M1 GR + Cosmology - 1b - Covariance and Gauge Invariance of Electromagnetism

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1 Outline

Here we continue with our review of special relativity, but now looking at Maxwell's electromagnetism. In outline:

- we start with Maxwell's equations and discuss how the source of the fields – the charge and current densities – transform under Lorentz boosts
- we introduce the electromagnetic 4-potential (a 4-vector) $\vec{A} \longrightarrow A^\mu = (\phi/c, \mathbf{A})$ from which the $\mathbf{E}$ and $\mathbf{B}$ fields may be obtained
- we also discuss the ambiguity of $\vec{A}$ coming from invariance of EM under a *gauge transformation*
  - this is useful as we will later see something mathematically very similar, though physically very different, in GR
- we then develop the classical dynamics of relativistic charged particles, starting with their Lagrangian, introducing the *canonical momentum* and the *canonical energy* (the Hamiltonian) and contrast these with their 'mechanical' counterparts
- we obtain the equations of motion (Euler-Lagrange equations); the energy-momentum relation; Hamilton's equations and the 'work equation' and finally the Hamilton-Jacobi equation.
- we next consider some aspects of electromagnetic *wave mechanics*
  - we review how Hamilton-Jacobi leads, following Dirac and Feynman's identification of the classical particle action with the phase of the QM wave-function to the relativistic (and, in the appropriate limit, the non-relativistic) Schrödinger equation for an electrically charged particle

- we introduce in the process the *gauge covariant derivative* – which bears some formal similarity to the covariant derivative in GR, though physically these are very different things
  - we discuss how the same equation describes *classical* electrically charged *scalar waves*

- returning to particles, we connect the covariant and non-covariant formalisms and show:
  - how the Faraday tensor contains the components of the **E** and **B** fields
  - how the familiar expressions for the Lorentz force law and rate of work emerge
  - how Maxwell's equations are expressed in terms of the Faraday tensor
    - the homogeneous equations coming from the *Bianchi identity* – something we will see later when we develop the field equations of GR
  - how charge conservation is 'built-in' to the structure of Maxwell's equations
  - how charge density and current transform under boosts

- we derive Liouville's theorem for charged particles which says that the phase-space density f of particles is constant along their orbits and how this applies for both canonical ($f(\mathbf{x}, \mathbf{P})$) and mechanical ($f(\mathbf{x}, \mathbf{p})$) densities

- finally we turn to the stress (or stress-energy) tensor
  - we show how the 4-stress for particles can be expressed in various equivalent ways in terms of density of particles in 3 or 4 dimensions or in 6-dimensional phase-space and how continuity (or conservation) of 4-momentum emerges in each case
  - we then turn to the 4-stress for radiation
    - we first follow the route of Maxwell and Poynting, obtaining the energy and momentum densities and then the momentum (Poynting) flux density and the momentum flux density of the field (Maxwell stress 3-tensor) – this gives a symmetric form for the stress tensor
    - we then obtain the stress tensor à la Noether from invariance of the Lagrangian density under time- and space-translations which, perhaps surprisingly, gives a different, and non-symmetric tensor

2 Maxwell's equations in terms of E and B.

The physical content of Maxwell's equations is illustrated in figure 1. In the 'rationalised' SI system they are

$$\begin{aligned} \nabla \cdot \mathbf{E} &= \rho/\epsilon_0 \qquad & \nabla \cdot \mathbf{B} &= 0 \\ \nabla \times \mathbf{E} &= -\dot{\mathbf{B}} \qquad & \nabla \times \mathbf{B} &= \mu_0 \mathbf{j} + \dot{\mathbf{E}}/c^2 \end{aligned} \tag{1}$$

where

- **E** and **B** are the electric and magnetic fields
- $\dot{\mathbf{E}}$ and $\dot{\mathbf{B}}$ are their partial derivatives with respect to time t
- ϵ_0 is the *permittivity of free space*
  - defined in terms of the electrostatic force between two charges
- μ_0 is the *permeability of free space*
  - defined in terms of the magnetostatic force between two current element (Biot & Savart)
- $c = 1/\sqrt{\epsilon_0 \mu_0}$
- ρ is the charge density
- $\mathbf{j}$ is the current density

