M1 GR + Cosmology - 1 - Review of Special Relativity

Nick Kaiser

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Contents

1 Course Preliminaries

1.1 Goals:

- understand in depth the Einstein field equations (derived in the most straightforward manner) and their physical content
- and understand the main applications and to be able to carry out some basic calculations relevant to observations with applications to
  - the solar system
  - stars and BHs
  - gravitational waves
  - gravitational lensing (time permitting)
- preparation for Benoit Semelin's semi-semester on cosmology

1.2 Recommended textbooks

- Bernard Schutz: *"A 1st Course in General Relativity"*
  - highly recommended – we will follow quite closely for GR – geometrical treatment (à la MTW)
- Ta-Pei Cheng: *"Relativity, Gravitation, and Cosmology"*
  - elementary introduction
- Hobson, Efstathiou and Lasenby *"General Relativity: An Introduction for Physicists"*
  - good coverage of cosmology also – appropriate level
- Sean Carroll *"Spacetime and Geometry"*
  - more advanced - notes also available on the web
- Tony Zee *"GR in a Nutshell"*
  - iconoclastic treatment - plus inspirational anecdotes
- Lenny Susskind – GR lectures available as on-line videos
- other excellent, but older, text are Misner, Thorne and Wheeler (MTW) *"Gravitation"* (highly geometric - little on cosmology) and Weinberg *"Gravitation and Cosmology"* (anti-geometrical view - little on black holes)
- other concise treatments (very good for consolidating/review but not for 1st exposure) are Dirac's *"General Relativity"* and the section of Landau and Lifshitz *"Classical theory of fields"*.

2 Founding principles of SR and essential physical implications

2.1 The founding principles

1. no absolute velocity
   - laws of physics are the same in different 'frames' with relative linear motion
   - so no way to measure how fast one is moving in an absolute sense
   - called *'Galilean relativity'*
   - embodied in Newton's concept of the family of equivalent *'inertial frames'* that have constant velocity with respect to each other and in which the same laws of motion apply
   - but note that the state of rotation *is* absolute
2. all observers measure the speed of light to be the same
   - very different from in Galilean relativity where velocities add
   - but implied by Maxwell's equations
     - c is constant of nature determined from constants ϵ_0 and μ_0 of electro- and magneto-statics
   - and confirmed experimentally
   - this was well known (Maxwell, Heaviside, Larmor, Lorentz, Poincaré etc) prior to 1905, but the prevailing 'world-view' was that there is some underlying physical aether (Maxwell's 'underworld') which would define a preferred frame and through which electromagnetic and other influences propagate, but that the laws of physics (like EM) do not allow one to determine that frame. Einstein's massive contribution was really a change in philosophical viewpoint (see John Bell's lecture "How to teach SR"); to embrace the non-existence of any absolute frame and elevate constancy of c to a principle.

2.2 Physical implications

2.2.1 Time dilation

- consider a simple "light clock" (illustrated in figure 1)
  - it is a rod of fixed $L_0 = 1\text{m}$
    - this is determined by atomic physics and the number of atoms in the rod
  - by (1), the length we measure if we move with the rod is independent of how we are moving
    - we call this the *proper length* of the rod
  - a photon (or small classical wave-packet) travels the length of the rod, reflects at the end and returns – for two "ticks" of the clock
  - so (2) implies that the *proper time* between ticks is $T_0 = L_0/c$
- now ask: what if the clock is moving relative to us (the 'lab frame') in direction perpendicular to its length – how long is the rod in our reference frame?
  - by (1) its length in our frame is still L_0
    - Since if it were different we could compare it with another standard rod that is at rest in the lab frame.
    - we could have the two rods make scratches on each other as they pass by
    - so we could unambiguously say which of the rods was shorter than the other - and the lab-frame and rod-frame observers would agree on this.
    - But that would violate (1) which says there is nothing special about either frame, so the the situation has to be symmetric
- now ask: what is the elapsed time – let's call it T – between ticks in the lab-frame?
- the answer is given by Pythagoras along which says that the distance $L = cT$ travelled is increased (see figure 1) and so $cT = \sqrt{L_0^2 + (vT)^2}$ which, solving for T, implies *relativistic time dilation*
  - $\boxed{T = \gamma(v)T_0}$
  - where the so-called *Lorentz γ-factor* is
  - $\boxed{\gamma \equiv 1/\sqrt{1 - v^2/c^2}}$
- limiting regimes for the Lorentz factor:
  - $\gamma = 1 + v^2/2c^2 + \ldots$ for $v \ll c$
  - $\gamma \to \infty$ for $v \to c$
- So *moving clocks run slow* by a factor $1/\gamma$
  - true for *all* clocks by (1)

2.2.2 Lorentz-Fitzgerald Length contraction

To see how *lengths* change between frames, we simply take our light clock and now have it move *parallel* to its length. Or, better still, have a double clock where the photons – or wave packets – start in the middle, travel out a unit proper distance to the ends, and then return to the centre as illustrated in figure 2.

- Q: what is interval between ticks if the rod is moving *parallel* to its length?
- A: must be the same: $T = \gamma T_0$, since a pair of orthogonal clocks must keep time with each other
- Q: what does this imply about the *length* of the rod in our frame?
- A: this is illustrated in figure 2 (which shows the world-lines – in a space-time diagram – for photons in a pair of light-clocks arranged end to end)

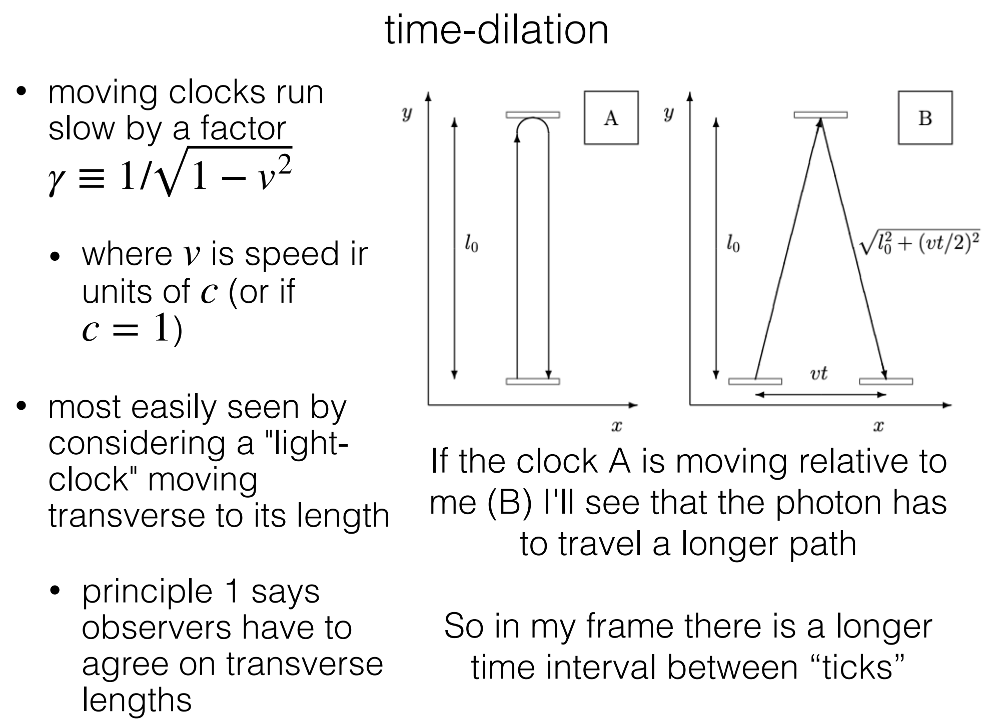


Figure 1: The time-dilation formula – which says that *moving clocks run slow* can be derived by considering a simple ‘light-clock’. From B’s perspective, the light has to travel further in making the round-trip, so constancy of c implies a longer elapsed time.

- the path on the right (in frame of reference in which the clock is moving) consists of two ‘legs’
- both are at 45-degrees by (2)
- if the first leg takes time t_1 the reflection point is at distance $r_+ = ct_1$ from the emission point
- but we can also compute this as $r_+ = L + vt_1$
- solving $ct_1 = L + vt_1$ for t_1 gives $t_1 = L/(c - v)$
- similarly, the time taken for the second leg is $t_2 = L/(c + v)$
- combining these gives $2T = t_1 + t_2 = L/(c - v) + L/(c + v) = 2Lc/(c^2 - v^2) = 2\gamma^2 L/c$
- so $L = cT/\gamma^2$ but $T = \gamma T_0$ from which we find the expression for *relativistic length contraction*:
- $\boxed{L = L_0/\gamma}$

• A metre ruler moving parallel to its length *is* length contracted in the lab-frame
  - Note we say ‘is’, not ‘appears’. The *appearance* is a bit more complicated as one has to allow for light propagation time.
  - this is calculable, but not particularly enlightening.

2.2.3 The ‘barn and pole’ paradox

• according to the lab-frame observer the moving clock is length contracted (as compared to the length of a standard clock at rest in the lab frame)

• but wouldn’t an observer moving with the first clock see the *lab-frame* clock to be length contracted?

• this is often posed as the ‘barn and pole’ or ‘car and garage’ paradox (as illustrated in figure 3)
  - a rod of some proper length rapidly approaches a barn of the same proper length
  - Q: is it possibly for the rod to be completely enclosed within the barn at some instant of time?
  - A1: from the barn’s perspective, yes: The rod is contracted so it fits with room to spare.

length contraction

- Considering a similar clock but moving parallel to its length shows that it's length must be less in the moving frame by a factor $1/\gamma$

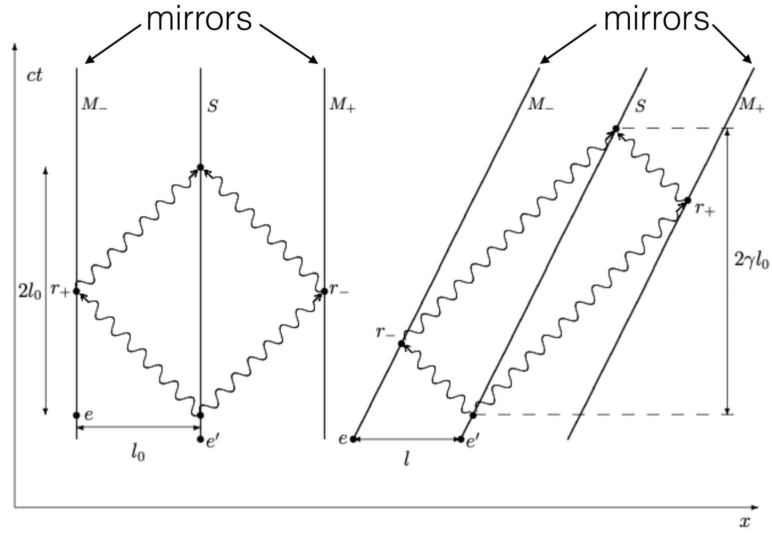


Figure 2: Relativistic length contraction can be derived by considering an identical clock to that used to derive time-dilation, but not moving *parallel* to its length. Here we show the *world-lines* of photons in a *space-time diagram*. In fact we are illustrating a slightly more elaborage clock consisting to a pair of clocks lying end to end. On the left, the wiggly lines are the photon paths in the 'clock-frame' and on the right in the 'lab-frame' (the frame in which the clocks are moving). The solid lines are the world lines of the centre and reflecting ends of the clock; these are vertical in the clock-frame but slope to the right in the lab-frame. Note that principle (2) tells us that the wiggly lines are at 45 degrees to the coordinate axes in both frames.

  - A2: from the rod's perspective, no: it is the barn that is contracted so it is shorter than the rod, so it is impossible for both ends of the rod to be in the barn at the same time

- Q: how is this resolved?

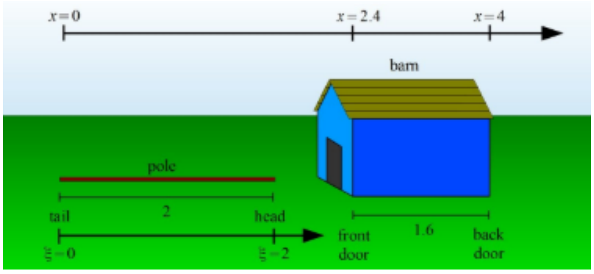
2.2.4 No universal simultaneity

To resolve the barn and pole paradox we may simply refer to figure (2).

- a pair of photons emitted by a source at the middle of the clock will reach the ends at the same time *in the clock's frame of reference* (left-hand figure)
- but in the lab-frame (right) the space-time diagram shows that those two events – labelled r_- and r_+ – are clearly not at the same coordinate time as recorded by observers in the lab frame – i.e. they are not simultaneous in the lab- (or barn-) frame
- the phrase *'at the same instant of time'* is ambiguous: different observers do not, in general agree on whether a pair of events happen simultaneously.
- this resolves the 'barn and rod' paradox
  - there are two events on the world-lines of the ends of the rod that are simultaneous from the barn's perspective and are both inside the barn
  - there are no two such events that are simultaneous as perceived by an observer moving with the pole

the "barn and pole" paradox

- the peculiarity of length contraction (from a Galilean perspective) is illustrated by the "barn and pole paradox"



- can the moving pole fit in the barn?
  - in the barn frame, yes. Since the pole is contracted
  - but in the pole's frame it is the barn that is contracted
- the resolution is that while there are two events on the pole ends that are both inside the barn *at the same time in the barn frame,* these two events are not simultaneous in the pole frame

Figure 3: The 'barn and pole paradox'.

3 Formalism of special relativity

3.1 Observers, reference frames and events

- a (inertial) *reference frame* can be realised by a lattice of rigid rulers carrying 'observers' at the intersection points carrying clocks that measure proper time
- the lattice must not be accelerating or rotating
  - this can be confirmed experimentally by the observers with accelerometers and/or gyroscopes
- the different observers' clocks are synchronised:
  - to do this, one observer is chosen to carry the 'master clock'
  - a neighbour observer sends a signal to the master when his, initially unsynchronised, clock reads t_1
  - on receipt the master sends back a signal encoding the reading t_0 of the master clock
  - this is received by the neighbour at time t_2, at which time the neighbour re-sets his clock to read $t_0 + (t_2 - t_1)/2$
  - the neighbour's clock is now synchronised
  - this process is then repeated to synchronise yet more distant clocks
- this provides a physical realisation of a coordinate system covering space-time
  - the marks on the rulers giving the spatial coordinates and the clocks giving the time
- in this arena we have the actual *events* that describe physics taking place
  - a point-like event is specified by its coordinates $(t, \mathbf{r})$
  - and we can have continuous families of events, such as the sequence of events defining the *world-line* of the end of the rod discussed above

3.2 The prototypical 4-vector: the displacement vector

- let there be two reference frames O and O' (i.e. grids made of rulers carrying observers holding clocks) that are in uniform rectilinear motion with respect to one another
- and let there be a pair of events A, B for which the observer O measures coordinate separations – or 'displacements' – $t = t_B - t_A$ and $\mathbf{r} = \mathbf{r}_B - \mathbf{r}_A$ and for which O' measures $t' = t'_B - t'_A$ and $\mathbf{r}' = \mathbf{r}'_B - \mathbf{r}'_A$

3.2.1 The components of a displacement 4-vector

- we let $(ct, \mathbf{r}) = (ct, x, y, z) = (x^0, x^1, x^2, x^3) = \{x^\alpha\}$
  - and we say that this set of 4 quantities (all with dimensions of length) are *the components of* the *displacement 4-vector* that is the separation between the two events
- while in the primed frame, we write $(ct', \mathbf{r}') = (ct', x', y', z') = (x^{0'}, x^{1'}, x^{2'}, x^{3'}) = \{x^{\alpha'}\}$
  - i.e. we put the prime on the index!

3.2.2 The displacement 4-vector itself

- we denote the actual 4-vector for the displacement from event A to event B by $\vec{x}$
  - we think about $\vec{x}$ as a concrete *geometrical entity*, which has a reality that is independent of the frame of reference in which its coordinates may be measured
  - just as we think of the events A and B as having a frame-independent existence
  - we might be tempted to identify $\vec{x}$ with it's components as $\vec{x} = x^\alpha$, but that would be sloppy, as we might equally well identify $\vec{x}$ with the primed components
  - and if you think about it you realise that one can't really *identify* $\vec{x}$ with its components, as there is more to the latter than $\vec{x}$, there is also the choice of frame (O in this case)
  - so to formalise the relation between a vector $\vec{x}$ and its components we write
  - $\boxed{\vec{x} \underset{\mathrm{O}}{\longrightarrow} x^\alpha}$
  - which we read as saying that $\vec{x}$ is the 4-vector that, in the particular reference frame O, has components x^α
  - though if the frame is obvious from the context we might simply write $\vec{x} \longrightarrow x^\alpha$
  - and we would write, for the *same* vector, $\vec{x} \underset{\mathrm{O}'}{\longrightarrow} x^{\alpha'}$

3.3 The Lorentz transformation matrix

- length contraction and time-dilation are *linear* in the sense that a 2m rod is shrunk by the same factor as a 1m rod
- this implies that there is a linear transformation relating x^α and $x^{\alpha'}$
  - $\boxed{x^{\alpha'} = \sum_\alpha \Lambda^{\alpha'}{}_\alpha x^\alpha}$
  - i.e. $\Lambda^{\alpha'}{}_\alpha$ is a 4x4 matrix – called the *Lorentz matrix*
  - whose components are only a function of the relative velocity of the two frames
  - and where the first index labels the rows and the second index labels the columns
  - so the above expression says that the components in the primed frame $x^{\alpha'}$, visualised as a column vector, are the matrix product of $\Lambda^{\alpha'}{}_\alpha$ and the column vector with components x^α
- let's specialise to the case where the velocity of O relative to O' is in the x-direction: $\mathbf{v} = (v, 0, 0)$
  - as in the above examples where O is taken to be the frame of the clock while O' is the 'lab-frame' in which the clock moves towards positive x'

- the coordinates in the y- and z-directions are unaffected so $x^{2'} = x^2$ and $x^{3'} = x^3$
- time-dilation and length contraction imply that
- $$\Lambda^{\alpha'}{}_{\alpha} = \begin{bmatrix} \gamma & \gamma v & & \\ \gamma v & \gamma & & \\ & & 1 & \\ & & & 1 \end{bmatrix}$$
- where the blank entries are zero and where we have set $c = 1$ (just replace $v \to v/c$ to get the transformation in units where $c \neq 1$)
- this is often called the *Lorentz boost matrix*
- this is the simplest case: velocity along the x-axis and no spatial rotation or reflection. These can all be included, but the simple form illustrates most of what we care about.
- let's check that this agrees with what we found before:
- **time dilation:**
  - Let O be the clock frame and consider two events at the same position a time T_0 apart
  - so $x^\alpha = (T_0, 0, 0, 0)$
  - performing the matrix multiplication gives $x^{\alpha'} = (\gamma T_0, \gamma v T_0, 0, 0)$
  - so in the lab frame O' the time interval is $T = x^{0'} = \gamma T_0$ in agreement with what we deduced
- **length contraction (1):**
  - now consider two events in the clock frame with the same time but different $x^1 = x$: i.e. $x^\alpha = (0, L_0, 0, 0)$
  - this could be the separation between the two events that are the reception, at the ends of the rod, of two photons emitted simultaneously from the centre of the rod
  - we now have, in the lab-frame, $x^{\alpha'} = (\gamma v L_0, \gamma L_0, 0, 0)$
  - so the x-separation in the lab frame $L = x^{1'} = \gamma L_0$
  - but this is *longer* than in the clock frame, when we know that the moving clock is shorter in the lab-frame – what has gone wrong?
- to see what went wrong, we can refer again to figure 2. The two events (simultaneous in the clock-frame on the left) might be r_- and r_+
- but these are clearly *not* simultaneous in the lab-frame (right panel)
  - and the displacement in x between r_- and r_+ is *not* what the lab-frame observer would call the distance between the mirrors
  - what the lab-frame observer *would* call the separation is that between two events on the world-lines of the mirrors that are simultaneous in the lab-frame
  - so we got caught out by the barn and pole paradox
  - or, equivalently, we used to the Lorentz matrix machinery to give the (correct) answer to the *wrong* question!
- **length contraction (2):**
  - to get the length – let's call it L, as yet unspecified – between the two ends of the rod in the lab frame O' we need to take $x^{\alpha'} = (0, L, 0, 0)$ and apply the *inverse* transformation $x^{\alpha'} \to x^\alpha$
  - the inverse is easy: it's the same as the matrix above, but with $v \to -v$
  - this gives, in the clock-frame, $x^\alpha = (-\gamma v L, \gamma L, 0, 0)$
  - the clock-frame spatial separation is $x^1 = x = \gamma L$ which must be the proper separation L_0, so we find that the lab-frame separation is $L = L_0/\gamma$ as we deduced before

- referring again to figure 2, the events in question might be those labelled e and e' (apologies – the prime does not denote the frame here)
- note that these two events do not occur at the same time in the clock-frame. But that is of no consequence: the world lines of the two ends of the rod in the clock-frame are vertical. So the x displacement from e to e' in the clock frame *is* what the clock-frame observer would call the length of the rod.

