}{\mu_0} F^{\mu\alpha} A_{\nu,\alpha} - j^\mu A_\nu \tag{180}$$

where we have used the inhomogeneous Maxwell's equations: $F^{\alpha\mu}{}_{,\alpha} = \mu_0 j^\mu$. The first term here is $\frac{1}{\mu_0} F^{\mu\alpha} A_{\nu,\alpha} = -\frac{1}{\mu_0} F^{\alpha\mu} A_{\nu,\alpha}$ which, when combined with $\frac{1}{\mu_0} F^{\alpha\mu} A_{\alpha,\nu}$ in (179), gives the gauge invariant product $\frac{1}{\mu_0} F^{\alpha\mu} F_{\alpha\nu}$.

Adding $\partial_\alpha C^{\mu\alpha}{}_\nu + j^\mu A_\nu = \frac{1}{\mu_0} F^{\mu\alpha} A_{\nu,\alpha}$ to (179) gives the symmetric stress-tensor

$$\boxed{T_{\mathrm{r}}^\mu{}_\nu \equiv \frac{1}{\mu_0} F^{\mu\alpha} F_{\nu\alpha} - \frac{1}{4\mu_0} \delta^\mu_\nu F^{\alpha\beta} F_{\alpha\beta}} \tag{181}$$

while changing its divergence, on the right hand side of (178), from $-j^\mu A_{\mu,\nu}$ to $-j^\mu A_{\mu,\nu} + \partial_\mu (j^\mu A_\nu) = -j^\mu F_{\mu\nu}$ (invoking charge conservation $j^\mu{}_{,\mu} = 0$) so, on raising the index ν,

$$\boxed{T_{\mathrm{r}}^{\mu\nu}{}_{,\mu} = -j^\mu F_\mu{}^\nu} \tag{182}$$

So $T_{\mathrm{r}}^{\mu\nu}$ is a symmetric, gauge-invariant tensor that depends only on the radiation fields in $F_{\mu\nu}$, and has a 4-divergence (182) with a source term which is gauge invariant also. Moreover, this source term is just the opposite to that which sources the divergence of the mechanical stress (??), so

$$\boxed{T_{\mathrm{r}}^{\mu\nu}{}_{,\mu} + T_{\mathrm{m}}^{\mu\nu}{}_{,\mu} = 0} \tag{183}$$

so whatever energy and momentum is given up by the radiation appears in the stress tensor for the matter and vice versa, the combination $T_{\mathrm{r}}^{\mu\nu} + T_{\mathrm{m}}^{\mu\nu}$ being conserved. The time component of the above equation expresses conservation of total (field plus mechanical) energy – in fact it is just Poynting's theorem – and the spatial components are the expression of Newton's law of action being equal and opposite to reaction.

If we compute the components of $T_{\mathrm{r}}^{\mu\nu}$ in terms of $\mathbf{E}$ and $\mathbf{B}$ (see appendix ??), we find that they are identical to what we found before (??) from Poynting's theorem and its analogue for momentum.

We arrived at (181) and (182) from Noether's theorem; i.e. by taking the derivative of the Lagrangian density with respect to the space-time coordinates. This actually led us to the canonical stress tensor, which we then had to massage to obtain the symmetric, gauge-invariant version. A much simpler alternative would have been to *postulate* (181) based on Poynting's theorem. Directly taking its derivative gives $T_{\mathrm{r}}^\mu{}_{\nu,\mu} = \frac{1}{\mu_0}(F^{\mu\alpha}{}_{,\mu} F_{\nu\alpha} + F^{\mu\alpha} F_{\nu\alpha,\mu} - \frac{1}{2} F^{\alpha\beta} F_{\alpha\beta,\nu})$. The first term on the right is, from Maxwell's equations, equal to $-j^\alpha F_{\alpha\nu}$, so the other terms must vanish. To show this we replace the dummy index μ by α in the second term, so the last two terms become $-\frac{1}{2\mu_0} F^{\alpha\beta}[2F_{\nu\alpha,\beta} + F_{\alpha\beta,\nu}]$ and invoking the definition of $F_{\alpha\beta} \equiv A_{[\alpha,\beta]}$ this is $-\frac{1}{2\mu_0} F^{\alpha\beta}[2A_{\nu,\alpha\beta} - (A_{\alpha,\nu\beta} + A_{\beta,\nu\alpha})]$ where by inspection $[\ldots]$ is symmetric under $\alpha \leftrightarrow \beta$ so this, when contracted with the anti-symmetric $F^{\alpha\beta}$ vanishes and we obtain (182).

D Additional material

D.1 Classical wave electrodynamics

As will be discussed in the following chapter, cosmologists are very keen on relativistic scalar fields. For the most part, they use real scalar fields, with Lagrangian densities like $\mathcal{L} = -\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}\mu^2\phi^2$ where is μ is a constant with units of spatial frequency.

Electric charge can be introduced by having a 2-component field, which can be represented as a complex field, with $\mathcal{L} = -\frac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi)^\star - \frac{1}{2}\mu^2\phi\phi^\star$. As it stands, that is rather sterile as the two field components are decoupled. But this field can, if we like, be coupled to Maxwell's electromagnetic field by replacing the ordinary derivatives ∂_μ by their gauge covariant counterparts D_μ. The field ϕ then has the same equation of motion as above, usually called the *Klein-Gordon equation*, with μ in place of $mc/\hbar$. And there is a charge and current density 4-vector which which looks superficially like the probability density and current for a QM wave function (but with $\vec{D}$ in place of ∇). But it must be kept in mind that the interpretation is very different; the field ϕ not a QM wave function. A full description of such fields does involve quantum mechanics, of course. There is a wave function, but it is not the field ϕ, rather it is a function – or rather functional – of ϕ. What the classical equations describe is the evolution of the expectation value of the field.