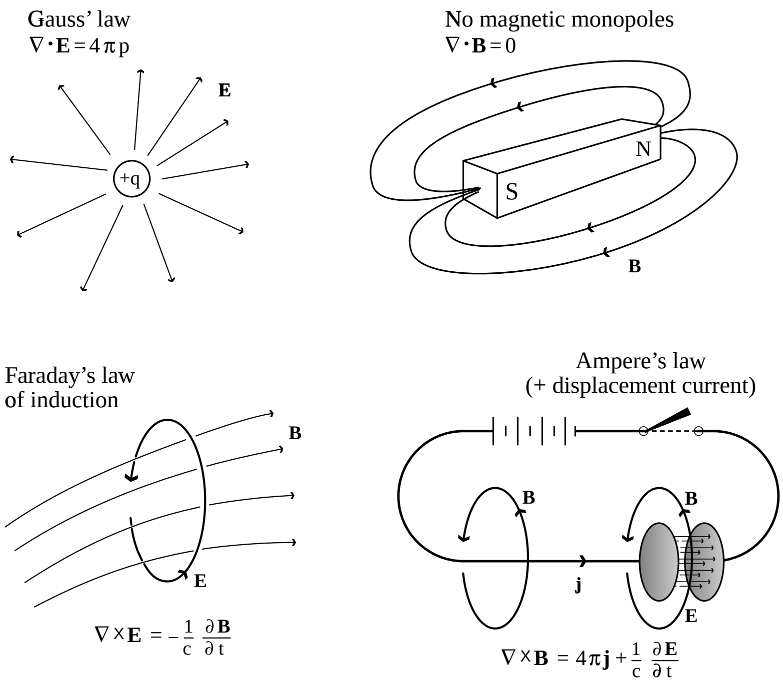

Figure 1: Maxwell's equations. These are in 'Gaussian units'.

The upper-right equation $\boldsymbol{\nabla} \cdot \mathbf{B} = 0$ and Faraday's law in the lower-left have no 'source term'. The are called the *homogeneous* Maxwell equations. Gauss's law (upper left) and the Ampère/Maxwell equation (lower-right) are driven by the electric charge and current density respectively. They are known as the *inhomogeneous* Maxwell equations.

Special relativity is 'built in' to Maxwell's electromagnetism. Most fundamentally from the fact that these equations permit wave-like solutions that propagate with speed c – determined by the empirical constants of electro- and magneto-statics – that is apparently independent of any frame of reference.

3 Comments on transformation of charge and current densities

- Lorentz contraction implies that *number* density n of a set of particles depends on the frame from which we view them
  - n is smallest in the rest-frame
- The electric charge of a particle is defined as that measured in the rest-frame
  - so q is a Lorentz invariant
  - and so charge density $\rho = qn$ transforms like n
- As illustrated in figure 2 this implies that the $\mathbf{E}$ and $\mathbf{B}$ fields must transform under a boost of reference frame.

4 Gauge Invariance in electromagnetism and the gauge principle

4.1 The electromagnetic 4-potential

- Maxwell's equation $\boldsymbol{\nabla} \cdot \mathbf{B} = 0$ implies
  - $\boxed{\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}}$
  - with $\mathbf{A}$ the *vector potential*,
- while $\boldsymbol{\nabla} \times \mathbf{E} + \dot{\mathbf{B}} = 0$ (together with the above) implies that $\mathbf{E} + \dot{\mathbf{A}}$ is the gradient of some scalar