- we see here that a proper understanding of the peculiarity of simultaneity in SR is crucial to avoid making errors
  - we will see this later when we consider how the density of particles transforms between different frames
- Q: use the Lorentz boost matrix to show how velocities add in SR
- The boost matrix transforms 4-vectors that describe displacements. But there are many other physical quantities that transform under boosts in the same way
  - examples we will see below are the 4-velocity, the 4-momentum, the electromagnetic potential $\vec{A}$, the 4-current of particles $\vec{n}$
- We say that any 4-component entity that transforms in the same way as a displacement is a *Lorentz 4-vector*

3.4 Transformation of the $x - t$ axes

- the events at unit distance in the rod-frame O along the x and t axes are $x^\alpha = (0, 1, 0, 0$ and $(1, 0, 0, 0)$ respectively
- so in the lab-frame O' these are $x^{\alpha'} = (\gamma v, \gamma, 0, 0)$ and $(\gamma, \gamma v, 0, 0)$ respectively
- so the O-frame axes appear 'sheared' in the O' frame, with angle between the two x-axes (or the two $t-$ axes $\theta = \tan^{-1}(v)$ (or $\theta = \tan^{-1}(v/c)$ if $c \neq 1$).
- figure 2 provides useful insight here:
  - in the clock frame the paths of the photons form a diamond – i.e. a square rotated by 45 degrees
  - in the lab-frame the paths are still at 45 degrees to the coordinate axes
  - but it is apparent that the square become a rectangle, having been stretched out along the upper-right diagonal direction and compressed along the other diagonal direction

3.5 Invariance of space-time 4-volume

- the transformation of 4-dimensional volume elements is $d^4x' = |\Lambda| d^4x$
- for a diagonal matrix the determinant (or the Jacobian of the transformation) is just the product of the diagonal elements.
- Here $|\Lambda|$ is the determinant of the upper-left 2x2 sub-matrix (times the other two diagonal elements, which are both unity)
- So $|\Lambda| = \gamma^2(1 - v^2) = 1$
- hence space-time volume is invariant under 'Lorentz boosts'
- as we might have guessed from the fact that times get dilated by a factor γ while lengths get contracted by a factor $1/\gamma$
- we can also see this from the distortion of the paths of the photons in figure 2
  - the square circuit in the clock-frame becomes a rectangle in the lab-frame
  - being stretched by a factor $\sqrt{(1+v)/(1-v)}$ along the upper-right direction and compressed by a factor $\sqrt{(1-v)/(1+v)}$ along the upper-left direction
  - but maintaining its area

3.6 The invariant squared interval

- consider $s^2 \equiv -t^2 + x^2 + y^2 + z^2$
- in the primed frame this is $(s')^2 = -(\gamma t + \gamma v x)^2 + (\gamma x + \gamma v t)^2 + y^2 + z^2$
- multiplying the factors out shows that $(s')^2 = s^2$: i.e. the squared interval is invariant with respect to Lorentz boosts
- this is similar to the squared length of a vector $l^2 = x^2 + y^2 + z^2$ in Euclidean space
- but here the squared interval can be positive, negative or zero
- we describe 4-vectors as being *space-like, time-like* or *null*
  - depending on whether they lie outside, inside or on the 'light-cone'
  - and we say, in this regard, that the space-time of special relativity is *'Minkowskian'*
- note that while *components of vectors* are frame dependent the light-cone structure is frame *independent*
- there is no direct analogue of this in Euclidean space
- we think of the two distinct cones whose intersection is some event as containing those events that are in the future and in the past of that event
  - but, interestingly, there is nothing in the formalism here – or in the nature of space-time itself – that tells us which is which

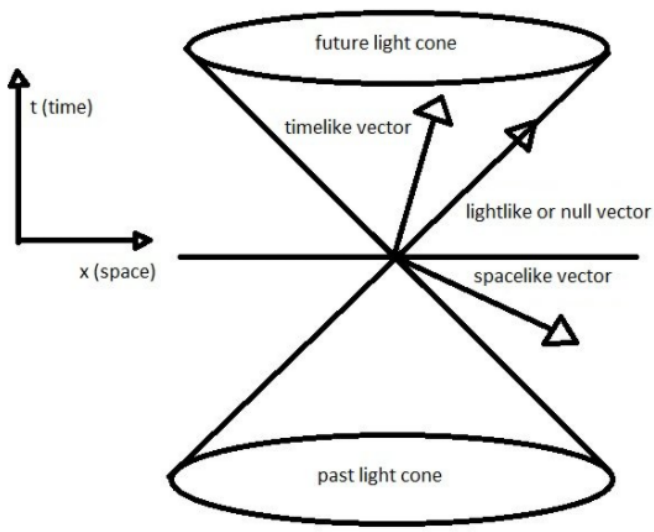


Figure 4: 4-vectors are classified as time-like, space-like or null depending on whether they lie inside, outside, or tangent to the so-called light cones. Time-like or null vectors may be future- or past-directed.

3.7 Notation: the metric, summation convention, and index raising and lowering

3.7.1 The Minkowski metric

- we can calculate the squared interval as
  - $\boxed{s^2 = \sum_\alpha \sum_\beta \eta_{\alpha\beta} x^\alpha x^\beta}$
  - where the 4x4 matrix $\eta_{\alpha\beta}$ – known as the *Minkowski metric* – has components
  - $\boxed{\eta_{\alpha\beta} = \text{diag}\{-1, 1, 1, 1\}}$

– and where, in matrix notation, the squared interval is

$$s^2 = [x^0, x^1, x^2, x^3] \begin{bmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{bmatrix} = [x^0, x^1, x^2, x^3] \begin{bmatrix} -x^0 \\ x^1 \\ x^2 \\ x^3 \end{bmatrix}$$

3.7.2 The Einstein summation convention

- we now introduce the *Einstein summation convention* which says that we drop the summation symbols so we can write $s^2 = \eta_{\alpha\beta} x^\alpha x^\beta$
  - henceforth *whenever* we have a pair of identical greek indices – one upstairs and one downstairs – summation is implied
  - we say that these are 'dummy' indices, and we can replace them by any other letter from the greek alphabet at will so, e.g., $\eta_{\alpha\beta} x^\alpha x^\beta = \eta_{\mu\beta} x^\mu x^\beta$
  - we can also write $s^2 = x_\alpha x^\alpha$ where $x_\alpha \equiv \eta_{\alpha\beta} x^\beta$

3.7.3 Covariant and contravariant vectors – metric as index raising/lowering operator

- multiplying x^α – a so-called '*contravariant vector*' – by $\eta_{\alpha\beta}$ effectively lowers the index to produce x_α which is called a '*covariant vector*'.
  - from $x^\alpha = (t, x, y, z)$ we get $x_\alpha = (-t, x, y, z)$ so the effect is simply to flip the sign of the time component
- we can also raise indices. This requires matrix multiplying x_α by the inverse of $\eta_{\alpha\beta}$, which we denote by $\eta^{\alpha\beta}$, so e.g. $x^\alpha = \eta^{\alpha\beta} x_\beta$
- but $\eta_{\alpha\beta}$ is its own inverse ($\eta^{\alpha\mu}\eta_{\mu\beta} = \delta^\alpha{}_\beta \equiv \text{diag}\{1,1,1,1\}$) so $\eta^{\alpha\beta} = \text{diag}\{-1,1,1,1\}$ also

3.7.4 The scalar product of two 4-vectors

- The metric can also be used to define the *scalar product* $\vec{v} \cdot \vec{u}$ of two vectors $\vec{v}$ and $\vec{u}$:
- $\vec{v} \cdot \vec{u} = \eta_{\alpha\beta} v^\alpha u^\beta = v^\alpha u_\alpha = v_\alpha u^\beta$
- in which notation, the invariant interval is the scalar product of the interval with itself
- $s^2 = \vec{x} \cdot \vec{x}$

3.8 What causes length contraction?

- The picture developed thus far regarding the frame dependence in special relativity is very much like that for passive rotations (i.e. rotations of the observer) in Euclidean space.
  - I.e. the world consists of objects like events, 4-momenta of particles etc..
  - these are real and frame independent entities
  - but the *coordinates* that we assign to events and the *components* of 4-momenta depend on the frame from which we view them
- this encourages the view that special relativity is *'just'* geometry
- in that world-view, the question "what causes length contraction?" may seem ill posed
  - one might be inclined to say that nothing *causes* length contraction
  - or perhaps one would say that it is just a consequence of the postulate that the speed of light is the same in all frames
- But perhaps a different view emerges if we try to think about what happens in a physical object while it is *in the process of becoming length contracted.* I.e. while it is being accelerated.

- to make this concrete consider a train composed of carriages that accelerates out of a station, with identical thrust being applied to each carriage, as illustrated in figure 5
  - perhaps it is a mag-lev train with the thrust being applied by electro-magnets
  - or perhaps the carriages are propelled by rocket motors

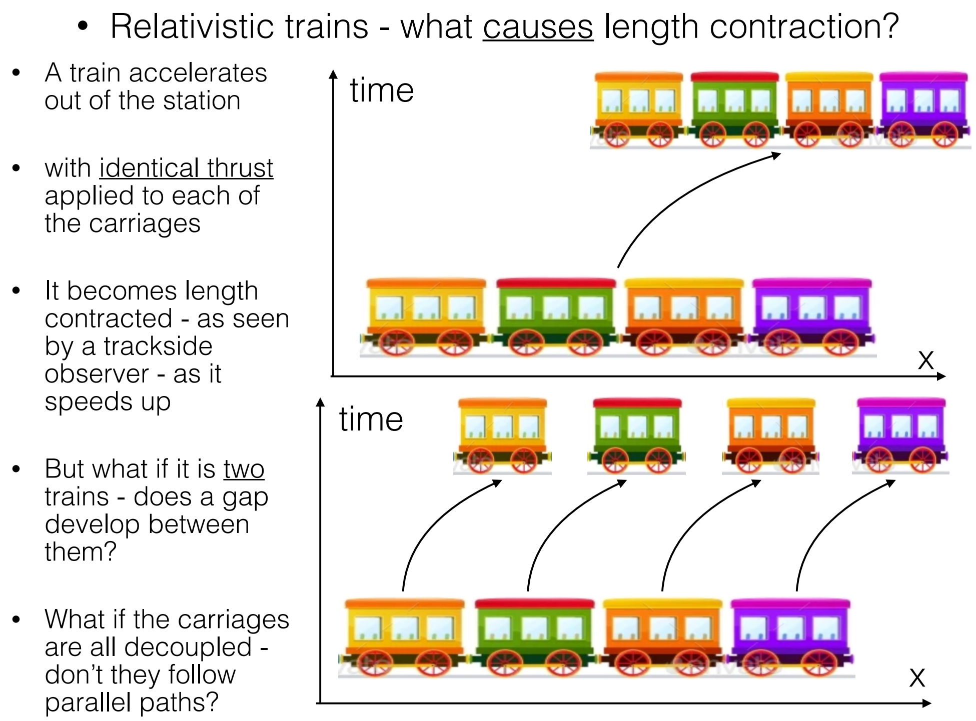


Figure 5: The 'trains paradox'. A train being accelerated out of a station will become length contracted (as seen in the frame of the track). But what about a train made of uncoupled carriages all being accelerated identically? The view that the phenomena of SR – time dilation and length contraction – are 'just' geometry might lead one to think that the train as a whole would still become length contracted. But consider the lower diagram: this suggests that the carriages follow paths in space-time that are identical, aside from their initial spatial displacement. This would suggest that gaps would develop between the carriages. If correct, this would say that the coupling between the carriages (in the train drawn in the upper diagram) plays a critical role in the contraction of the train.

- We would all agree – and this would surely be correct – that a track-side observer would perceive the train as a whole to be length contracted
- But what if the train were actually two trains that were initially lying nose-to-tail?
- As they accelerate, each train will become length contracted, but what about the mid-point of the combined system? Do the two trains remain touching each other? Or does a gap develop?
- Many physicists, at least when put on the spot, seem to feel intuitively that no gap would develop.
- But an alternative view would lead one to question this:
  - consider a space-time diagram of the paths of the two trains in the track-side observer frame
  - it would seem that the centres of the two trains – being identical – would move along identical paths
  - in which case the distance between the centres of the trains would remain the same in the frame of the track-side observer
  - so if, as is surely correct, the trains individually contract, a gap must develop

- similarly, if the ‘train’ consisted of de-coupled carriages, each carriage would length contract, but, since the carriages move along identical paths that are simply displaced from one another by their initial separation, the overall length of the train would not contract
- if correct, this seems to say that the contraction of the train when the carriages *are* coupled, is, in fact, caused in some way by the couplings between the carriages.
- In which case the question posed above is not at all ill posed.

- What do you think happens? Do gaps develop?
- If you think a gap does develop between the two trains, put yourself in the position of the driver of the trailing train
  - he sees the leading train get ahead of him
  - from his perspective, the leading train has a greater acceleration
  - if we are dealing with rocket driven trains, would he perceive the leading train to be burning its fuel more rapidly? If so, why?

4 The 4-velocity and 4-momentum

4.1 The 4-velocity

- consider a particle moving along a path in space-time
- in the lab-frame O we could write this path as $x^\alpha(\lambda)$ where λ parameterises the sequence of positions along the path
- a particularly useful parameterisation is to use the *proper time* as registered by a clock that the particle carries
  - this is an example of what is called, in GR, an *affine transformation*
- we can make a boost into the instantaneous ‘rest-frame’ of the particle (this is known as the momentarily comoving reference frame or MCRF)
  - so a small interval of the path becomes $dx^\alpha = (d\tau, 0, 0, 0)$
  - and so $ds^2 = -d\tau^2$ which is minus the squared time interval recorded by the particle
  - i.e. the *proper time*
- we can divide dx^α by $d\tau$ and call this the *4-velocity*
- $\boxed{U^\alpha \equiv dx^\alpha/d\tau}$
- but since ds^2 (and hence $d\tau$ also) is invariant under Lorentz boosts it follows that U^α transforms under such boosts exactly as the 4-vector displacement between a pair of events
  - i.e. the 4-velocity $U^\alpha = dx^\alpha/d\tau$ is a 4-vector
- there is an important implicit assumption that the clock is insensitive to any acceleration of the observer
  - one would not want to use a pendulum clock – whose period depends on the acceleration – here
  - the kind of clock postulated here is one that ticks at the same rate as a clock carried by an inertial observer who is instantaneously co-moving with the observer in question
- another complication is that this parameterisation of the path by proper time breaks down for a massless particle like the photon – for which the proper time for a finite motion is zero (we’ll remedy that presently)
- that said, in the MCRF, the 4-velocity is $U^\alpha = (1, 0, 0, 0)$

- so the norm of the 4-velocity – which is frame invariant – is
- $\boxed{U^\alpha U_\alpha = -1}$
- and the components of the 4-velocity in the lab-frame – for the case $\mathbf{v} = (v, 0, 0)$ – are
- $\boxed{U^{\alpha'} = \Lambda^{\alpha'}{}_\alpha U^\alpha = (\gamma, \gamma v, 0, 0)}$

4.2 The 4-momentum

- the non-relativistic 3-momentum of a particle is just the mass times the 3-velocity
- similarly, we can *define* the *4-momentum*, p^α as the (proper) mass m times the 4-velocity:
- $\boxed{p^\alpha \equiv m U^\alpha}$
  - the proper mass is Lorentz invariant (you measure it by pushing a particle in its rest-frame with a known force and measuring how it accelerates - the ratio is the the proper mass)
  - a word on notation is in order – many books and papers use m_0 for the proper mass and take m to be γm_0 (the famous equation $E = mc^2$, we will see, adopts that convention)
- Lorentz invariance of m means p^α, like U^α, is also a 4-vector.
- this implies $p^{\alpha'} = (\gamma m, \gamma m \mathbf{v})$ - so the spatial part of the 4-momentum, in the lab-frame, is
- $\boxed{\mathbf{p} = \gamma m \mathbf{v}}$
  - or γ times what you would have written down as the normal Newtonian momentum
  - which we will refer to as the *relativistic 3-momentum*, or just the (3-)momentum
  - Q: why do we call $\mathbf{p}$ *defined* in this way *the* momentum?
  - A: the reason is that it is the the *relativistic* $\mathbf{p}$ that is conserved in collisions

4.3 The 'cricketers on trains' thought experiment:

- To see why it is the relativistic $\mathbf{p} = \gamma m \mathbf{v}$ that is conserved (and not the Newtonian $\mathbf{p} = m\mathbf{v}$) consider two parallel railway lines with separation $2D$ and 2 carriages travelling in opposite directions as illustrated in figure 6.
- as they are about to pass each other, two cricketers on the trains throw identical balls (of proper mass m) out of the carriages, perpendicular to the tracks with some (small) velocity u in their frames
- the throws were carefully timed so that the balls bounce off each other elastically and return to the cricketers
- in the frame of a track-side observer all this is symmetrical as shown in the upper diagram
- but now look at this from the frame of one of the cricketers B (lower diagram)
- note that both A and B agree on the distance between the trains, since this is perpendicular to their motion
- B sees his ball move a distance $2D$ (out and back) in time $T = 2D/u$
  - and it's moving slowly, so he assigns it a momentum $p = mu$ and so the change of its Newtonian momentum when it recoils is $\Delta p_N = 2p = 2mu$
- the out and back flight of A's ball takes the same time T in A's frame
- but from B's perspective, the time for A's ball's return trip is time dilated: $T' = \gamma T$
  - where γ is the Lorentz factor for A (in B's frame)
  - and which is also essentially the γ factor for A's ball in B's frame (since the transverse velocities are small compared to the relative motion of the trains)

- so in B's frame the transverse component of A's ball's velocity is $v_x = 2D/T' = u/\gamma$
- so when it bounces, the change $\Delta p_N = 2p = 2mv_x$ of Newtonian momentum of A's ball is *smaller* than that of his ball by a factor $1/\gamma$ – so Newtonian momentum is *not* conserved
- the change of the *relativistic* momentum of A's ball is $2m\gamma v_x$ which is the same as $2mu$ which is the same as the relativistic momentum of B's ball (in B's frame) since, by assumption, $u \ll 1$.
- thus we can say that the *inertial mass* – that which we measure using of the change of velocity imparted by a given impulse of momentum – for this transversely moving mass, is γm

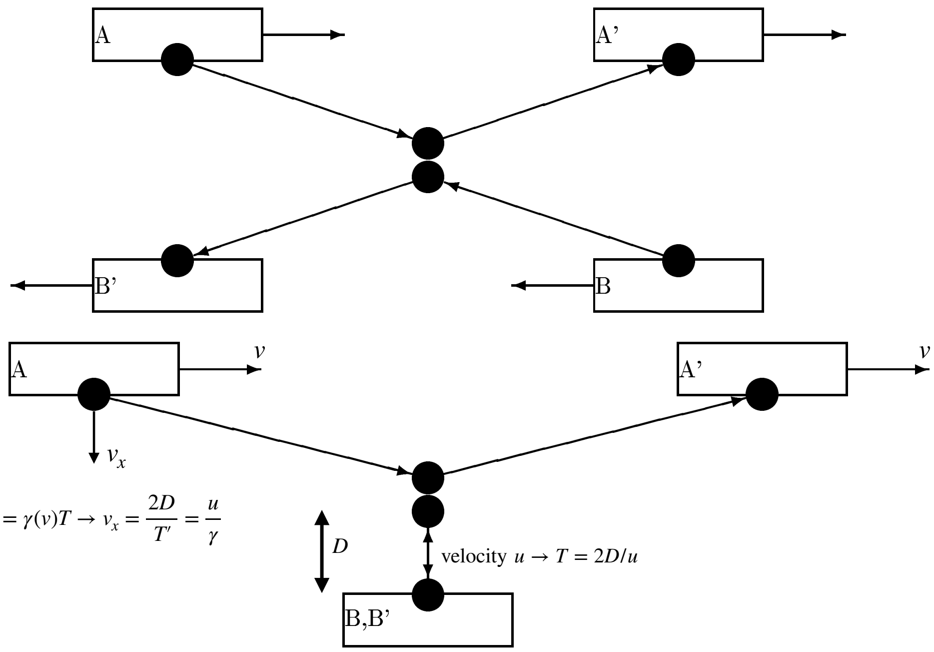


Figure 6: Conservation of relativistic 3-momentum in collisions. Two observers A and B pass each other on rapidly moving carriages and as they do so they bounce balls off each other, exchanging momentum. The upper panel shows the symmetric situation in the center of mass frame. The lower panel shows the situation from B's point of view. Now B assigns a longer time interval to the pair of events A and A' than to B, B' while transverse distances are invariant so it follows that the transverse velocity he assigns to A's ball is smaller than that of his own by a factor $1/\gamma(v)$ (where v – assumed to be much greater than u – is A's speed relative to B). Thus, in B's frame mu_x is not conserved, but γmu_x is conserved in the collision.

4.4 Equivalence of mass and energy

- the time component of the 4-momentum is $p^0 = mU^0 = \gamma mc$
- for small velocity we can expand $\gamma = 1 + v^2/2c^2 + \ldots$ so
- $\boxed{p^0 c = mc^2 + mv^2/2 + \ldots}$
- we call $E = p^0 c \equiv \gamma mc^2$ the *relativistic energy:* which consists of a *rest-mass energy* $E_0 = mc^2$ plus an extra component associated with the motion that, to lowest order for small velocities, coincides with the usual Newtonian *kinetic energy*
- and in terms of which we can write the 4-momentum as
- $\boxed{\vec{p} \longrightarrow (E/c, \mathbf{p})}$
- but there is a bit more to mass-energy equivalence than this
- what Einstein famously argued is that if a system looses energy ΔE then its inertial mass will decrease by an amount $\Delta m = \Delta E/c^2$ and that mass and energy are interchangeable and that by tapping into the rest-mass energy very large amounts of energy can be released as in H-bombs.

- one argument he gave for this (after famous 1905 paper laying the foundations for SR) is illustrated in figure 7
  - interestingly, in making this case, he argued that the photons are like wave packets with field $\propto \cos(k(x-ct))$ and, from quantum mechanics, momentum $p = \hbar k$, and energy $\Delta E/2 = pc$
  - he then considers this in a frame (primed frame) in which the particle has some velocity $v \ll c$ parallel to the direction of the photons and then, by applying the appropriate boost to x and t in $\cos(k(x-ct))$ he gets the boosted wave vectors $k' = k(1 \pm v + v^2/2 + \ldots)$
  - which is one way to show how the momentum of the photons get Doppler shifted
  - what is also interesting is that he seemed to have some discomfort with this argument and wrote that he worried that maybe god "is laughing at me and pulling me by the nose"

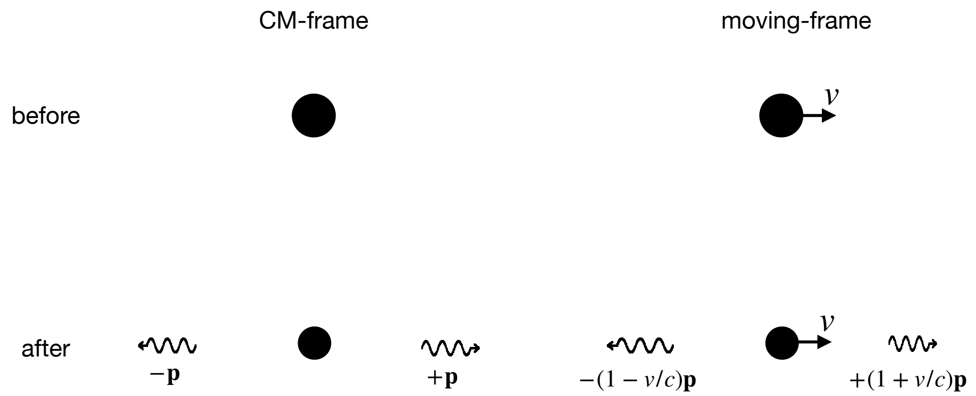


Figure 7: On the left is shown an object of a certain mass m in its rest frame which emits two massless particles (photons) with equal and opposite momenta. The same thing is shown on the right in a frame in which the object is initially moving to the right with speed v. Since the the object remains at rest in the CM frame, neither does its speed change in the moving frame. In the CM frame the two photons have equal and opposite momentum, so total momentum is conserved. But in the moving frame the right-going photon is blue-shifted and has a greater (absolute) momentum than the left-going photon. The two photons in the moving frame have a non-zero net momentum, which must be balanced by a change in the momentum of the object. But as its speed did not change this implies its mass must have decreased.

4.4.1 The relativistic energy-momentum relation

- as with any 4-vector, the 'norm' or squared length of the 4-velocity is frame invariant:
  - so $\vec{p} \cdot \vec{p} = \eta_{\mu\nu} p^\mu p^\nu = p^\mu p_\nu$ is a Lorentz invariant
  - it's value is most easily determined in the rest-frame of the particle (where $\gamma = 1$, so $p^\alpha = (m, 0, 0, 0)$) to be $-m^2$
  - this gives the *relativistic energy-momentum relation:*
  - $\boxed{E^2 = p^2 + m^2}$
    * or $E^2 = p^2 c^2 + m^2 c^4$ in units where $c \neq 1$
  - i.e. the allowed combinations of E and $\mathbf{p}$ lie on a 3D hyperbola in 4D $E, \mathbf{p}$ space
    * as illustrated in figure 8
    * this is known as the *mass shell*
    * and where, since $E = p^0 = \gamma m$ is positive, we take the positive root

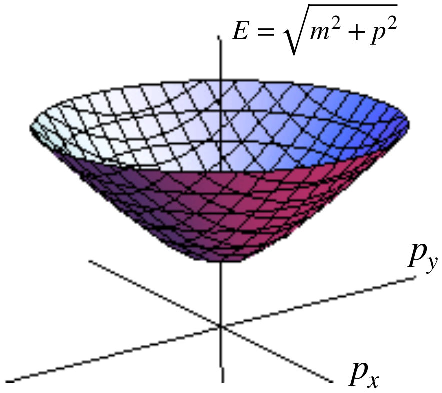


Figure 8: Relativistic energy-momentum relation for a particle of mass m. The energy, for a given momentum $\mathbf{p}$, lies on a hyperboloid.

4.4.2 Conservation of total 4-momentum and the invariant mass

- consider a collision between two particles of masses m_1 and m_2
- we can always choose a frame in which they have equal and opposite relativistic 3-momenta $\mathbf{p}_1 = -\mathbf{p}_2 = \mathbf{p}$
  - this is called the *center of momentum frame*
  - in which the 4-momenta of the particles are
  - $$\begin{array}{c} \vec{p}_1 \underset{\text{CM}}{\longrightarrow} (E_1, \mathbf{p}_1) = (\sqrt{m_1^2 + |\mathbf{p}|^2}, +\mathbf{p}) \\ \vec{p}_2 \underset{\text{CM}}{\longrightarrow} (E_2, \mathbf{p}_2) = (\sqrt{m_2^2 + |\mathbf{p}|^2}, -\mathbf{p}) \end{array}$$
- conservation of momentum implies that their momenta are equal and opposite after the collision
- and if the collision is elastic, $|\mathbf{p}_1|$ and $|\mathbf{p}_2|$ are unchanged
  - which we can think of as following from time reversal symmetry
  - though this would not hold for inelastic collisions – where time-reversal symmetry is violated
- the result of an elastic collision is simply to reverse the 3-momenta
  - so the particles are moving away from each other after the collision
- and to rotate the axis along which they move
  - so, for given initial $\vec{p}_1$ and $\vec{p}_2$ (or equivalently given $\mathbf{p}_1$ and $\mathbf{p}_2$) the possible collisions form a 2-parameter family (determined by the direction of one of the out-going particles)
- boosting these individual 4-momenta to some other frame moving with velocity $\mathbf{v} = (v, 0, 0)$ wrt to the centre of momentum frame we have
- $$\begin{array}{c} \vec{p}_1 \longrightarrow (\gamma E_1 + \gamma v p_x, \gamma p_x + \gamma v E_1, p_y, p_z) \\ \vec{p}_2 \longrightarrow (\gamma E_2 - \gamma v p_x, -\gamma p_x + \gamma v E_2, -p_y, -p_z) \end{array}$$
- and adding these gives the total energy-momentum 4-vector
- $\boxed{\vec{p}_{\text{tot}} = \vec{p}_1 + \vec{p}_2 \longrightarrow (\gamma(E_1 + E_2), -\gamma\mathbf{v}(E_1 + E_2))}$
- i.e. just what one would find for boosting the 4-momentum of a single particle with
- $\boxed{\vec{p} \underset{\text{CM}}{\longrightarrow} (E_1 + E_2, \mathbf{0})}$
- the sum of the energies in the CM frame ($E_1 + E_2$ here) is called the *invariant mass*

  - it is the minimum value for the total energy (as the total energy is evidently larger in any frame other than the CM frame)

- this all applies to situations were particles are emitted (as in figure 7) or where particles decay into other particles

- in a particle collider, the summed energy of the ingoing particles (in the CM frame) gives the mass of the most massive product that can me created

4.4.3 The 4-momentum for massless particles

- the 4-velocity is defined as $U^\alpha = dx^\alpha/d\tau$ but for two neighbouring points on the world-line of a mass-less particle $d\tau^2 = -ds^2 = (dx^0)^2 - |d\mathbf{x}|^2 = 0$

- so U^α is ill defined

- the 4-momentum, however, is well-defined

- we can (and should!) think of a massless particle as the limit as $m \to 0$ of a massive particle
  - for finite $\mathbf{p}$ such particles have asymptotically identical energy $E = |\mathbf{p}|$
  - and follow asymptotically identical paths
  - so we have $p^\alpha = mU^\alpha = mdx^\alpha/d\tau$
  - while $d\tau \to 0$ (for given dx^α) $d\tau/m$ remains finite

5 Photons and electromagnetic wave packets in special relativity

5.1 The 4-gradient of a scalar as a covariant 4-vector

- contravariant vectors transform as $x^{\alpha'} = \Lambda^{\alpha'}{}_{\alpha} x^\alpha$ and the same is true for infinitesimal 4-vectors $dx^{\alpha'}$ and dx^α

- covariant vectors are transformed by multiplying by the inverse of the Lorentz matrix (obtained simply by flipping the sign of v) since $x^\alpha x_\alpha$ is Lorentz invariant
  - (as an aside) another way to write the rule for contravariant vectors is to consider $x^{\alpha'}$ to be a function (linear in SR) of the coordinates in another frame: $x^{\alpha'} = x^{\alpha'}(x^\alpha)$ from which, applying the chain rule, $dx^{\alpha'} = (\partial x^{\alpha'}/\partial x^\alpha)dx^\alpha$
  - so $\Lambda^{\alpha'}{}_{\alpha} = \partial x^{\alpha'}/\partial x^\alpha$
    * note that the α index is 'downstairs' in both expressions

- if there is a field ϕ that is Lorentz invariant
  - as for any field that represents any scalar physical quantity

- then the change in ϕ between two points in space-time separated by dx^α is
  - $d\phi = (\partial\phi/\partial x^\alpha)dx^\alpha$ must be Lorentz invariant also

- that means that $\phi_{,\alpha} \equiv \partial\phi/\partial x^\alpha$ must transform like (i.e. be) a covariant vector
  - and the same is true, in a somewhat more abstract sense, for the partial derivative *operator* $\partial/\partial x^\alpha$

- as an illustrative example, consider ϕ to be the phase of a planar monochromatic electromagnetic wave (or perhaps a nearly monochromatic wave-packet) with potential $A = A_0 \cos(\phi)$ with $\phi = -\omega t + \mathbf{k} \cdot \mathbf{x}$ with $\omega = |\mathbf{k}|$
  - figure 9 shows a space-time $x - t$ diagram showing lines of constant phase
  - the gradient of the phase $\phi_{,\alpha}$ is the arrow pointing in the direction of increasing phase
  - which way do wavefronts move in space as a function of time?

– also indicated, again as an arrow, is the corresponding contra-variant vector $\phi^{,\alpha} = \eta^{\alpha\beta}\phi_{,\beta}$

- in this case $\phi_{,\alpha} = (-\omega, \mathbf{k})$
  – note the highly suggestive parallel with the (covariant) 4-momentum $p_\alpha = (-E, \mathbf{p})$
  – bearing in mind that for a photon $E = \hbar\omega$ and $\mathbf{p} = \hbar\mathbf{k}$

- a better way to visualise a covariant vector like the gradient of the phase $\phi_{,\alpha}$ – sometimes called a '*1-form*' – is not as an arrow, but as a collection of a lot of little closely packed surfaces of constant ϕ separated by multiples of some constant value (e.g. $\Delta\phi = 2\pi$)
  – the 'length' of the gradient vector is inversely proportional to the spacing in space-time coordinates
  – i.e. the density of the surfaces

- then, with a contravariant vector dx^α visualised as an arrow, $\phi_{,\alpha}dx^\alpha$ is the difference in phase – or whatever scalar field we are dealing with – between one end of the vector and the the other. Or equivalently the number of surfaces that the vector punctures.

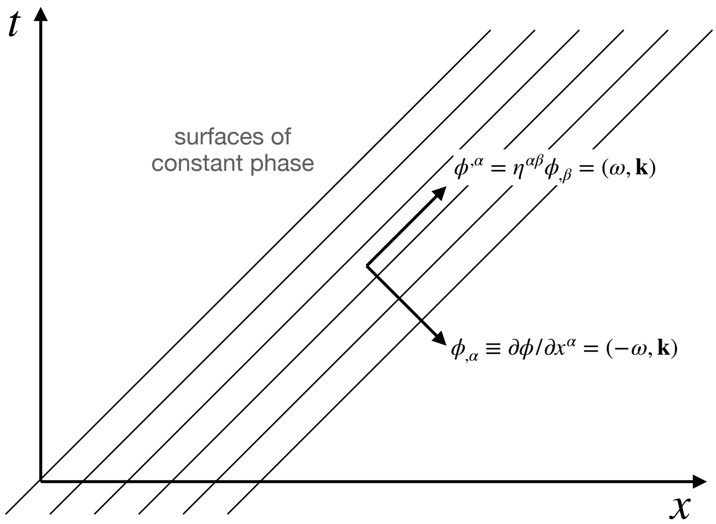


Figure 9: Solid lines are surfaces of constant phase for a nearly monochromatic null wave packet which is propagating to positive x with increasing time t. The covariant vector $\phi_{,\alpha} = \partial\phi\partial x_\alpha$ points 'backwards' in time. Its contra-variant counter-part $\phi^{,\alpha}$ is parallel to the 4-momentum of the packet.