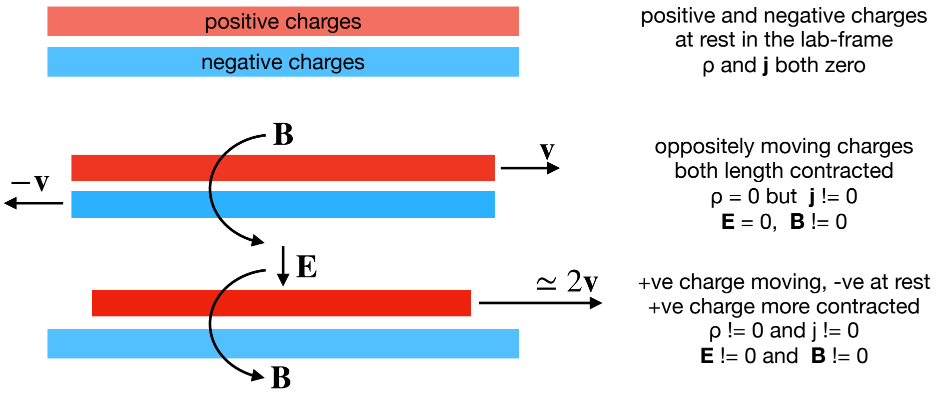

Figure 2: We can make an electric current by combining two oppositely moving 'tubes' of charge of opposite sign (centre). In the lab (or CoM) frame both are slightly contracted (for small velocity) by the same amount, so there is no charge density and, by Gauss's law, no $\mathbf{E}$ field. In the frame of the negative charges (bottom) there is a net positive charge density and hence a non-vanishing electric field. Note that a positive charge above the tubes that is at rest in the frame of the negative charges would experience a downward force. In the CoM frame (centre) the charge is moving to the left and we would say it feels a $q\mathbf{v} \times \mathbf{B}$ force. In the lower frame we would say the charge is at rest and suffers and electric force $q\mathbf{E}$.

  - $\boxed{\mathbf{E} = -\boldsymbol{\nabla}\varphi - \dot{\mathbf{A}}}$
  - where φ is called the *scalar potential*

- the scalar and vector (or magnetic) potentials are the components of
  - $\boxed{\vec{A} \longrightarrow A^\mu = (\phi/c, \mathbf{A})}$
  - which, as we will see later, is a Lorentz 4-vector

4.2 Invariance of electromagnetic fields under a gauge transformation

- The potentials φ and $\mathbf{A}$ are not unique as $\mathbf{E}$ and $\mathbf{B}$ are invariant under what is called a *gauge transformation* of the 4-potential
- $\boxed{A_\mu \to A'_\mu = A_\mu + \xi_{,\mu}}$
- where $\xi(\vec{x})$ is an arbitrary function of space-time, so $\xi_{,\mu}$ are the covariant components of a 4-vector[1]
- one can see this for the magnetic field since if we write
  - $$\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial_x & \partial_y & \partial_z \\ A_x & A_y & A_z \end{vmatrix} = \begin{bmatrix} A_{z,y} - A_{y,z} \\ A_{x,z} - A_{z,x} \\ A_{y,x} - A_{x,y} \end{bmatrix}$$
  - we see that each component under a gauge transformation vanishes (the change in the x-component, for example, being $B'_x - B_x = \xi_{,zy} - \xi_{,yz}$, which vanishes by virtue of commutativity of partial derivatives
- and similarly, we find that the change in the $\mathbf{E}$ field involves antisymmetric combinations of partial derivatives – but now with, for instance, $\xi_{,tx} - \xi_{,xt} = 0$ appearing in the change of the x component of $\mathbf{E}$

[1] In these notes we use the convention that the flat-spacetime metric (used for raising and lowering indices on 4-vectors and tensors) is $\eta_{\mu\nu} = \mathrm{diag}\{-1, 1, 1, 1\}$. A spacetime 4-displacement $\vec{x}$ has 'contravariant' components $x^\mu = (ct, x, y, z) = (ct, \mathbf{x})$ and 'covariant' components $x_\mu = \eta_{\mu\nu}x^\nu = (-ct, x, y, z) = (-ct, \mathbf{x})$, displaying the Einstein summation convention. The gradient operator $\vec{\partial}$ has covariant components $\partial_\mu = (c^{-1}\partial_t, \partial_x, \partial_y, \partial_z) = (c^{-1}\partial_t, \boldsymbol{\nabla})$, where $\partial_t = \partial/\partial t$, $\partial_x = \partial/\partial x$ etc., and $X_{,\mu}$ denotes $\partial_\mu X$. We will, however, assume the units of length and time are such that the speed of light $c = 1$, and we may be sloppy and sometimes drop the c.

5 Classical particle electrodynamics

5.1 The Lagrangian and the action

Let's guess that the relativistic Lagrangian for a particle of proper mass m and charge q moving with velocity $\dot{\mathbf{x}}$ in an electromagnetic potential A_μ is

$$L(\mathbf{x}, \dot{\mathbf{x}}, t) = -mc^2/\gamma - q\varphi + q\dot{\mathbf{x}} \cdot \mathbf{A} \tag{2}$$

where, as usual, $\gamma \equiv 1/\sqrt{1 - |\dot{\mathbf{x}}|^2/c^2}$ is the Lorentz boost factor.