5.2 The Doppler Shift

- consider a photon with energy in the 'lab' or 'observer' frame $E_{\rm obs}$ and moving with 3-direction $\hat{\mathbf{n}}(\theta, \phi)$
- the 4-momentum is $p^\alpha = E_{\rm obs}(1, \cos\theta, \sin\theta\cos\phi, \sin\theta\sin\phi)$
- let the emitter be moving with velocity v in the $+x$ direction
- applying a boost show that the p^0 in the emitter frame (i.e. the energy in the emitter frame) is $E_{\rm em} = \gamma(1 - v\cos\theta)E_{\rm obs}$
- so the *red-shift* is $1 + z = \lambda_{\rm obs}/\lambda_{\rm em} = \nu_{\rm em}/\nu_{\rm obs} = E_{\rm em}/E_{\rm obs} = \gamma(1 - v\cos\theta)$
  – if the source is moving in the same direction as the photon then $\cos\theta = 1$ and $1 + z = \sqrt{(1 - v)/(1 + v)}$
    * this is greater than unity (red-shift) for a source moving away from the observer
  – while for a source moving perpendicular to the direction the observer sees the photon coming from ($\cos\theta = 0$) this is also a redshift: $1 + z = \gamma$
    * this is known as the *transverse Doppler redshift*

- A puzzle:
  - imagine a rocket travelling along the $y-$axis which emits a photon as it passes $y = 0$ whose energy is measured by an observer who is sitting somewhere on the $+x$ axis.
  - the above formula says the observed energy is less than that emitted: $E_{\rm obs} = E_{\rm em}/\gamma$
    - but the rocket had to convert some rest-mass into radiation
    - and that rest-mass was moving relative to the observer
    - and so had lab-frame energy γmc^2: i.e. *greater* than mc^2
    - if energy is conserved shouldn't we see a *blue-shift?*
  - what is going on?
  - consider the (rather fanciful) thought experiment illustrated in figure 10

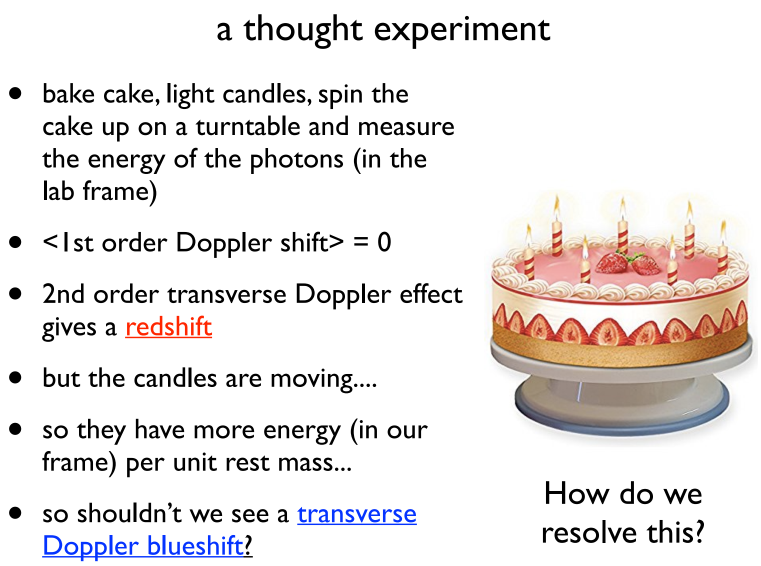


Figure 10: If we have a moving sources the photons we receive have a first order Doppler shift (of order v/c) which may be positive or negative – and which will average to zero if the collection of sources is not moving as a whole. The second order effect is the transverse Doppler *redshift*. But in the situation shown here the average photon energy would be *blue-shifted*.

5.3 Aberration

- consider a source emitting unit energy photons with angles (polar and azimuthal) θ, ϕ
  - i.e. $p^\alpha = (1, \cos\theta, \sin\theta\cos\phi, \sin\theta\sin\phi)$
- if the source is moving speed v along the $+x$ axis, boost p^α into the lab-frame
  - $p^{\alpha'} = (\gamma(1 + v\cos\theta), \gamma(v + \cos\theta), \sin\theta\cos\phi, \sin\theta\sin\phi)$
- find expressions for $\cos\theta'$ and $\sin\theta'$ in terms of $\cos\theta$, $\sin\theta$ and v
- what is $\cos\theta'$ for photons emitted in the equatorial direction in the emitter frame?
- show that for $\gamma \gg 1$ these photons are strongly 'beamed' and that they emerge, in the lab-frame, at direction $\theta' = 1/\gamma$

5.4 Compton scattering

See TD.

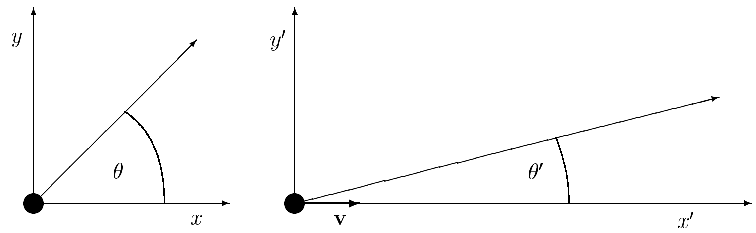


Figure 11: A photon (arrow) emitted from a source (moving to the right) in the source-frame (left) and observer-frame (right). For highly relativistic particles the photons tend to emerge tightly 'beamed' in the forward direction

6 Transformation of volumes and densities

6.1 Spatial volumes and space-density of particles

- Consider a cubical $1m^3$ volume in some reference frame O that contains, say, 1 million particles
  - so the density of particles in that frame is 1 per cm^3
- if we observe that volume from a relatively moving frame O' we find it to be length contracted
  - so the density of particles in the frame O' is larger (by a factor γ)
- But consider instead a cubical volume at rest in the primed frame
- the un-primed observer will find this to be length contracted in his frame
- so that observer will conclude – by the same line of argument – that the density of particles is *higher* in the un-primed frame, not lower
- Q: which, if any, of these contradictory conclusions is correct?
- A: the answer depends on how the *particles* in question are moving
  - if the particles are at rest in the frame O then the former conclusion is correct
  - to see why, consider the 4 pairs of events that define the corners of the cube in O' at a certain time in that frame
    * where the separation of each pair lies along the direction of relative motion
  - the spatial separation between these events in O will be larger by a factor γ
  - they will also have a non-vanishing temporal separation
    * but as the particles are at rest in O, this is irrelevant,
  - this is illustrated in figure 12
- We can conclude that an observer that is moving with respect to a 'dust' of particles will see their density to be *higher* than the density that would be measured by an observer sharing the same rest-frame as the particles
- and that the density – which we shall denote by n – is enhanced by exactly the same amount as the *energy* is enhanced in a relatively moving frame
  - from which it follows that n/E is a *Lorentz-invariant*
  - as is $E \times d^3r$
  - the latter following from the fact that $n \times d^3r$ is the number of particles, which is automatically invariant
- Q: the density of particles transforms under boosts in the same way as E, which is the time component of a 4-vector. What would the spatial parts of such a 4-vector represent?

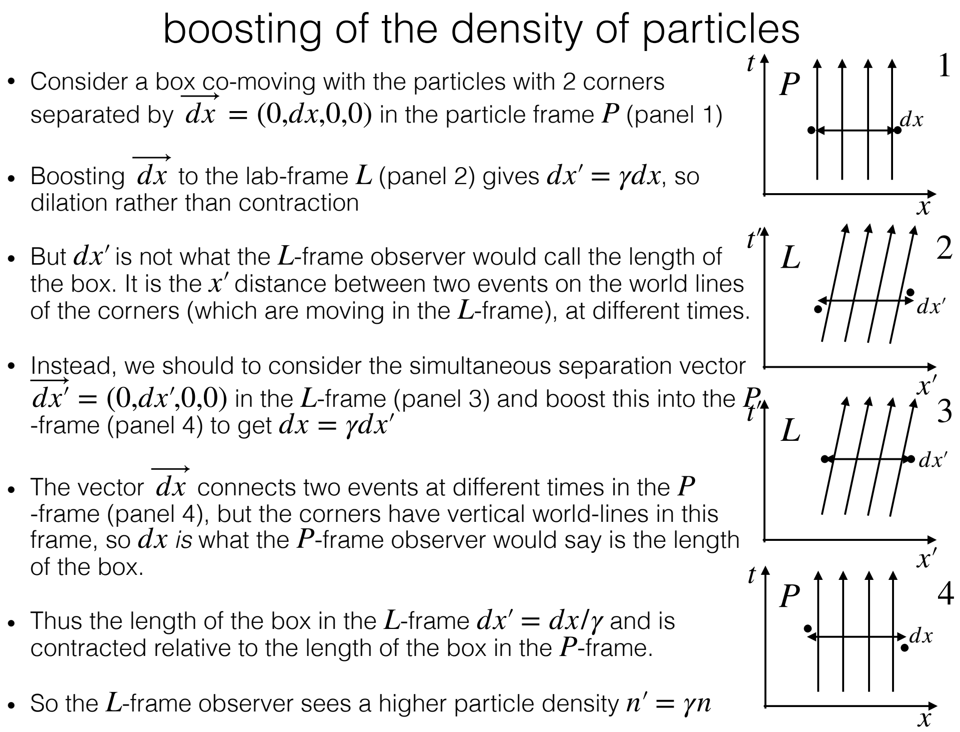


Figure 12: Illustration of transformation of volumes under boosts. The barn-and-pole paradox is at play here. The top panels shows how one might mistakenly conclude that the density of particles seen from a moving frame would be decreased, rather than enhanced. To get the right answer we need to consider the transformation of a pair of events that are simultaneous in the lab-frame L.

6.2 Transformation of momentum space volume

- Consider now a set of particles (as illustrated in figure 13) with a certain space-density in the frame O that have a very small range of velocities so they have (relativistic) 3-momenta $\mathbf{p} = (dp_x, dp_y, dp_z)$ that lie in an infinitesimal cubical volume of 3-momentum space volume Δp^3 centred on a 'fiducial' particle with $\mathbf{p} = \mathbf{p}_0 = (0, 0, 0)$
- Recalling the relativistic hyperbolic energy momentum surface $E^2 = p^2 + m^2$ it is evident that these have (to first order in $|\mathbf{p}|$, which is an infinitesimal) $E = p^0 = m$
- An observer in the lab-frame O' moving in the (minus) x-direction will find the fiducial particle to have 4-momentum $p_0^{\alpha'} = (\gamma m, \gamma mv, 0, 0)$
  - and will see the general particle to have 4-momentum $p^{\alpha'} = (\gamma(m + vdp_x), \gamma(mv + dp_x), dp_y, dp_z)$
  - taking the difference we have $d\mathbf{p}' = \mathbf{p}' - \mathbf{p}'_0 = (\gamma dp_x, dp_y, dp_z)$ so they inhabit a volume
  - $\boxed{\Delta p'^3 = \gamma \Delta p^3}$
  - and the density of particles per unit momentum-space volume is correspondingly *decreased* (by a factor $1/\gamma$) as compared to that in the rest-frame of the particles
- this is in contrast to the *space-density*
  - where, as we saw, a moving observer sees the particles to have a density *enhanced* by a factor γ
  - $\boxed{n' = \gamma n}$

6.3 Phase-space density and phase-space volume invariance

- A useful description for particulate matter is in terms of the *phase-space density* defined by
  - $\boxed{d^6 N = f(\mathbf{r}, \mathbf{p}) d^3 r d^3 p}$

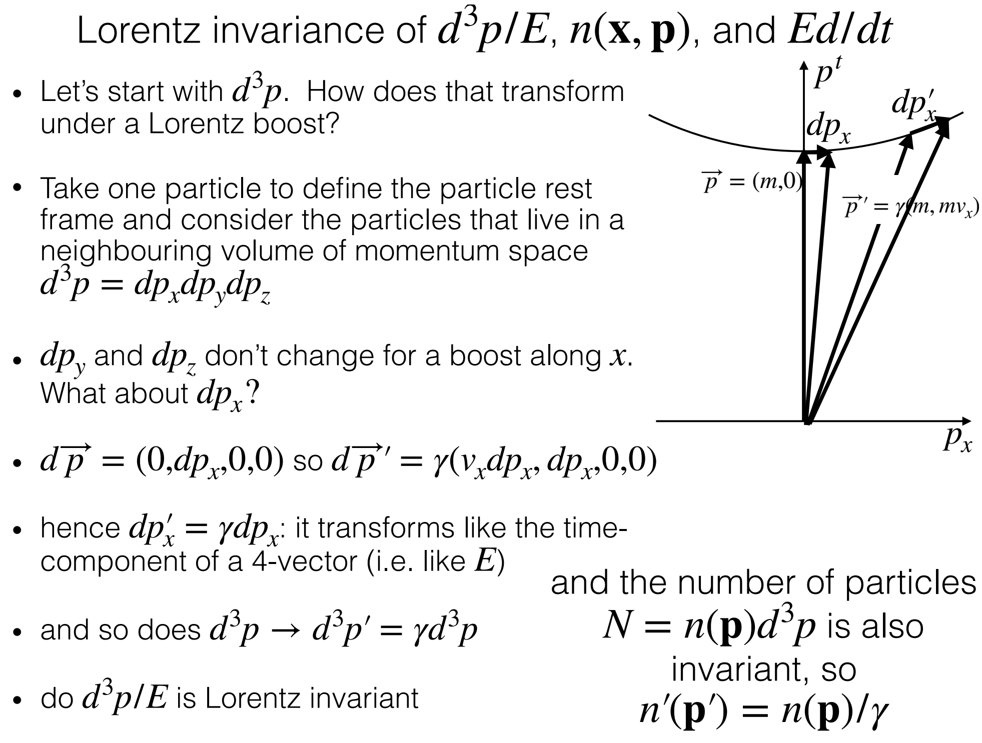


Figure 13: Transformation of 3-momentum volume elements. We consider here particles that have a range of 3-momenta d^3p centred on the origin $\mathbf{p} = 0$ in the frame of the particles. The difference of momentum between such particles is evidently purely space-like. In the lab-frame these particles lie on the hyperboloidal 'mass-shell' and occupy a 3-momentum volume that is larger by a factor γ: i.e. $d^3p' = \gamma d^3p$. That implies that the *density* in 3-momentum-space is decreased.

  - where d^3r is a spatial volume element, d^3p is a (relativistic) momentum-space volume element and d^6N the number of particles in the 6D phase-space volume element d^3rd^3p

- we can define these quantities – since they refer to particles with only an infinitesimal range of momenta – in the rest frame of these d^6N particles
- the results of the two above sub-sections furnish the remarkable result
  - $\boxed{d^3r'd^3p' = d^3rd^3p}$
  - the phase-space volume is Lorentz invariant!
- additionally, since d^6N is a number of particles – albeit an infinitesimal one – and therefore also something everyone has to agree on regardless of their reference frame we find
  - $f(\mathbf{r}, \mathbf{p})$ is Lorentz invariant also!
- relativists love to work exclusively with 4-vectors (and, as we shall see, 4-tensors) but, as we have seen, there is great utility to work in the "3+1" formalism used here also with the phase-space density in three and six dimensions (as here)
  - what is often done is to cast the equations into a form where something like d^3p never appears by itself
  - the collisional Boltzmann equation that is used extensively in cosmology and particle physics is usually written, for example, as integrals over d^3p/E (times other quantities that are also Lorentz invariant)
  - the d^3p/E notation is used so the reader can see immediately that everything is legal and Lorentz-frame invariant
  - though numerical calculations are usually performed in a specific reference frame

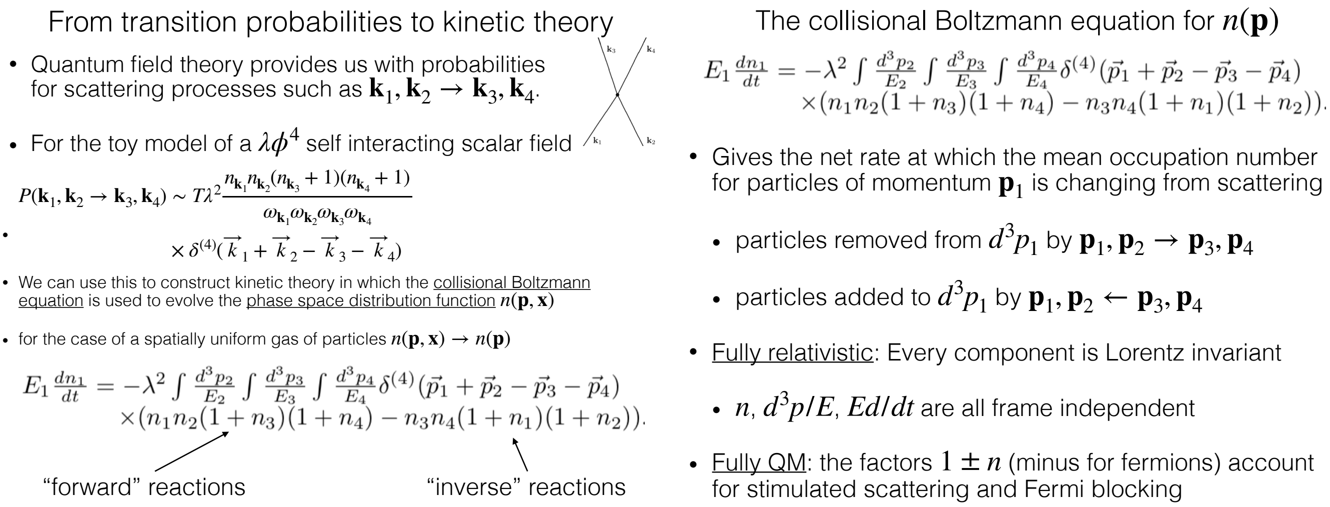


Figure 14: Illustration of how we obtain the collisional Boltzmann equation (as used by cosmologists for e.g. big-bang nucleosynthesis calculations. Quantum field theory gives probabilities for scattering transitions (here for a 'toy' model of a self-interacting scalar field). More generally this gives an invariant differential scattering cross-section. This always includes an energy and momentum conserving 4-dimensional Dirac δ-function. Stimulated emission and Fermi blocking factors also emerge very simply from field theory. This results in a deterministic equation for evolving the phase-space density (here for the case that the space-density is uniform). This is constructed and written in a way that is manifestly Lorentz-invariant using, for instance, the fact that d^3p and E both transform like the time-component of a 4-vector.