The 'dot' operator here always denotes derivative with respect to coordinate time t (not proper time). In Maxwell's equations, $\dot{\mathbf{A}}$, for instance, is the partial derivative $\partial \mathbf{A}/\partial t$. Here we are applying it to the position $\mathbf{x}$ of a particle, so $\dot{\mathbf{x}} = d\mathbf{x}/dt$, the total derivative.

That's in 3+1 notation. We may also write this in 4-notation as

$$\boxed{L(\mathbf{x}, \dot{\mathbf{x}}, t) = -mc^2/\gamma + q\dot{x}^\mu A_\mu.} \tag{3}$$

where $\dot{x}^\mu = d\partial_t(ct, \mathbf{x})/dt = (c, \dot{\mathbf{x}})$ and $A_\mu = (-\varphi/c, \mathbf{A})$.

At first sight this might seem like a bad choice as it is evidently not gauge invariant, nor is it Lorentz invariant. (Q: why is this *not* a Lorentz scalar?)
On the positive side:

- it generates an *action* $S = \int dt L$ with differential
  - $\boxed{dS = -mc^2 d\tau + A_\mu dx^\mu}$
  - which *is* a Lorentz scalar (the proper time interval here being $d\tau = dt/\gamma$)
- and, as we shall see, it gives the empirically determined Lorentz force law
  - $\boxed{d\mathbf{p}/dt = q(\mathbf{E} + \dot{\mathbf{x}} \times \mathbf{B})}$
  - where $\mathbf{p} = \gamma m \mathbf{v}$ is the relativistic momentum we encountered previously
- and work equation (giving the rate at which the field does work on the particle)
  - $\boxed{dE/dt = \mathbf{E} \cdot \dot{\mathbf{x}}}$
  - where $E = \gamma mc^2$ is the energy

5.2 The canonical and mechanical 3-momenta

The *'canonical'* or *'generalised'* momentum is defined, in general, by

$$\boxed{\mathbf{P} \equiv \partial L/\partial \dot{\mathbf{x}}} \tag{4}$$

which here is given, from (2), by

$$\boxed{\mathbf{P} = \mathbf{p} + q\mathbf{A}} \tag{5}$$

where

$$\boxed{\mathbf{p} \equiv \gamma m \dot{\mathbf{x}}} \tag{6}$$

is the usual relativistic 3-momentum. We will call the latter the *mechanical momentum*.

Note that the canonical momentum is not gauge invariant.

5.3 The Euler-Lagrange equation

The *Euler-Lagrange equation* obtained from the requirement that the particle path extremises the action is, in general,

$$\boxed{d\mathbf{P}/dt = \partial L/\partial \mathbf{x}} \tag{7}$$

so here, since the only dependence on $\mathbf{x}$ in (3) is in $A_\mu(\mathbf{x}, t)$, the *generalised force* $\partial L/\partial \mathbf{x}$ has components

$$\boxed{dP_i/dt = q\dot{x}^\mu A_{\mu,i}} \tag{8}$$

Thus a particle in a *spatially uniform* potential (i.e. one for which $A_{\mu,i} = 0$) has constant $\mathbf{P}$.

5.4 $d\mathbf{p}/dt$ and the Faraday tensor

From (5), the rate of change of the mechanical momentum is $d\mathbf{p}/dt = d\mathbf{P}/dt - qd\mathbf{A}/dt$. The *convective derivative* of $\mathbf{A}$ here is $d\mathbf{A}/dt = \dot{\mathbf{A}} + (\dot{\mathbf{x}} \cdot \boldsymbol{\nabla})\mathbf{A}$ or, in components, $dA_i/dt = A_{i,t} + \dot{x}_j A_{i,j} = \dot{x}^\mu A_{i,\mu}$.

Using (8) for dP_i/dt gives

$$\boxed{dp_i/dt = d(P_i - qdA_i)/dt = q\dot{x}^\mu F_{\mu i}} \tag{9}$$

where

$$\boxed{F_{\mu\nu} \equiv A_{[\mu,\nu]} \equiv A_{\mu,\nu} - A_{\nu,\mu}} \tag{10}$$

is the definition of the *Faraday tensor*. It is an antisymmetric rank-2 Lorentz 4-tensor whose 6 ~~independent~~ components are, as we will see shortly, the 6 components of the $\mathbf{E}$ and $\mathbf{B}$ fields.