7 Continuity of particle number, energy and momentum

7.1 Introductory remarks

- The goal of this section is to familiarise ourselves with the '*stress-energy*' 4-tensor $T_{\mu\nu}$ for particulate matter
- it is a generalisation and extension of the the 3-dimensional *stress-tensor* T_{ij} from elastics and fluid mechanics that describes the transport of 3-momentum
  - the definition of T_{ij} is the flux density of the ith component of momentum in the jth direction
- the stress-energy tensor plays a critical role in GR as it is how matter appears in the Einstein field equations
  - it is through the stress-energy tensor that *matter tells space-time how to curve*.
- A critical feature of $T_{\mu\nu}$ is that it has vanishing *4-divergence*
  - this vanishing of 4 quantities expresses the conservation – or continuity – of energy and the 3-components of momentum
  - these laws follow directly from the invariance of space-time under displacement in time and in the 3-components of space
- Many GR texts consider primarily the case of the stress-energy for an *ideal fluid*
  - this is a historical throwback to the early days of GR where the focus was on relativistic stars
  - but most applications of GR are to matter that is not very well described as a fluid; examples are the dark matter, neutrinos and stars in stellar systems – which are all described by phase-space density (or 'distribution functions') $f(\mathbf{r}, \mathbf{p})$ – or the dark energy (and perhaps the dark matter if it is the axion or an axion-like field) which, if it does indeed appear on the 'matter-side' of Einstein's equation

- fluids emerge as a special case, when scattering is highly efficient and establishes local thermal equilibrium and the macroscopic behaviour can be described in terms of a space-density and a velocity field $\mathbf{v}(t, \mathbf{r})$
- it therefore behooves us to broaden the discussion to include these possibilities

- We will start, as something of a warm-up exercise, with the derivation of the law of continuity of particle density $n^\alpha{}_{,\alpha} = 0$ where the particle flux (or current) 4-vector is n^α

- We will then move on to construction of $T_{\mu\nu}$ for a distribution of particles in phase-space and demonstrate its conservation laws; followed by the specialisation to a fluid; and then finally we consider $T_{\mu\nu}$ for a scalar field – arguably the simplest type of matter. The latter demonstrates many of the features for other fields such as the EM field and also has several extremely important applications in cosmology (early- and late-time inflation).

7.2 Particle number continuity equation

- conservation of particles in phase space is expressed in the *Vlasov equation:*
  - $\boxed{\partial f/\partial t + \boldsymbol{\nabla}^{(6)} \cdot (f\dot{\mathbf{x}}^{(6)}) = 0}$
    * where $\mathbf{x}^{(6)} \equiv (\mathbf{r}, \mathbf{p})$ denotes particle coordinates in phase-space
    * and $f(\mathbf{r}, \mathbf{p}, t)$ is the phase-space density
    * and $\boldsymbol{\nabla}^{(6)} \equiv (\boldsymbol{\nabla}_\mathbf{r}, \boldsymbol{\nabla}_\mathbf{p})$ is the 6D partial derivative,
    * and $\dot{\mathbf{x}}^{(6)}$ denotes the rate of change of $\mathbf{x}^{(6)}$ with respect to coordinate time (*not* proper time)
  - this comes from the fact that the rate of change of number of particles in a 6-dimensional phase-space volume element comes from the sum over all 6 directions of the difference between the fluxes of particles across the two opposite faces

- or, splitting the 6 terms in $\boldsymbol{\nabla}^{(6)} \cdot (f\dot{\mathbf{x}}^{(6)})$ into space and momentum parts
  - $\boxed{\partial f/\partial t + \boldsymbol{\nabla}_\mathbf{r} \cdot (f\dot{\mathbf{r}}) + \boldsymbol{\nabla}_\mathbf{p} \cdot (f\dot{\mathbf{p}}) = 0}$
  - we are using here the fact that at any point in 6D phase space there is a unique 6-velocity
    * this is rather like a *fluid* in 3D where at a macroscopic level there is a velocity *field* $\mathbf{v}(\mathbf{r})$
  - the reason for this is that the particles in question obey Hamilton's equations; a pair of 1st order differential equations giving $\dot{\mathbf{r}}$ and $\dot{\mathbf{p}}$ as functions of $\mathbf{r}$ and $\mathbf{p}$
    * so at any point $(\mathbf{r}, \mathbf{p})$ the 'phase-velocity' $(\dot{\mathbf{r}}, \dot{\mathbf{p}})$ is fully determined
  - a fluid in 3D, however, is quite different from what we are describing here where we have a *distribution* of momenta, and therefore velocities, at any point in 3-space
  - though, of course, at a microscopic level a gaseous or liquid fluid is actually composed of atoms or molecules which, at any point in space, there is a distribution of velocities, just as we are considering here

- this is useful, but also useful is the continuity equation for the density of particles in 3D

- to obtain this we simply integrate $\partial f/\partial t + \boldsymbol{\nabla}^{(6)} \cdot (f\dot{\mathbf{x}}^{(6)}) = 0$ over $\mathbf{p}$
  - though in doing so we will assume, for simplicity, that the particles all have the same proper mass m

- noting that the term involving $\boldsymbol{\nabla}_\mathbf{p}$ integrates to zero assuming $f \to 0$ as $\mathbf{p} \to \infty$ we have
  - $\boxed{\partial n/\partial t + \boldsymbol{\nabla} \cdot (n\overline{\mathbf{v}}) = 0}$
  - where we have dropped the subscript $\mathbf{r}$ on $\boldsymbol{\nabla}_\mathbf{r}$
  - and where we have defined
    * the *space number density* $\boxed{n \equiv \int d^3p\, f}$ and

    * the *mean velocity* $\boxed{\overline{\mathbf{v}} \equiv \int d^3p\, \dot{\mathbf{r}} f / \int d^3p\, f}$

  - and we have used the commutativity of integrating over $\mathbf{p}$ and taking time or space derivatives:
    * so $\int d^3p\, \partial_t f = \partial_t \int d^3p f$
    * and $\int d^3p\, \boldsymbol{\nabla} \cdot (f\dot{\mathbf{r}}) = \boldsymbol{\nabla} \cdot \int d^3p\, f\dot{\mathbf{r}}$
  - the above expression says that the time rate of change of the density n at fixed position $\partial n/\partial t$ is minus the 3-divergence of the particle 3-current $n\overline{\mathbf{v}}$

- equivalently, and more compactly, we can write this as the vanishing of a single *4-divergence*
  - $\boxed{n^\alpha{}_{,\alpha} = 0}$
  - this being simply shorthand for $\partial_t n^0 + \boldsymbol{\nabla} \cdot \mathbf{n} = 0$

- where the components of n^α are $(n^0, \mathbf{n}) = (n, n\overline{\mathbf{v}})$, and which we call the particle *4-current*
  - or, equivalently, $n^\alpha = \int d^3p\, (1, \mathbf{p}/E)\, f$, since $\mathbf{v} = \dot{\mathbf{r}} = \mathbf{p}/E$
  - or, equally, as
  - $\boxed{\vec{n} = \int \frac{d^3p}{E} f\vec{p}}$
  - where, as usual, the 4-momentum is $\vec{p} \longrightarrow (E, \mathbf{p})$

- this last expression makes it clear that $\vec{n}$ does indeed transform as a 4-vector since, as we have seen, both d^3p/E and $f(\mathbf{r}, \mathbf{p}, t)$ are Lorentz invariants

- note that as we have integrated over 3-momentum $\vec{n}$ is a 4-vector *field* $\vec{n}(\vec{x})$

- the vanishing 4-divergence $n^\alpha{}_{,\alpha} = 0$ is a compact and concise expression of particle conservation
  - Schutz derives this for a 'dust': i.e. a collection of particles where all the particles in a given region of space have the same velocity. I.e. the *velocity dispersion* vanishes – so we have a *pressureless fluid*
  - that is rather restrictive - the derivation here allows an arbitrary distribution of velocities at each point in space

- defining the *covariant derivative operator* $\vec{\nabla} \longrightarrow \partial/\partial x^\alpha$ we can write this continuity or conservation law as

- $\boxed{\vec{\nabla} \cdot \vec{n} = 0}$

- this is useful, but is really a warm-up exercise for the next section

7.3 The stress tensor and continuity of energy and momentum

- Let's now multiply the fundamental equation expressing conservation of particles by components of p^α and again integrate over all momenta
  - this is called 'taking the first moment' – what we did before was take the zeroth moment

- and for simplicity (we'll relax this presently) assume that there are no forces acting on the particles, so $\dot{\mathbf{p}} = 0$ and hence $\partial f/\partial t + \boldsymbol{\nabla} \cdot (f\dot{\mathbf{r}}) = 0$

- multiplying by the energy $p^0 = E$ and integrating gives
  - $\int d^3p\, E(\partial f/\partial t + \boldsymbol{\nabla} \cdot (f\dot{\mathbf{r}})) = 0$

- or
  - $\frac{\partial}{\partial t} \int d^3p\, Ef(\mathbf{r}, \mathbf{p}) + \boldsymbol{\nabla} \cdot \int d^3p\, E\mathbf{v} f(\mathbf{r}, \mathbf{p}) = 0$

- or
  - $\boxed{\partial(n\langle E\rangle)/\partial t + \nabla \cdot (n\langle E\mathbf{v}\rangle) = 0}$
- where $n = \int d^3p\, f$ as before and $\langle X\rangle \equiv \int d^3p\, fX / \int d^3p f$ defines the average of X over particles
- this says that the rate of change of energy density – the number density times the mean energy per particle – is minus the divergence of the *energy flux density* $n\langle E\mathbf{v}\rangle$
  - this is analogous to the charge current $n\langle q\mathbf{v}\rangle$, which is equal to the number density of the particles times the average of their charge times their velocity and gives the rate per unit area at which charge is being transported
  - here $n\langle E\mathbf{v}\rangle$ gives the rate of transport of energy per unit area
- or equivalently, it expresses conservation (or continuity) of energy
- similarly, multiplying by the 3-momentum $\mathbf{p}$ and integrating gives
  - $\int d^3p\, \mathbf{p}(\partial f/\partial t + \nabla \cdot (f\dot{\mathbf{r}})) = 0$
- or
  - $\frac{\partial}{\partial t} \int d^3p\, \mathbf{p} f(\mathbf{r}, \mathbf{p}) + \nabla \cdot \int d^3p\, \mathbf{p}\mathbf{v} f(\mathbf{r}, \mathbf{p}) = 0$
- or
  - $\boxed{\partial(n\langle \mathbf{p}\rangle)/\partial t + \nabla \cdot (n\langle \mathbf{p}\mathbf{v}\rangle) = 0}$
- this is the relativistic generalisation of Newton's $\mathbf{F} = m\mathbf{a}$
  - as it says that the rate of change of momentum (density) = force (density)
- since it says the rate of change of the space density of relativistic momentum $n\langle \mathbf{p}\rangle$ is minus the divergence of the momentum flux density $n\langle \mathbf{p}\mathbf{v}\rangle$
  - the latter being the rate at which momentum is being transported per unit area
  - this is a 3-tensor: $n\langle p_i v_j\rangle$ being the rate at which the $i^{\rm th}$ component of momentum is being transported in the $j^{\rm th}$ direction
  - and is also the *definition* of *pressure*
  - Q: if pressure is the flux of momentum, what direction is momentum flowing in in a pressurised balloon? In a compressed spring?
- so this is the expression of conservation of 3-momentum
- replacing $\mathbf{v} \to \mathbf{p}/E$ we can combine these four conservation laws into the vanishing of a 4-divergence
  - $\boxed{T^{\alpha\beta}{}_{,\beta} = 0}$
- where the *stress-energy 4-tensor* is
  - $\boxed{T^{\alpha\beta} \equiv \int \frac{d^3p}{E} p^\alpha p^\beta f(\mathbf{r}, \mathbf{p})}$
- which can also be expressed symbolically as
  - $\boxed{\mathbf{T} \equiv \int \frac{d^3p}{E} \vec{p}\vec{p} f(\mathbf{r}, \mathbf{p})}$
  - where $\vec{p}$ denotes the 4-vector with components $\{p^\alpha\}$
- this has profound significance in GR

  - there is a 4-tensor $G^{\alpha\beta}$ – which is of a purely geometrical nature and is formed from the curvature tensor – that obeys an identical continuity equation - Einstein's insight was to equate these as the prescription for how matter tells space-time how to curve

- it is restrictive to impose $\dot{\mathbf{p}} = 0$, and also to impose conservation of particles – since high energy collisions can create and annihilate particles and transmute particles from one type to another

- but the conservation law $T^{\alpha\beta}{}_{,\beta} = 0$ is of more general applicability

  - if we allow for collisions between the particles then we get a 'collision term', so
  - $\boxed{\partial f/\partial t + \boldsymbol{\nabla}^{(6)} \cdot (f\dot{\mathbf{x}}^{(6)}) = (\partial f/\partial t)_{\rm coll}}$
  - but if the collisions individually conserve 4-momentum then $\int d^3p\, p^\alpha (\partial f/\partial t)_{\rm coll} = 0$
    * think of 3-momentum space as divided into a fine grid of cubical cells with label $\mathbf{p}$
    * a collision where two particles with 3-momenta $\mathbf{p}_1$ and $\mathbf{p}_2$ scatter off one another and emerge with 4-momenta $\mathbf{p}_1'$ and $\mathbf{p}_2'$ depletes f in the cells $\mathbf{p}_1$ and $\mathbf{p}_2$ and enhances f in the cells $\mathbf{p}_1'$ and $\mathbf{p}_2'$
    * if the rate at which collisions of this kind is $R_{12\to1'2'}$ then this gives a contribution to the collision term $\delta(\partial f/\partial t)_{\rm coll}(\mathbf{p}) = R_{12\to1'2'}(\delta(\mathbf{p}-\mathbf{p}_1') + \delta(\mathbf{p}-\mathbf{p}_2') - \delta(\mathbf{p}-\mathbf{p}_1) - \delta(\mathbf{p}-\mathbf{p}_2))$
    * where $\delta(\mathbf{p}-\mathbf{p}')$ is zero unless $\mathbf{p}$ and $\mathbf{p}'$ refer to the same cell
    * multiplying $\delta(\partial f/\partial t)_{\rm coll}(\mathbf{p})$ by $\mathbf{p}$ and integrating over all $\mathbf{p}$ one finds that this has no effect since $\mathbf{p}_1' + \mathbf{p}_2' = \mathbf{p}_1 + \mathbf{p}_2$
    * and similarly for the energy
    * so collisions of this type have no effect on the conservation law $T^{\alpha\beta}{}_{,\beta} = 0$
    * the total collision term is, of course, the sum over all possible types of collisions
    * but the conclusion that $T^{\alpha\beta}{}_{,\beta} = 0$ is unchanged remains valid
  - this is not restricted to number conserving collisions – the same holds true for number violating reactions and for reactions involving different types of particles
    * in such case the number conservation law is violated, of course,
    * but energy and momentum conservation – being more fundamental – still hold
  - also, if we have forces such as electromagnetic forces acting on the particles then energy and momentum for the particles alone is naturally no longer conserved
    * but there is also a stress-energy tensor for the electromagnetic field
    * and the total stress-energy – particles *plus* fields – obeys the continuity equation
    * this guarantees that Newton's 3rd law (equal and opposite reaction) is obeyed

- we assumed above a 'phase-fluid' where at each point in phase-space there is a unique 6-velocity $(\dot{\mathbf{r}}, \dot{\mathbf{p}})$

- if one has particles with different charge-to-mass ratios moving under the influence of EM fields then each 'species' will have a unique velocity and the total stress-energy is the sum over the different 'species' of particles

- To summarise, we can write the stress-energy tensor as

$$T^{\alpha\beta} = \left[\begin{array}{c|c} T^{00} & T^{0i} \\ \hline T^{i0} & T^{ij} \end{array}\right] = \int \frac{d^3p}{E}\, f p^\alpha p^\beta = \int \frac{d^3p}{E}\, f \left[\begin{array}{c|c} E^2 & E\mathbf{p} \\ \hline E\mathbf{p} & \mathbf{pp} \end{array}\right] = \left[\begin{array}{c|c} n\langle E\rangle & n\langle \mathbf{v}E\rangle \\ \hline n\langle \mathbf{p}\rangle & n\langle \mathbf{vp}\rangle \end{array}\right] \tag{1}$$

  - in the left column we have the *densities*, of energy and momentum respectively,
  - and in the right column we have the energy and momentum *flux densities*

- The stress-energy tensor is sometimes called the energy-momentum tensor

  - we normally name 4-quantities by the spatial part (e.g. we call $\vec{p} = (E, \mathbf{p})$ the *4-momentum*)
  - and following that convention we would call $T^{\alpha\beta}$ the *4-stress*
  - since the 3×3 spatial part T_{ij} in the lower-right segment is conventionally called the stress

7.4 Transformation of the stress tensor under a boost

- From the definition $T^{\alpha\beta} \equiv \int (d^3p/E) p^\alpha p^\beta f$ and from the invariance of both d^3p/E and f we have
- $T^{\alpha'\beta'} \equiv \int (d^3p/E) p^{\alpha'} p^{\beta'} f = \int (d^3p/E)(\Lambda^{\alpha'}{}_\alpha p^\alpha)(\Lambda^{\beta'}{}_\beta p^\beta) f = \Lambda^{\alpha'}{}_\alpha \Lambda^{\beta'}{}_\beta T^{\alpha\beta}$
- so we simply apply (matrix multiply by) a Lorentz matrix for each index
- the same is true for any other tensor
  - Q: how does the metric tensor $\eta^{\alpha\beta}$ transform under Lorentz boosts?

8 Perfect fluids

8.1 Stress-energy for a perfect fluid

- if we boost from the lab-frame into the frame where the *momentum density* $n\langle \mathbf{p} \rangle$ (and therefore also the energy flux density $n\langle \mathbf{v}E \rangle$) vanish – the so-called 'co-moving' frame
- and if collisions between the particles are sufficient to render the 3D pressure tensor $T^{ij} = n\langle \mathbf{vp} \rangle$ isotropic: so $T^{ij} = P\delta^{ij}$
- then the 4D stress tensor is $T^{\alpha\beta} = \mathrm{diag}\{\rho, P, P, P\}$
  - where $\rho \equiv n\langle E \rangle$ is the energy density
- we can also write this in a frame independent form as $T^{\alpha\beta} = (\rho + P)U^\alpha U^\beta + P\eta^{\alpha\beta}$
  - where U^α is the 4-velocity of an observer co-moving with the fluid element
  - i.e. the observer who measures zero momentum density locally
- where ρ and P are the energy density and pressure as measured by such an observer – i.e. the 'proper' energy density and pressure
  - note that the *energy* density ρ is *not* the (proper) mass density; since it contains the kinetic energy in addition to rest-mass
- in passing from $T^{\alpha\beta} = \mathrm{diag}\{\rho, P, P, P\}$ to $T^{\alpha\beta} = (\rho+P)U^\alpha U^\beta + P\eta^{\alpha\beta}$ we have used a line of argument that we will use a lot more
  - we first derive an equality in some specific reference frame
  - we express that in terms of 4-vectors and 4-tensors
  - and can then be confident that this expresses a frame-invariant relationship

8.2 $T^{0\alpha}{}_{,\alpha} = 0$ is the first law of thermodynamics

- Consider a fluid with some flow velocity field $\mathbf{v}$
- boost into the frame of the fluid at some point in space
  - at that point, $U^\alpha = (1, 0, 0, 0)$
  - i.e. the fluid three velocity at that point is $\mathbf{v} = 0$
- in the neighbourhood of that point, $\mathbf{v}$ will vary, to lowest order, linearly with distance
  - i.e. $v_i = \theta_{ij} r_i$
  - where θ_{ij} is the expansion rate 3-tensor (it has units of 1 / time)
- it is not difficult to show that a comoving volume element here has $\dot{V}/V = \theta_{ii}$
  - θ_{ii} being the 3-divergence of the velocity field

- while, to linear order in $\mathbf{v}$, $T^{0\alpha} = (\rho, (\rho + P)\mathbf{v})$
  - you can find this either by boosting $T^{\alpha\beta} = \text{diag}\{\rho, P, P, P\}$
  - or, more easily, from the formula $T^{\alpha\beta} = (\rho + P)U^\alpha U^\beta + P\eta^{\alpha\beta}$ with $U^\alpha = \gamma(1, \mathbf{v})$
- equating the 4-divergence $T^{0\alpha}{}_{,\alpha}$ to zero gives
  - $\dot{\rho} = -(\rho + P)\theta_{ii}$ plus terms that are grow linearly with distance and which can be ignored (i.e. this formula is a good approximation for finite volumes, provided they are small in linear size compared to the length-scale over with $\rho + P$ varies)
  - or $\dot{\rho} = -(\rho + P/c^2)\dot{V}/V$
    - where we have put in the factor c^2 needed since P has units of density times velocity squared
  - for $P = 0$ this simply says that $\rho V =$ constant – i.e. mass-energy is conserved
  - for $P \neq 0$ there is an extra decrease in density
  - or equivalently an additional decrease in the mass $\Delta M = -P\Delta V/c^2$
  - which is therefore equivalent to a decrease in energy $E = mc^2$ of $\Delta E = -P\Delta V$
    - just as one expects from the 1st law of thermodynamics (plus mass-energy equivalence from SR)
      - $dE = -PdV$ being easily understood for an expanding volume element through the loss of energy of particles bouncing off the walls
    - it is interesting how this comes about in terms of the stress-energy tensor where we see from $T^{0\alpha} = (\rho, (\rho + P)\mathbf{v})$ that a moving element of fluid has an energy flux density with a contribution from the pressure P in addition to the mass-density times velocity

9 Relativistic scalar fields

Another extremely important type of matter in cosmology is the relativistic scalar field. Such fields are invoked to drive inflation in the early universe – the 'inflaton' field – and late time inflation – where the dark energy may be the 'quintessence' field – and the axion field or ultra-light axion-like fields are candidates for the dark matter.

All of the above is hypothetical. The only relativistic scalar field known to exist is the Higgs field. But as Zel'dovich comments in his monograph *'My Universe'*, "once the genie was out of the bottle there was no putting it back".

Below we first review some aspects of elementary mechanics that we will need; we next consider the dynamics of a simple solid-state lattice made of beads, rods and springs and which turns out to be mathematically isomorphic to the relativistic scalar field and derive the continuity equations for energy and wave-momentum for such a lattice. We then make the transition to the relativistic scalar field.

9.1 Review of some elements of mechanics

9.1.1 The Lagrangian and the Euler-Lagrange equations:

- The Lagrangian for a simple mechanical system with a single degree of freedom – say a mass on a spring, with displacement q – is $L(q, \dot{q}, t)$
- it is the kinetic energy minus the potential energy ($L = K - V$) and from it, by asserting that the variation of the *action* $\delta S \equiv \delta \int dtL = 0$ we get the equation of motion (Euler-Lagrange equation) $d(\partial L/\partial \dot{q})/dt = \partial L/\partial q$.
  - draw a path $q(t)$ on the $t - q$ plane and a neighbouring path $q(t) + \delta q(t)$
  - write δS as an integral wrt time of $\delta L = (\partial L/\partial q)\delta q + (\partial L/\partial \dot{q})\delta \dot{q}$
  - integrate the second term in δS by parts
  - to obtain $\delta S = \int dt\, \delta q[(\partial L/\partial q) - d(\partial L/\partial \dot{q})/dt]$
  - so at the extremum of S the quantity $[\ldots]$ must vanish, giving the EL equation

– $\boxed{d(\partial L/\partial \dot{q})/dt = \partial L/\partial q}$

- with $K = m\dot{q}^2/2$, so $\partial L/\partial \dot{q} = m\dot{q}$, we have $m\ddot{q} = -dV/dq$
  - or Newton's law $F = -dV/dq = ma$
- this is readily generalisable to multiple degrees of freedom q_i

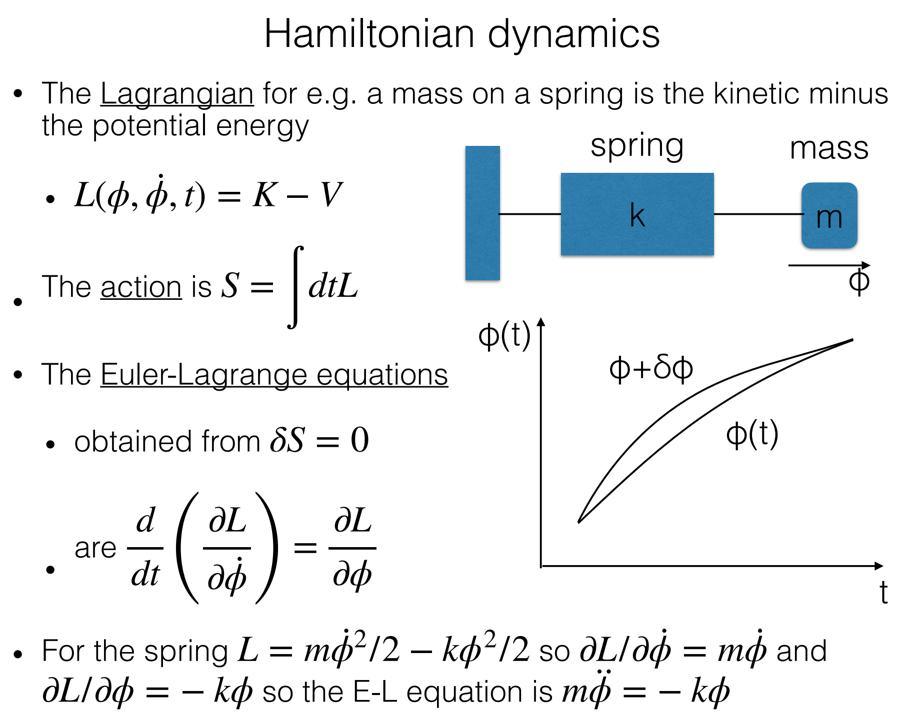


Figure 15: Hamiltonian dynamics for a system with 1 degree of freedom (dof).

9.1.2 Energy conservation:

- for any particular solution of the E-L equations, the Lagrangian can be considered a function of time alone: $L(t) = L(q(t), \dot{q}(t), t)$, whose (total) time derivative is
- $\frac{dL}{dt} = \frac{\partial L}{\partial q}\frac{dq}{dt} + \frac{\partial L}{\partial \dot{q}}\frac{d\dot{q}}{dt} + \frac{\partial L}{\partial t}$
- using the equation of motion to replace $\partial L/\partial q \rightarrow d(\partial L/\partial \dot{q})/dt$ and $dq/dt \rightarrow \dot{q}$ we see that the first two terms on the RHS are the total derivative $d/dt(\dot{q}\partial L/\partial \dot{q})$ so, combining this with the total derivative on the LHS,
- $\boxed{\frac{d}{dt}\left(\dot{q}\frac{\partial L}{\partial \dot{q}} - L\right) = -\frac{\partial L}{\partial t}}$
- so if L does not depend explicitly on time (i.e. $L = L(q, \dot{q})$) the RHS vanishes and the quantity
- $\boxed{H \equiv \dot{q}\partial L/\partial \dot{q} - L}$
- called *Hamiltonian*, does not depend on time
- this is an example of how a *symmetry*
  - here the Lagrangian for the system being independent of time
  - implies a *conservation law* – in this case constancy of the energy H
  - which for $L = K - V$ gives $H = K + V = m\dot{q}^2/2 + V$

Euler-Lagrange equations

- The action is $S[\phi(t)] = \int dt\, L(\phi, \dot\phi, t)$
- so for 2 paths $\phi(t)$ and $\phi(t) + \delta\phi(t)$
- $\delta S = \int dt \left[\frac{\partial L}{\partial \phi}\delta\phi + \frac{\partial L}{\partial \dot\phi}\delta\dot\phi\right]$
- integrating the second term by parts gives
- $\int dt \frac{\partial L}{\partial \dot\phi}\delta\dot\phi = \left[\frac{\partial L}{\partial \dot\phi}\delta\phi\right] - \int dt\, \delta\phi\, \frac{d}{dt}\left(\frac{\partial L}{\partial \dot\phi}\right)$
- but the boundary term [….] vanishes as the two paths start and end at the same points, hence
- $\delta S = \int dt\, \delta\phi \left[\frac{\partial L}{\partial \phi} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot\phi}\right)\right] = 0$
- so, since $\delta\phi$ is arbitrary, this implies

momentum

force

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot\phi}\right) = \frac{\partial L}{\partial \phi}$$

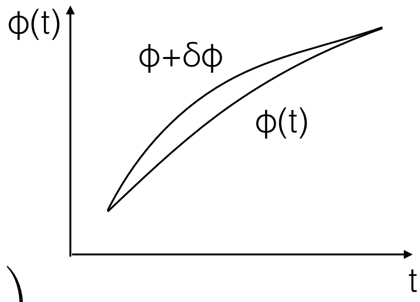


Figure 16: Euler-Lagrange equations for 1 dof in more detail.

9.1.3 Hamilton's equations:

- we define $p \equiv \partial L/\partial \dot q$
- and consider the total differential of the energy $H = \dot q p - L$ (considered as a function of time for a particular solution)
  - the differential of the first term is $\dot q dp + p d\dot q$
  - while that of the second is $-(\partial L/\partial q)dq - (\partial L/\partial \dot q)d\dot q - (\partial L/\partial \dot t)dt$
  - but the second term is $-pd\dot q$, so the terms involving $d\dot q$ cancel and we have
  - $\boxed{dH = \dot q dp - (\partial L/\partial q)dq - (\partial L/\partial \dot t)dt}$
- so the *Hamiltonian* H can be considered a function of p, q and t, with
  - $dH = (\partial H/\partial p)dp + (\partial H/\partial q)dq + (\partial H/\partial t)dt$
  - implying
    - $\boxed{\dot p = -\partial H/\partial q}$
    - $\boxed{\dot q = \partial H/\partial p}$
  - which are *Hamilton's equations:* two coupled 1st order equations that can be used in place of the single 2nd order Euler-Lagrange equation
  - and that $\partial H/\partial t = -\partial L/\partial t$, which we have seen is also the total time derivative of the energy (i.e. of H) considered as a function of time, so
    - $\boxed{dH/dt = \partial H/\partial t}$
- this is also generalisable to a system with N degrees of freedom q_i

The Hamiltonian H and energy conservation

- For a particular solution $\phi(t)$ we can think of $L(\phi, \dot{\phi}, t)$ as being just a function of time $L(t) = L(\phi(t), \dot{\phi}(t), t)$
- If we differentiate this with respect to time we get
- $\dfrac{dL}{dt} = \dfrac{\partial L}{\partial \phi}\dfrac{d\phi}{dt} + \dfrac{\partial L}{\partial \dot{\phi}}\dfrac{d\dot{\phi}}{dt} + \dfrac{\partial L}{\partial t}$
- but since the EL equation is $\partial L/\partial \phi = d(\partial L/\partial \dot{\phi})/dt$, and $d\phi/dt = \dot{\phi}$, the first two terms on the right are the total time derivative $d(\dot{\phi}p)/dt$ where we have *defined* $p \equiv \partial L/\partial \dot{\phi}$
- so defining $H \equiv \dot{\phi}p - L$ we have $dH/dt = -\,\partial L/\partial t$ and so, if the Lagrangian has no explicit time dependence, H is constant
- for the mass on a spring $H = m\dot{\phi}^2/2 + k\phi^2/2$, which is the kinetic plus potential energy, and is, unsurprisingly, constant
- but if m or k were time dependent we'd have $dH/dt \neq 0$

Figure 17: The Hamiltonian and energy conservation.

Hamilton's equations

- We won't be needing these much, but if we take the differential of $H = \dot{\phi}p - L$ we get
- $dH = \dot{\phi}dp + pd\dot{\phi} - \dfrac{\partial L}{\partial \phi}d\phi - \dfrac{\partial L}{\partial \dot{\phi}}d\dot{\phi} - \dfrac{\partial L}{\partial t}dt$
- but as $p \equiv \partial L/\partial \dot{\phi}$ the 2nd and 4th terms cancel and, with $\partial L/\partial \phi = dp/dt = \dot{p}$ in the 3rd, we have
- $dH = \dot{\phi}dp - \dot{p}d\phi - \dfrac{\partial L}{\partial t}dt$
- so $H = H(\phi, p, t)$ and the coefficients of dp and $d\phi$ give us Hamilton's equations
- $\dot{\phi} = \partial H/\partial p \quad \text{and} \quad \dot{p} = -\,\partial H/\partial \phi$

Figure 18: Hamiltons equations.

9.2 The 'scalar elasticity' model for a scalar field:

9.2.1 The discrete lattice model:

- Lagrangian analysis of a classical scalar field $\phi(\mathbf{x})$ starts with a simple discrete 'lattice' model
- we consider a 1D 'lattice' or array of particles tethered to a base by springs and with additional springs coupling neighbouring particles as indicated in the figure
  - the kinetic energy T is the sum of the kinetic energies of the particles $m(\dot{\phi}_i)^2/2$
  - and the potential energy V is the sum of squares of the differences of the displacements between neighbouring particles $K'(\phi_{i+1} - \phi_i)^2/2$
  - plus the potential energy of the particles in their confining potential well $= K\phi_i^2/2$
- this is a straightforward generalisation of the theory of a single particle with displacement q to a system of N particles with displacements $q \to \phi_i$
  - we now have N Euler-Lagrange equations
  - following the same steps as above
    * i.e. writing out dL/dt as partial derivatives with respect to ϕ_i, $\dot{\phi}_i$ and t, and invoking the EL equations,
- we find that invariance of L with respect to time – which we will assume henceforth – implies conservation of the energy $\sum_i \dot{\phi}_i(\partial L/\partial \dot{\phi}_i) - L$

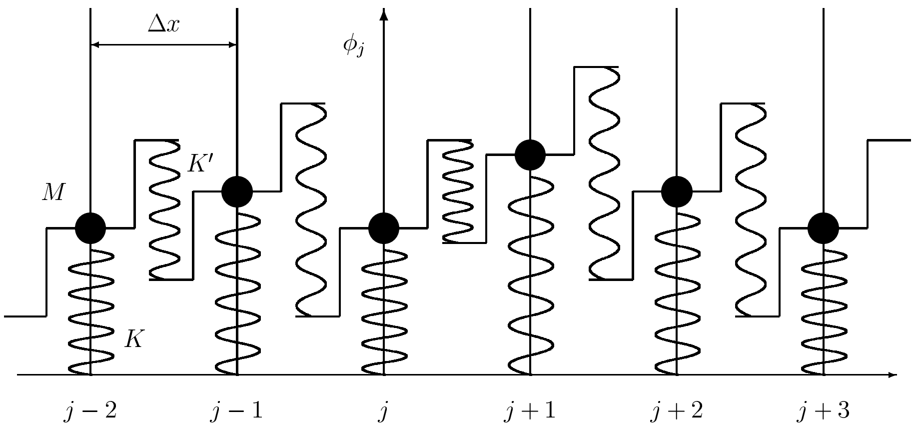


Figure 16.1: A lattice of coupled oscillators. Massive beads are constrained to move in the vertical direction by rods, and tethered to the base by springs with spring constant K. Neighboring particles are also coupled by springs of spring constant K'. The displacement of the jth bead is ϕ_j.

Figure 19: A 1D lattice model for a scalar field.

9.2.2 The continuum limit:

- taking the limit as the spacing $\to 0$
  - or, equivalently, considering disturbances that vary slowly compared to the lattice separation
  - and replacing the index i by a continuous variable x
- we have a *continuous* system with a displacement *field* $\phi(x,t)$
- described by a Lagrangian *density* $\mathcal{L}(\dot{\phi}, \phi', \phi) = (A\dot{\phi}^2 - B\phi'^2 - C\phi^2)/2$
  - where $\dot{\phi}$ and ϕ' are partial derivatives wrt time and space respectively
  - where A, B and C are positive constants,
  - with A determined by the masses of the particles (and their spacing),

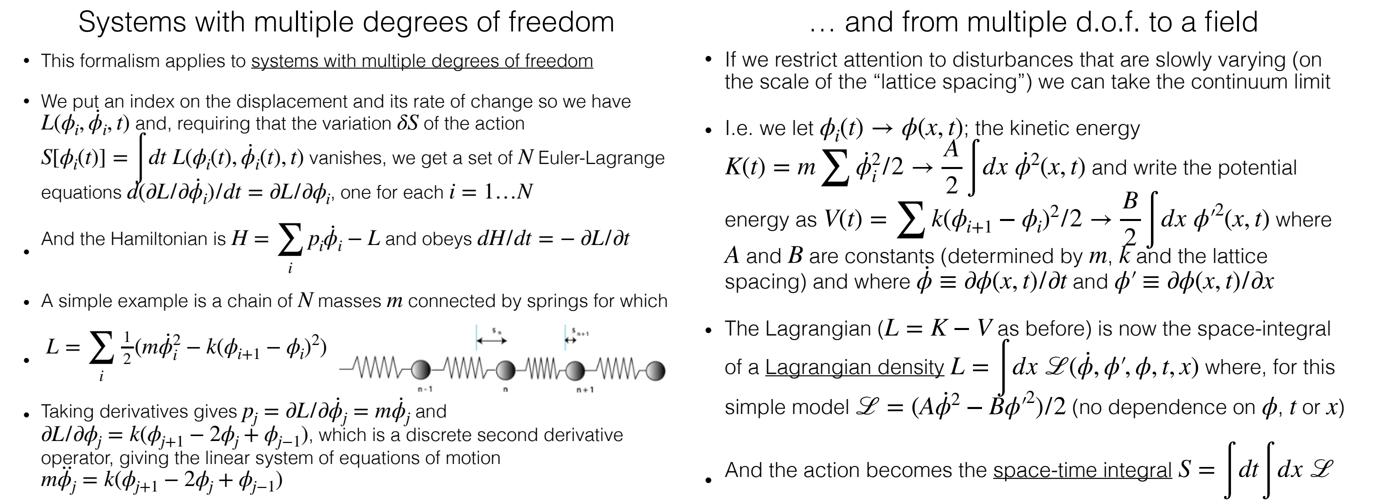


Figure 20: Hamiltonian systems with multiple degrees of freedom and transition to a continuous field.