5.5 The Hamiltonian and the energy-momentum relation

The *Hamiltonian* is defined, in general, to be

$$\boxed{H \equiv \dot{\mathbf{x}} \cdot \mathbf{P} - L(\mathbf{x}, \dot{\mathbf{x}}, t)} \tag{11}$$

which here, from (2) and (5), is $H = \dot{\mathbf{x}} \cdot (\mathbf{p} + q\mathbf{A}) + mc^2/\gamma + q\varphi - q\dot{\mathbf{x}} \cdot \mathbf{A}$.

The terms involving $\mathbf{A}$ cancel while $\dot{\mathbf{x}} \cdot \mathbf{p} + m/\gamma = \gamma m(|\dot{\mathbf{x}}|^2 + c^2/\gamma^2) = \gamma mc^2$ or

$$\boxed{H = \gamma mc^2 + q\varphi} \tag{12}$$

Despite its appearance, H is formally only a function of $\mathbf{x}$, $\mathbf{P}$ and t since the differential of (11) is

$$dH = \dot{\mathbf{x}} \cdot d\mathbf{P} - (\partial L/\partial \mathbf{x}) \cdot d\mathbf{x} - (\partial L/\partial t)dt \tag{13}$$

the terms $\mathbf{P} \cdot d\dot{\mathbf{x}} - (\partial L/\partial \dot{\mathbf{x}}) \cdot d\dot{\mathbf{x}}$ having cancelled by virtue of the definition (4) of $\mathbf{P}$.

To express H explicitly in terms of $\mathbf{x}$ and $\mathbf{P}$ we can note that the definition $\mathbf{p} \equiv \gamma m\dot{\mathbf{x}}$ implies $|\mathbf{p}|^2 + m^2c^2 = m^2(\gamma^2|\dot{\mathbf{x}}|^2 + c^2) = \gamma^2 m^2 c^2$, or, defining $E \equiv \gamma mc^2$, which is the mechanical (i.e. rest mass plus kinetic) energy,

$$\boxed{E^2 = c^2|\mathbf{p}|^2 + m^2c^4} \tag{14}$$

which is the familiar relativistic energy momentum relation.

But according to (12) $E = \gamma mc^2 = H - q\varphi$, so, along with $\mathbf{p} = \mathbf{P} - q\mathbf{A}$ from (5), the energy-momentum relation, in terms of H and $\mathbf{P}$, is

$$(H - q\varphi)^2 = c^2|\mathbf{P} - q\mathbf{A}|^2 + m^2c^4. \tag{15}$$

and consequently

$$\boxed{H(\mathbf{x}, \mathbf{P}, t) = \sqrt{m^2c^4 + c^2|\mathbf{P} - q\mathbf{A}|^2} + q\varphi} \tag{16}$$

with the dependence of H on $\mathbf{x}$ and t coming *via* $\mathbf{A}(\mathbf{x}, t)$ and $\varphi(\mathbf{x}, t)$.

5.6 Hamilton's equations and dH/dt

Inspection of the coefficients of $d\mathbf{P}$ and $d\mathbf{x}$ in (13) provide us, in general, with *Hamilton's equations*

$$\boxed{\dot{\mathbf{x}} = \frac{\partial H}{\partial \mathbf{P}}} \quad \text{and} \quad \boxed{\dot{\mathbf{P}} = -\frac{\partial H}{\partial \mathbf{x}}} \tag{17}$$

where to obtain the latter we have invoked the Euler-Lagrange equation $\dot{\mathbf{P}} = \partial L/\partial \mathbf{x}$.

Equation (13) for dH is valid for arbitrary $d\mathbf{x}$ and $d\mathbf{P}$. If these differentials are those for an actual particle obeying the equation of motion the first two terms $\dot{\mathbf{x}} \cdot d\mathbf{P} - \dot{\mathbf{P}} \cdot d\mathbf{x}$ in (13) cancel – to confirm this, simply divide by dt – and we have the equation for the evolution of the Hamiltonian which, in general, is

$$\boxed{dH/dt = -\partial L/\partial t} \tag{18}$$

and here is, from (3),

$$\boxed{dH/dt = q\dot{x}^\mu A_{\mu,t}} \tag{19}$$

so in a *static* potential (i.e. one for which $A_{\mu,t} = 0$) the Hamiltonian – which we will sometimes call the *canonical energy* – is constant.