  - B being determined by the spring constant of the connecting springs
  - and C depending on the strength of the base springs
  - so the first term is the kinetic energy density $\mathcal{K} = A\dot\phi^2/2$
    * the total kinetic energy being $K = \int dx\, \mathcal{K}$
  - and the other terms are the minus potential energy density $\mathcal{V} = (B\phi'^2 + C\phi^2)/2$
    * the potential energy being $V = \int dx\, \mathcal{V}$

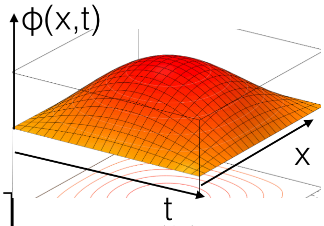
- the action is now the 1+1 space-time integral $S = \int dtdx\mathcal{L}$
- to get the equations of motion implied by $\delta S = 0$ we imagine the 2-D *surface* $\phi(x, t)$ lying above the $x - t$ plane
- and a vertically displaced surface $\phi(x, t) + \delta\phi(x, t)$
- to obtain $\delta S = \int dtdx[\delta\dot\phi\partial\mathcal{L}/\partial\dot\phi + \delta\phi'\partial\mathcal{L}/\partial\phi' + \delta\phi\partial\mathcal{L}/\partial\phi]$
- integrating the first two terms by parts gives
- $\delta S = -\int dtdx\delta\phi[\partial_t(\partial\mathcal{L}/\partial\dot\phi) + \partial_x(\partial\mathcal{L}/\partial\phi_{,x}) - \partial\mathcal{L}/\partial\phi]$
- requiring this vanish for arbitrary displacement $\delta\phi$ gives the Euler-Lagrange equations:
  - $\partial_t(\partial\mathcal{L}/\partial\dot\phi) + \partial_x(\partial\mathcal{L}/\partial\phi') - \partial\mathcal{L}/\partial\phi = 0$
  - or, for $\mathcal{L} = (A\dot\phi^2 - B\phi'^2 - C\phi^2)/2$, the equation of motion is $A\ddot\phi - B\phi'' + C\phi = 0$
- this is a linear, but dispersive, wave equation which allows travelling wave solutions like $\phi(x, t) = \phi_0\cos(\omega t - kx)$ with *dispersion relation* $A\omega^2 = Bk^2 + C$
  - this is very similar to the relativistic energy-momentum relation $E^2 = p^2 + m^2$ with $E \to \hbar\omega$ and $p \to \hbar k$
  - it is also very similar to the dispersion relation for waves in a cold plasma: $\omega^2 = c^2k^2 + \omega_p^2$

9.2.3 Time translational invariance:

- above we derived the conservation of energy for the system $L(\dot q, q)$ by considering the *total time derivative* dL/dt
  - considering L to be, for a particular solution $q = q(t)$, $\dot q = \dot q(t)$, a function of time t

The Euler-Lagrange equations for $\phi(x,t)$

- To obtain the field equations we now vary the surface $\phi(x,t)$
  $\phi(x,t) \to \phi(x,t) + \delta\phi(x,t)$ and demand that the variation in S vanish
  - in doing this we set $\delta\phi = 0$ on the time boundaries and either assume periodic boundary conditions on x or that the field tend to zero as $x \to \pm\infty$. This gives



- $$0 = \delta S = \int dt \int dx \left[\frac{\partial \mathscr{L}}{\partial \dot\phi}\delta\dot\phi + \frac{\partial \mathscr{L}}{\partial \phi'}\delta\phi' + \frac{\partial \mathscr{L}}{\partial \phi}\delta\phi \right]$$
- Integrating the first two terms by parts gives
  $0 = \delta S = \int dt \int dx\, \delta\phi \left[-\frac{d}{dt}\frac{\partial \mathscr{L}}{\partial \dot\phi} - \frac{d}{dx}\frac{\partial \mathscr{L}}{\partial \phi'} + \frac{\partial \mathscr{L}}{\partial \phi} \right]$ where now d/dt and d/dx denote derivatives at constant x and t respectively (i.e. they are really partial derivatives - but not the same as e.g. $\partial\mathscr{L}(\dot\phi, \phi', \phi, x, t)/\partial t$ which denotes the time derivative holding $\dot\phi, \phi', \phi$ and x fixed)
- Requiring $\delta S = 0$ for arbitrary $\delta\phi$ gives $\frac{d}{dt}\frac{\partial \mathscr{L}}{\partial \dot\phi} + \frac{d}{dx}\frac{\partial \mathscr{L}}{\partial \phi'} = \frac{\partial \mathscr{L}}{\partial \phi}$ which, for the model $\mathscr{L} = (A\dot\phi^2 - B\phi'^2)/2$, is the wave equation $A\ddot\phi - B\phi'' = 0$

Figure 21: The Euler-Lagrange equations for a continuous field.

- here we do something very similar: we consider the Lagrangian density $\mathcal{L}(\dot\phi, \phi', \phi)$, for a particular solution $\phi = \phi(x,t)$, with associated $\dot\phi(x,t)$ and $\phi'(x,t)$, to be a function $\mathcal{L}(t,x)$ of time and space
- considering the *partial* derivative of this function with respect to t holding x fixed:
  - $\partial_t\mathcal{L} = (\partial\mathcal{L}/\partial\dot\phi)\partial_t\dot\phi + (\partial\mathcal{L}/\partial\phi')\partial_t\phi' + (\partial\mathcal{L}/\partial\phi)\partial_t\phi$
- and applying the equation of motion to replace $\partial\mathcal{L}/\partial\phi$ by $\partial_t(\partial\mathcal{L}/\partial\dot\phi) + \partial_x(\partial\mathcal{L}/\partial\phi')$
- we see (noting that $\partial_t\phi' = \partial_x\dot\phi$) that the RHS is ∂_t of $\dot\phi\partial\mathcal{L}/\partial\dot\phi$ plus ∂_x of $\dot\phi\partial\mathcal{L}/\partial\phi'$
- so, combining the two time derivative terms, we have a continuity equation $\partial_t\mathcal{E} + \partial_x\mathcal{F} = 0$
  - where $\mathcal{E} \equiv \dot\phi\partial\mathcal{L}/\partial\dot\phi - \mathcal{L}$
  - and $\mathcal{F} \equiv \dot\phi\partial\mathcal{L}/\partial\phi'$
- but with $\mathcal{L} = \mathcal{K} - \mathcal{V} = (A\dot\phi^2 - B\phi'^2 - C\phi^2)/2$ we have $\mathcal{E} = \mathcal{K} + \mathcal{V} = (A\dot\phi^2 + B\phi'^2 + C\phi^2)/2$ which is evidently the *energy density*
- so $\partial_t\mathcal{E} + \partial_x\mathcal{F} = 0$ expresses continuity of energy
  - with $\mathcal{F} = \dot\phi\partial\mathcal{L}/\partial\phi'$, which is equal to $-B\dot\phi\phi'$, being the *energy flux density*
  - the rate of change of $\mathcal{E}$ being minus the spatial 1-divergence of $\mathcal{F}$
- If we integrate this over position x the term involving $\mathcal{F}$ becomes a surface term, which we can assume to be zero if the fluctuations of the field are within some limited domain and we have the law of *conservation of energy:* $d(E \equiv \int dx\mathcal{E})/dt = 0$.
  - this conservation law being a consequence of the symmetry that the Lagrangian density does not depend explicitly on time t (i.e. the mass- and spring-coefficients A, B and C are constant)
- regarding the energy flux density, note that for a wave packet with ϕ being $\cos(kx - \omega t)$ times some extended 'envelope-function' $\mathcal{F}$ has the same sign as the wave-vector k, which makes sense since the group velocity is k/ω (see problem).

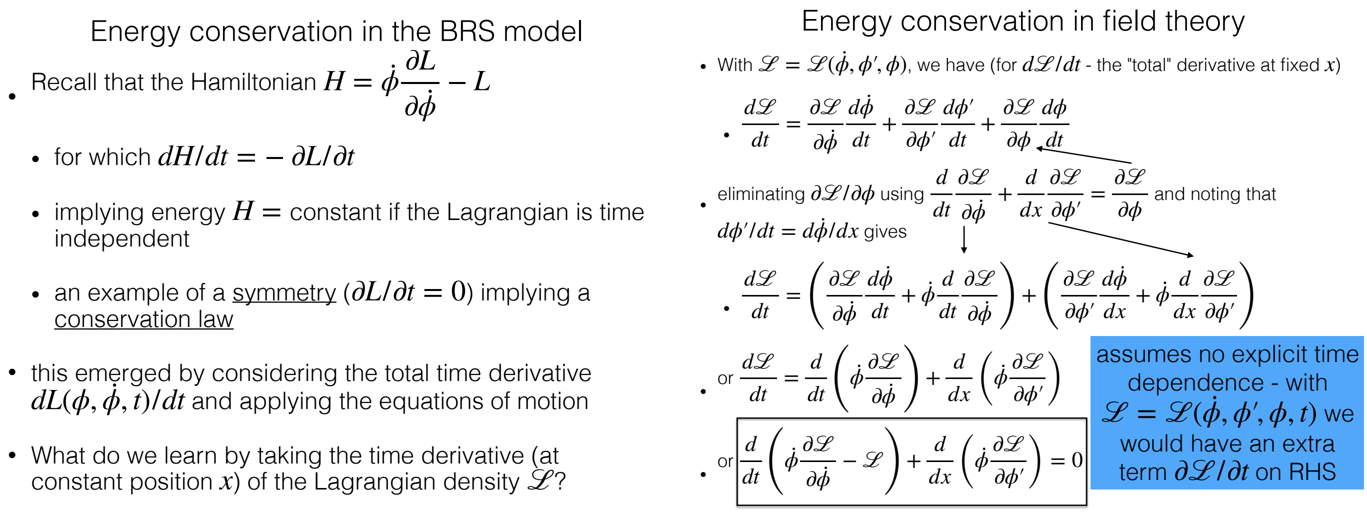


Figure 22: Energy conservation in field theory.

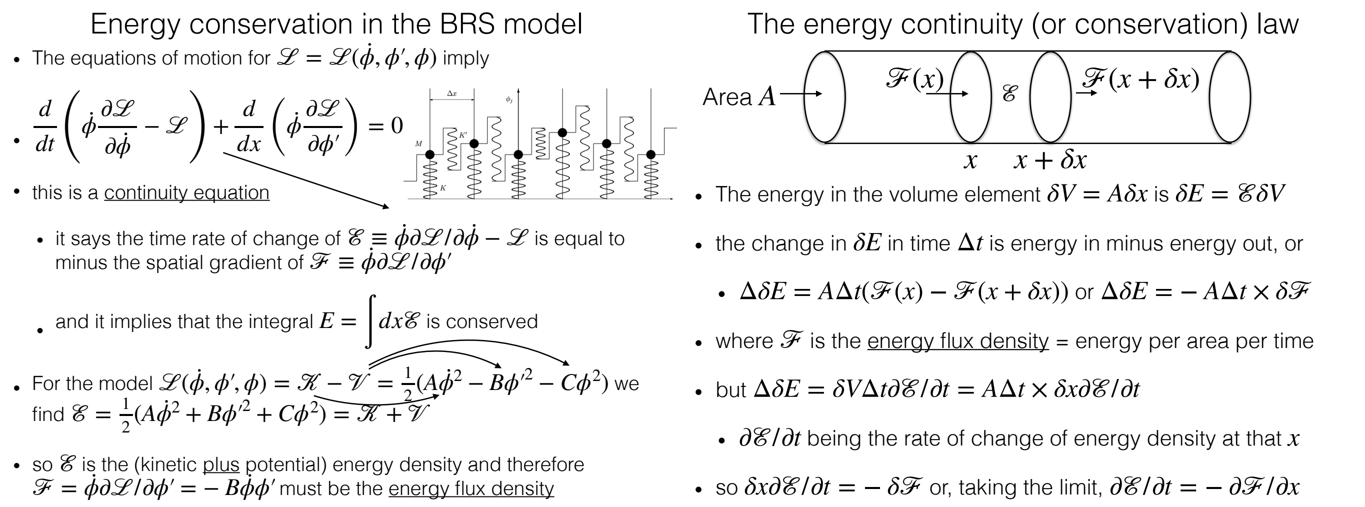


Figure 23: Energy conservation in field theory.

9.2.4 Spatial translational invariance:

- we have assumed that all of the masses, all of the potentials and all of the connecting springs are identical
  - so, in the continuum limit, A, B and C, and hence also $\mathcal{L}$, are independent of position x
- this invariance of $\mathcal{L}$ with respect to spatial displacements implies that there is another conserved quantity – let's call it the *wave-momentum*
  - and if we were to make a 3-D lattice we would have 3 conserved quantities
- it is trivial to obtain the continuity equation. We just need to switch $x \leftrightarrow t$ (and swap $\phi' \leftrightarrow \dot{\phi}$) in the previous section to obtain $\partial_t(-\phi'\partial\mathcal{L}/\partial\dot{\phi}) + \partial_x(\mathcal{L} - \phi'\partial\mathcal{L}/\partial\phi') = 0$
  - or $\partial_t\mathcal{M} + \partial_x\mathcal{P} = 0$
  - so the *wave-momentum density* is $\mathcal{M} = -\phi'\partial\mathcal{L}/\partial\dot{\phi}$
  - and the *wave-momentum flux density* is $\mathcal{P} = \phi'\partial\mathcal{L}/\partial\phi' - \mathcal{L}$
- The momentum density is $\mathcal{M} = -A\dot{\phi}'\phi'$. This is just A/B times the energy flux density $\mathcal{F}$.
  - So, like $\mathcal{F}$, it is positive for a wave packet propagating toward positive x (i.e. with positive k). This seems sensible.

- On the other hand, it is not difficult to see that the total wave-momentum is *not* the same as the normal momentum (which is just the sum of the mass times velocity). Both normal and wave-momentum are conserved. But their conservation arises from different symmetries. The normal momentum is conserved because of the homogeneity of *space*. The Lagrangian is the same no matter where in the universe the system is located. The wave momentum is conserved as a result of the properties of the lattice being independent of location on the lattice. One could imagine a lattice in which e.g. C were to vary with position. This would then *not* conserve wave momentum. But the normal momentum would still be conserved.

- The continuity equation is very similar to Newton's law that rate of change of momentum is equal to the force.
  - Here we have that the rate of change of wave-momentum *density* is (minus) the 1-divergence of the a force *density*.
  - or, equvalently, the pressure gradient

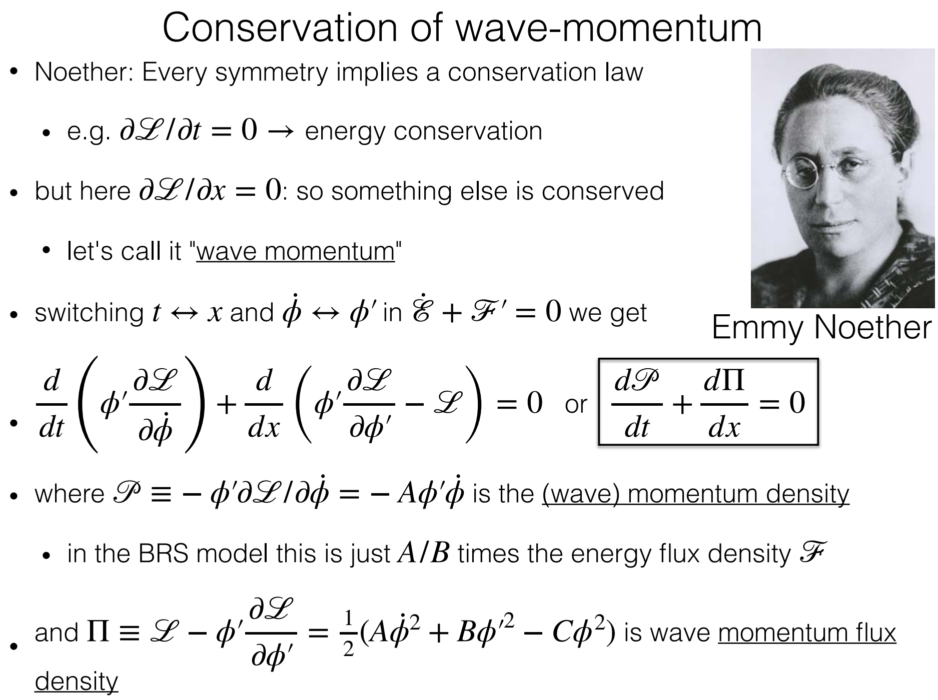


Figure 24: Conservation of wave momentum.

9.2.5 Some questions:

- Q: what are the *phase*- and *group*- velocities for this system?

- Q: for a wave packet of finite extent show that if we integrate the rate of change of the energy- or momentum-density over all space the spatial derivative term vanishes. Obtain thereby a relation between total energy and momentum for wave-packet. What does this remind you of?

- Q: Show that for long wavelength fluctuations – such that $B\phi_{,xx} \ll C\phi$ or $\lambda^2 \gg B/C$ – one can make a change of variables $\phi(x,t) = (\psi(x,t)e^{imt} + \text{c.c})/2$ where c.c. denotes complex conjugation and $m \equiv \sqrt{C/A}$ and where $\psi(x,t)$ is slowly varying with time-scale for variation $\tau \sim (\sqrt{AC}/B)\lambda^2$ and in terms of which the Euler-Lagrange equations become (to leading order in $B/(C\lambda^2) \ll 1$) $i\partial\psi/\partial t = \frac{B}{2\sqrt{AC}}\partial^2\psi/\partial x^2$ and the momentum is $p = i\sqrt{CA}\int dx\ (\psi^*\partial\psi/\partial x + \text{c.c.})/4$ plus rapidly oscillating terms whose time average vanishes. What do these equations remind you of?

- Q: Regarding the previous question, does it seem strange that the original field ϕ has only one real degree of freedom while the complex field ψ has two?

- Q: generalise the theory to allow the coefficient C to vary smoothly with position. How does that change the continuity equation for momentum? (Hint: be careful – what we called $\mathcal{L}_{,x}$ was the partial derivative of $\mathcal{L}$ considered as a function of x and t. That was fairly unambiguous since $\mathcal{L}$ did not have an explicit x dependence. Here you have $\mathcal{L}(\phi, \phi_{,t}, \phi_{,x}, x)$. You should probably denote what we called $\mathcal{L}_{,x}$ above as $\delta\mathcal{L}/\delta x$ to avoid confusion.) What does this modification imply for the rate of change of momentum of a wave-packet? Hint: You might want to draw the analogy with EM waves propagating in a plasma with varying plasma frequency (where, for instance, radio waves can be reflected from the ionosphere).

- Q: what if we had a *finite* lattice like this on a skate-board with a wave-packet propagating along it carrying energy and momentum in the $+x$ direction say. When the packet reaches the end, it will reflect and the sign of the momentum will flip. Would we see the skate-board start to move from the recoil?

9.3 Transition to a relativistic scalar field

9.3.1 Lagrangian, action, equations of motion and stress-tensor

- the transition to a relativistic scalar field is, formally at least, straightforward.
- first we make x a 3-vector $\mathbf{x}$
  - this means we will now have 4 conserved quantities
  - the energy and 3 components of the wave-momentum
- next we demand that the constants A and B be equal – this means that the group velocity is asymptotically unity – the speed of light in our units – for high momentum (high wave-number)
- and we take them to be unity – this being effectively a re-scaling or re-definition of the field strength so that e.g. the kinetic energy density is just $\phi_{,t}^2/2$
- finally we set $C = m^2/\hbar^2$ where $\hbar$ is the reduced Planck constant $h/2\pi$
  - while this looks a bit 'quantum-mechanical', $\hbar$ is simply introduced as a parameter; we are still dealing with a classical (but of course wave-mechanical) field theory
  - in reality one should describe such a field quantum mechanically. That is quite easy here as we're dealing with a set of independent simple harmonic oscillators. Each of these has a wave function – a function of the displacement – and we can define creation and destruction operators and generate occupation number eigenstates etc. The equations here, in contract, describe the evolution of the *expectation value* of the field in such a state which, according to Ehrenfest's theorem, is described by the classical equations of motion.
- with this choice of constants, the dispersion relation is $\omega^2 = |\mathbf{k}|^2 + m^2/\hbar^2$, just as for a relativistic particle
  - note again the close resemblance to the dispersion relation $\omega^2 = c^2k^2 + \omega_p^2$ for an electromagnetic wave in a cold plasma with *plasma frequency* ω_p
  - or, with $E = \hbar\omega$ and $\mathbf{p} = \hbar\mathbf{k}$, the relativistic energy-momentum relation $E^2 = |\mathbf{p}|^2c^2 + m^2c^4$
  - in what follows, we will use $\mu \equiv m/\hbar$, which measures mass, but has units of frequency
- switching to 4-vector notation, so $(t, \mathbf{x}) \to \{x^\alpha\}$ and $(\dot{\phi}, \boldsymbol{\nabla}\phi) \to \{\phi_{,\alpha}\}$ we achieve some significant simplification:
  - the Lagrangian density is
    * $\boxed{\mathcal{L}(\phi_{,\alpha}, \phi) = -\frac{1}{2}(\phi_{,\alpha}\phi^{,\alpha} + \mu^2\phi^2)}$
    * which is manifestly Lorentz invariant
  - the action is

* $\boxed{S = \int d^4x \mathcal{L}(\phi_{,\alpha}, \phi)}$
  * which is also invariant as space-time volume elements like d^4x are invariant
- and $\delta S = 0$ gives the Lorentz invariant Euler-Lagrange equation:
  * $\boxed{\phi^{,\alpha}{}_{,\alpha} - \mu^2\phi = 0}$
  * or
  * $\boxed{\Box\phi = \mu^2\phi}$
  * which is called the *Klein-Gordon equation*
- the invariance of $\mathcal{L}$ with respect to the 4 space-time coordinates x^α implies the conservation laws (or continuity equations) $T^{\alpha\beta}{}_{,\beta} = 0$ where the stress energy tensor is
  * $\boxed{T^{\alpha\beta} = -\phi^{,\alpha}\partial\mathcal{L}/\partial\phi_{,\beta} + \mathcal{L}\eta^{\alpha\beta}}$
  * or
  * $\boxed{T^{\alpha\beta} = \phi^{,\alpha}\phi^{,\beta} + \mathcal{L}\eta^{\alpha\beta}}$

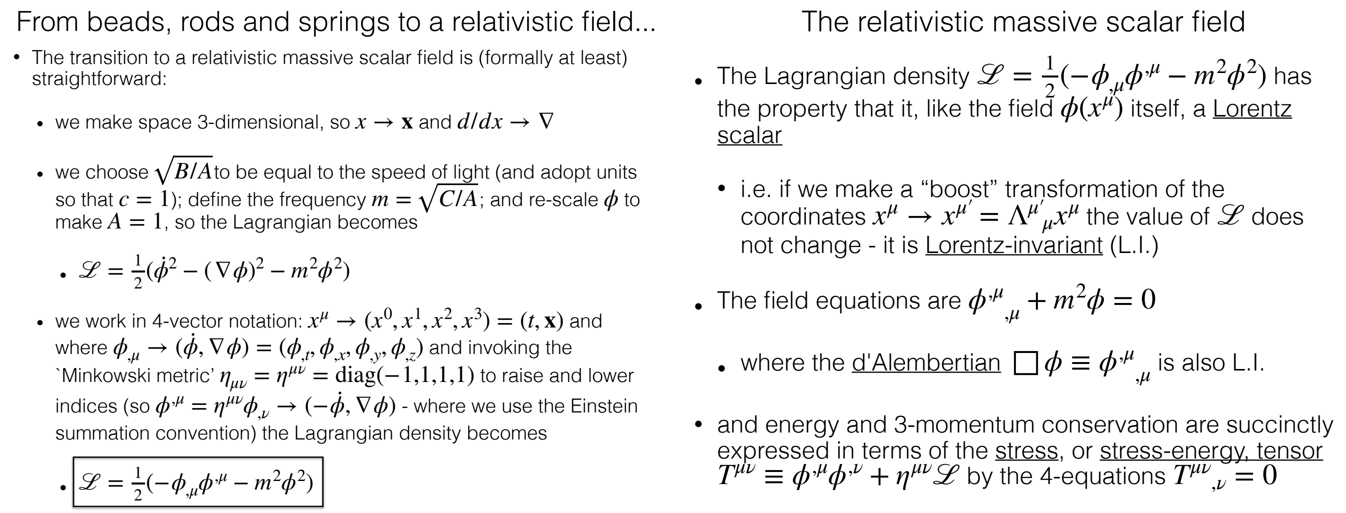


Figure 25: Energy conservation in field theory.

9.3.2 What does it mean?

- we started with a simple, and conceptually straightforward, model for a 'solid-state' lattice where the field was simply the physical displacement of the masses and the energy was the kinetic and potential energies
- wave packets on such a lattice, we noticed, had properties rather similar to that of relativistic particles
- the relativistic scalar field is mathematically identical – the formal transition being essentially a matter of choosing appropriate constants for the three terms in the Lagrangian density
- but we don't normally think of there being an underlying physical lattice on which the fields we observe are a physical displacement
  - though apparently Maxwell *did* think of the EM fields as propagating through an *aether* that had some microphysical mechanism communicating disturbances. He talked of the 'hidden underworld' in which 'the medium [...] may have rotatory as well as vibratory motion'.
  - 'I didn't really get rid of *action at a distance*' he is also supposed to have said, 'I just replaced a big action by lots of little actions'.
- the field ϕ is probably best visualised as a displacement in some abstract space, but the elastic lattice model is nonetheless of great help in giving a mathematically precise but conceptually unchallenging analogy

- the theory thus developed is that of a classical field
- the free field can be decomposed into independent harmonic oscillators which can be quantised in the usual way (exactly as we would for phonons) and this leads to bosonic particles, and that allows one to construct, for example, coherent states – analogous to the coherent light from a laser – which validates most of the classical concepts used above

9.3.3 More general relativistic field theories

- it is possible to construct variants of the Klein-Gordon theory while maintaining the attractive properties of relativistic invariance
- one possibility is to replace the potential energy term $\mu^2\phi^2/2$ by a more general function of the field, often denoted by $V(\phi)$
  - this changes things radically: the 'free-field' theory described above has equations of motion that are linear in ϕ so we can add solutions and plane wave solutions, wave-packets etc., can propagate without interacting with each other
  - with a 'non-harmonic' potential $V(\phi)$ there are wave-interactions at the classical level and scattering of particles in the quantised field
- another possibility is to have multiple fields
  - one can have multiple scalar fields and theories in which the Lagrangian is the sum of the individual free-field Lagrangians plus interaction term(s) involving the various fields – again allowing scattering of particles and also such phenomena as spontaneous symmetry breaking
  - a 2-component (or complex) scalar field can represent a charged field if coupled to the electromagnetic field
- more radical are proposals to modify the so called 'kinetic term' $-\phi_{,\alpha}\phi^{,\alpha}$ in the Lagrangian density, replacing it, for instance, by some function of this scalar
  - though this terminology may seem rather odd since, in the elastic analogy it is only the time derivatives that one would consider to be kinetic energy; the spatial gradients of ϕ contributing to the potential energy.

9.3.4 Applications of the scalar field

- scalar fields exists (the Higgs field)
- they can describe *axions*, which are one candidate for the dark matter
- or *ultra-light scalar fields* which are another popular candidate
- the above are applications where the field is effectively free
- scalar fields are also invoked to explain accelerated expansion
  - in the early universe, where it is called *inflation*, driven by an *inflaton* field
  - or in the late universe, where the field is often called *quintessence*
  - these applications typically, but not necessarily, invoke non-harmonic potentials $V(\phi)$
  - it is typically assumed that, within the region of the universe of interest, the scalar field is homogeneous, so one can ignore the $(\nabla\phi)^2$ term in the stress-tensor. One then finds that the energy density is $\rho = \dot{\phi}^2/2 + V(\phi)$ and the pressure is $P = \dot{\phi}^2/2 - V(\phi)$. This allows the pressure to be *negative*, and this drives accelerated expansion.
- scalar fields are in some sense the simplest kind of matter, and understanding them is a useful warm up for understanding e.g. *vector fields* like electromagnetism

- there are also many beautiful similarities between scalar fields and particles both in how they behave and in the form of their stress-energy, which appears as the source driving Einstein's equations
  - Q: show that if one has a random sea of Klein-Gordon waves then the stress energy tensor becomes identical to that for particles obtained above. Show that the phase-space density $f(\mathbf{p})$ is replaced by the power-spectrum of the waves: $f(\mathbf{p}) \rightarrow P_\phi(\mathbf{k})$.

10 Additional material:

10.1 Liouville's theorem

- Liouville's theorem says that for collisionless particles moving in some potential the phase-space density along the particle trajectories is constant
  - so the 6-fluid element around a fiducial particle may get distorted
    * the space and momentum space volumes may both change
  - but their product remains the same
    * so if their space density increases their momentum- or velocity-space density must go down
- there are various ways of deriving this
  - one way is to start with $\partial f/\partial t + \boldsymbol{\nabla}^{(6)} \cdot (f\dot{\mathbf{x}}^{(6)}) = 0$
  - or $\partial f/\partial t + \boldsymbol{\nabla}_\mathbf{x} \cdot (f\dot{\mathbf{x}}) + \boldsymbol{\nabla}_\mathbf{p} \cdot (f\dot{\mathbf{p}}) = 0$
  - or $\partial f/\partial t + \dot{\mathbf{x}} \cdot \boldsymbol{\nabla}_\mathbf{x} f + \dot{\mathbf{p}} \cdot \boldsymbol{\nabla}_\mathbf{p} f = -f[\boldsymbol{\nabla}_\mathbf{x}\dot{\mathbf{x}} + \boldsymbol{\nabla}_\mathbf{p}\dot{\mathbf{p}}]$
    * but the left-hand side is the *convective derivative* df/dt along the path of the 6D fluid element
    * and, from Hamilton's equations, $\dot{\mathbf{x}} = \boldsymbol{\nabla}_\mathbf{p} H$ and $\dot{\mathbf{p}} = -\boldsymbol{\nabla}_\mathbf{x} H$, so the RHS vanishes, hence we have Liouville's theorem for collisionless particles:
  - $\boxed{df/dt = 0}$
  - this says that the phase-space density f is constant for any element of the 'phase-fluid'
  - i.e. the phase-space density is an *incompressible fluid*
- it is closely related to the fact that d^3xd^3p has the same units as $\hbar^3$ – we can think of their being one quantum state per volume in phase-space (or two, allowing for spin), and occupation number being adiabatically conserved
- and also to the fact that for a particle moving in a slowly varying potential the phase-space area within the orbit $\oint q\, dp = \oint p\, dq$ is an *adiabatic invariant*.
- one particularly important consequence of this for relativistic particles is that if we define I_ν to be the energy per unit frequency per unit area per unit solid angle – this is called the 'brightness' – then I_ν/ν^3 is constant along rays.
- this comes about as follows
  - phase-space density $f(\mathbf{r}, \mathbf{p})$ is, by definition, the number of photons per unit spatial volume per unit momentum-space volume
  - or $f = d^6N/d^3rd^3p$ = number/volume/momentum-volume
  - multiplying by the photon energy $h\nu$ gives
  - $h\nu \times f$ = energy/volume/mom-vol
  - and multiplying by c gives
  - $c \times h\nu \times f$ = energy/area/time/mom-vol
  - with $d^3p = d\Omega p^2 dp = (h/c)^3 d\Omega\nu^2 d\nu$
  - if we define I_ν to be the energy/time/area/solid-angle/frequency then
  - $I_\nu = c^4h^2\nu^3 f$ so constancy of f requires constancy of I_ν/ν^3

10.2 Invariance of power

- while we will not need it much here, another useful Lorentz invariant quantity is power
  - for example the power radiated by a collection of moving charges
- this is Lorentz invariant as it is the ratio of the energy emitted in some period of time divided by that time – both of which transform as the 0th components of 4-vectors
  - so we can use Larmor's (non-relativistic) formula to calculate the total power radiated in the instantaneous rest-frame of the accelerated particles
  - and use this to give the total power in the observer or lab-frame
  - though if we want to describe the angular distribution of the power we need to worry about aberration

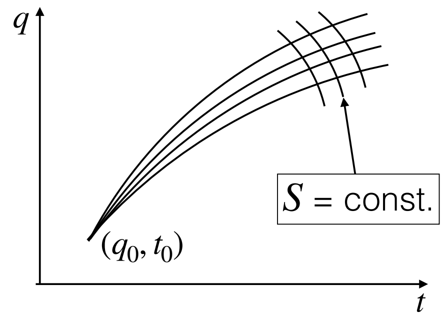
10.3 Summary of relativistic invariants

- Special relativity is challenging because fundamental concepts like space, time, energy and momentum become frame-dependent
- many problems, however, can be solved, or the physics be simply understood. by appealing to Lorentz invariants
- and it is helpful to express physical relations, as much as possible, in terms of Lorentz invariant quantities
- we have seen several:
  - space-time volume elements like d^4x
  - the proper separation ds^2 and the squared length p^2 of any 4-vector p^α
  - several invariants associated with the phase-space density:
    * anything that counts numbers (like $d^6N = f(\mathbf{r}, \mathbf{p})d^3rd^3p$, $d^3N = nd^3r$)
    * d^3p/E (and also n/E and Ed^3r)
    * 6-dimensional volume elements d^3rd^3p
    * and therefore the number of quantum states $d^2rd^3p/\hbar^3$ (with e.g. the factor two for different spin-states)
    * and f itself (which we also found to be invariant along particle trajectories for collisionless particles)
    * I_ν/ν^3
  - the power radiated by a system of charges

10.4 From Hamilton and Jacobi to Dirac and Feynman

Hamilton-Jacobi equations

- consider ensemble of particles starting from same place q_0 at t_0 with a range of initial momenta

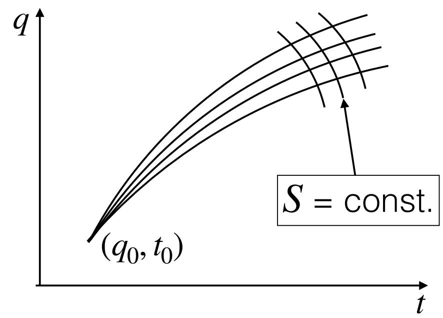


- $S = \int dt L(q, \dot{q})$
- $\delta S = \int dt \left[\frac{\partial L}{\partial q} \delta q + \frac{\partial L}{\partial \dot{q}} \delta \dot{q} \right]$
- $= \int dt (\dot{p} \delta q + p \delta \dot{q})$
- $= \int dt \frac{d(p\delta q)}{dt} = [p \delta q]_t \quad \rightarrow \quad \boxed{p = \frac{\partial S}{\partial q} \qquad E = \frac{\partial S}{\partial t}}$
- $dS = \frac{\partial S}{\partial q} dq + \frac{\partial S}{\partial t} dt = L dt$ -> $\frac{\partial S}{\partial t} = \frac{\partial S}{\partial q} \dot{q} - L = p\dot{q} - L = H$

Figure 26: Hamilton-Jacobi equations.

classical vs. quantum (Dirac, Feynman…)

- Classical Hamilton-Jacobi: consider ensemble of particles starting from same place q_0 at t_0 with a range of initial momenta
  - the action is then $S[q(t)] = S(q, t)$
  - and the Hamilton-Jacobi equations are
    - $p = -\partial S / \partial q$ and $E = -\partial S / \partial t$
- Quantum description:
  - wave-function $\psi(q, t) \sim e^{iS(q,t)/\hbar}$
  - times slowly varying envelope



- explains $\delta S = 0$ principle (constructive interference)
- suggests E, p operators: $E \rightarrow E_{\rm op} = i\hbar\partial/\partial t$, $p \rightarrow p_{\rm op} = i\hbar\partial/\partial x$
  - since then e.g. $E_{\rm op}\psi = i\hbar\partial(e^{iS/\hbar})/\partial t = -(\partial S/\partial t)\psi = E\psi$
- Feyman path integral approach -> propagator
- quantum wave-function -> classical (particles) as $\hbar \rightarrow 0$

Figure 27: Hamilton-Jacobi equations.