5.7 Covariant equations of motion for the canonical and mechanical 4-momenta

The equations for the rate of change of energy and momentum (either the mechanical or canonical versions) are succinctly combined here in 4-notation. But they are not in a relativistically covariant form because $\dot{x}^\mu$ is not a Lorentz 4-vector, and neither is dp_i/dt a component of a 4-vector, even though p_i is. The non-covariance stems from the derivative with respect to coordinate time. To cast the equations of motion into a covariant form, in which all the quantities appearing are either Lorentz 4-vectors, 4-tensors we need to replace d/dt by $d/d\tau$ where τ is proper time.

Defining the 4-velocity $u^\mu = (\gamma c, \gamma \dot{\mathbf{x}}) = \gamma \dot{x}^\mu$ and the *canonical 4-momentum* $P^\mu \equiv (H/c, \mathbf{P})$ we have, for the 4-vector $dP^\mu/d\tau = \gamma dP^\mu/dt$:

$$\boxed{dP^\mu/d\tau = qu^\nu A_\nu{}^{,\mu}} \tag{20}$$

which is not gauge invariant.

While, for the *mechanical 4-momentum* $p^\mu \equiv mu^\mu = (\gamma mc, \gamma m\dot{\mathbf{x}})$,

$$\boxed{dp^\mu/d\tau = qu^\nu F_\nu{}^\mu.} \tag{21}$$

which is gauge invariant.

Note that if we 'dot' $dp^\mu/d\tau$ with p_μ we have $p_\mu dp^\mu/d\tau = qp_\mu u^\nu F_\nu{}^\mu = qmu^\mu u^\nu F_{\nu\mu}$ which is the trace of the product of a symmetric tensor $(qmu^\mu u^\nu)$ and an anti-symmetric one $(F_{\nu\mu})$ and so vanishes, so $d\vec{p}/dt$ is orthogonal to $\vec{p}$, something we could have inferred directly from the fact that, according to (14), the norm of the 4-momentum is $\vec{p}\cdot\vec{p} = p_\mu p^\mu = -m^2$ which is constant.

5.8 Gauge invariance of particle electrodynamics

The Lagrangian (3) is not gauge invariant (it changes if $A_\mu \to A_\mu + \xi_{,\mu}$). Neither is the action $S[x(t)] = \int dt L(x, \dot{x}, t)$ nor are the canonical momentum $\mathbf{P}$ and energy H (their equations of motion (20) not being gauge invariant).

But the Faraday tensor $F_{\mu\nu}$ *is* gauge invariant since $F'_{\mu\nu} = A'_{[\mu,\nu]} = A_{[\mu,\nu]} + \xi_{,\mu\nu} - \xi_{,\nu\mu} = A_{[\mu,\nu]} = F_{\mu\nu}$. And so the equation of motion for the 'mechanical' 4-momentum $dp^\mu/d\tau = qu^\nu F_\nu{}^\mu$ is gauge invariant and consequently the 4-momenta p^μ and the paths $x^\mu(\tau)$, which are obtained by integrating the $dp^\mu/d\tau = md^2x^\mu/d\tau^2 = qF_\nu{}^\mu dx^\nu/d\tau$, are also gauge invariant.

The gauge dependence of the action S can be seen directly from the definition of the particle paths as those which extremise the action. For a variation of a path $\mathbf{x}(t) \to \mathbf{x}(t) + \delta\mathbf{x}(t)$ the variation of the action is

$$\delta S = \int_{t_1}^{t_2} dt \left(\delta\mathbf{x}\cdot\frac{\partial L}{\partial \mathbf{x}} + \delta\dot{\mathbf{x}}\cdot\frac{\partial L}{\partial \dot{\mathbf{x}}}\right) = \left[\delta\mathbf{x}\cdot\frac{\partial L}{\partial \dot{\mathbf{x}}}\right]_{t_1}^{t_2} + \int_{t_1}^{t_2} dt\delta\mathbf{x}\cdot\left(\frac{\partial L}{\partial \mathbf{x}} - \frac{d}{dt}\frac{\partial L}{\partial \dot{\mathbf{x}}}\right) \tag{22}$$

where we have integrated by parts to eliminate $\delta\dot{\mathbf{x}}$.

In the integral here we have, from (3), the generalised force $\partial L/\partial x_i = q\dot{x}^\mu A_{\mu,i}$ and the generalised momentum $\partial L/\partial \dot{x}_i = \gamma m\dot{x}_i + qA_i$ whose time derivative is $d(\partial L/\partial \dot{x}_i)/dt = d(\gamma m\dot{x}_i)/dt + q\dot{x}^\mu A_{i,\mu}$. Consequently

$$\delta S = \left[\delta\mathbf{x}\cdot\frac{\partial L}{\partial \dot{\mathbf{x}}}\right]_{t_1}^{t_2} - \int_{t_1}^{t_2} dt\delta x_i \left[\frac{d}{dt}\frac{m\dot{x}_i}{\sqrt{1-|\dot{\mathbf{x}}|^2/c^2}} - q(cF_{0i} + \dot{x}_j F_{ji})\right] \tag{23}$$

where the integrand – whose vanishing for extremal paths implies the Euler-Lagrange equation – is manifestly gauge invariant. The 'boundary term' $[\delta\mathbf{x}\cdot(\partial L/\partial\dot{\mathbf{x}})]$, on the other hand, is gauge-dependent.

5.9 The Hamilton-Jacobi equation

Requiring that δS vanish for two paths that begin and end at the same end points – so that the boundary term above vanishes – gives the equations of motion. On the other hand, if we consider different paths obeying the equations of motion – so the integral term vanishes – and assume that these have the same starting point but have different end points, we have $\delta S = \delta\mathbf{x}\cdot\partial L/\partial\dot{\mathbf{x}} = \delta\mathbf{x}\cdot\mathbf{P}$, so $\mathbf{P}$ is evidently the rate at which S changes with position (at fixed t_2):

$$\boxed{\mathbf{P} = \partial S/\partial \mathbf{x}} \tag{24}$$

And, since $dS = Ldt = (\partial S/\partial t)dt + (\partial S/\partial \mathbf{x}) \cdot d\mathbf{x} = (\partial S/\partial t + \mathbf{P} \cdot \dot{\mathbf{x}})dt$, it follows that $\partial S/\partial t = L - \mathbf{P} \cdot \dot{\mathbf{x}} = -H$, so

$$\boxed{H = -\partial S/\partial t} \tag{25}$$

the two above expressions being the space and time components of the covariant expression

$$\boxed{P^\mu = \partial^\mu S} \tag{26}$$

Using (24) and (25) to replace $\mathbf{P}$ and H in the energy-momentum relation (15) gives us the *Hamilton-Jacobi equation*:

$$\boxed{(\partial S/\partial t + q\varphi)^2 = c^2 |\partial S/\partial \mathbf{x} - q\mathbf{A}|^2 + m^2 c^4} \tag{27}$$

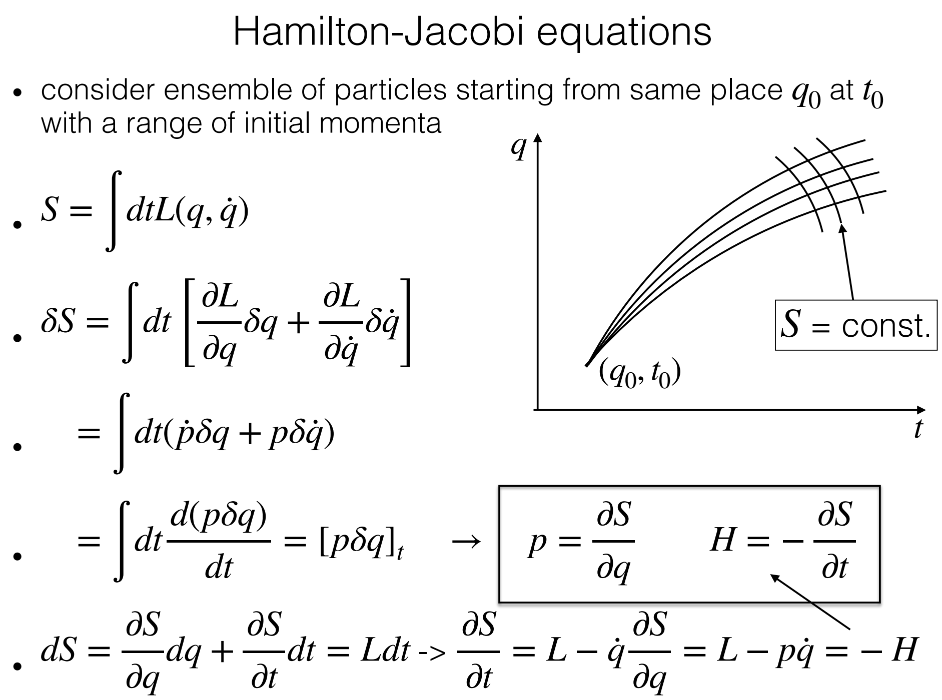

Figure 3: The Hamilton-Jacobi equations. To obtain the Euler-Lagrange equations we demand that δS vanish for two paths that begin and end at the same points. To obtain the H-J equations we consider a family of paths that start at the same position and consider how the action S varies as a function of the final q, t. The result is that the momentum is the rate of change of S with position and the energy is (minus) the rate of change of S with time. For a charged particle it is the 'canonical' momentum $\mathbf{P} = \mathbf{p} + q\mathbf{A}$ and the 'canonical energy' (i.e. the Hamiltonian) $H = E + q\varphi$ that appear here. Dirac realised that the quantum mechanical wave function may be obtained simply by taking the exponential of i times the classical action divided by the reduced Planck's constant: $\psi \propto e^{iS/\hbar}$.

6 Wave-mechanics of a charged field

Historically, it was J.J. Thomson's discovery that cathode rays (whose very name indicates that they were at first thought to be some kind of radiation) carried electric charge that convinced most physicists that electrons were really particles. He got the Nobel prize for that in 1906. But it wasn't long before the wave-nature of electrons was discovered, leading to J.J.'s son G.P. getting the same prize for discovering electrons were really waves. The wave-like nature of matter like electrons emerged with the development of quantum mechanics, and the Schrödinger wave equation was considered to describe the wave-function for a single particle, whose squared modulus gives the probability to find the particle. In the process, the relativistic version of Schrödinger's equation was dropped in favour of the non-relativistic version which, while more limited in scope, has a conserved probability density and current. Subsequently, quantum field theory was developed, in which, for bosonic particles, one starts with a classical field (obeying the relativistic Schrödinger equation) and which can be considered, in the limit that interactions are negligible, to be a collection of de-coupled harmonic oscillators, which may be quantised in the usual manner using non-relativistic QM, to obtain energy eigenstates (whose wave-functions are functions of the field - not of space). Interactions are then added perturbatively to obtain amplitudes – and hence probabilities – for scattering processes. This is known as '2nd quantisation'.

6.1 The quantum mechanical wave-function from Hamilton and Jacobi

We saw that for a collection of particles that start from the same position with a range of initial momenta their action $S = \int dtL$ has partial derivatives with respect to final times and positions given by $H = -\partial S/\partial t$ and $\mathbf{P} = \partial S/\partial \mathbf{x}$. It follows that the action, in the vicinity of some space-time point $(t_0, \mathbf{x}_0)$, which we take, temporarily, to be the origin of our coordinate system, is

$$S = S_0 - Ht + \mathbf{P} \cdot \mathbf{x} + \ldots \tag{28}$$

where $\ldots$ denotes terms that are quadratic or higher order in $\mathbf{x}$ and t..

According to the formulation of QM of Dirac and Feynman, the quantum mechanical amplitude to be at $(t, \mathbf{x})$, or, in other words, the *wave-function* $\psi(t, \mathbf{x})$, is proportional to $e^{iS/\hbar}$.

So here the wave-function is

$$\psi \propto e^{i(-Ht+\mathbf{P}\cdot\mathbf{x})/\hbar} \tag{29}$$

leading one to $i\hbar\partial_t\psi = H\psi$ and $-i\hbar\boldsymbol{\nabla}\psi = \mathbf{P}\psi$ and therefore the identification of H with the operator $i\hbar\partial_t$ (or, equivalently, of $P^0 = H/c$ with $-i(\hbar/c)\partial^t = -i\hbar\partial^0$) and of $\mathbf{P}$ with the operator $-i\hbar\boldsymbol{\nabla}$.

In general, there may be more than one classical path leading to $(t_0, \mathbf{x}_0)$, in which case there will be interference in the total wave-function (this being, according to Feynman, the sum over all paths). Putting that aside, and using $P^\mu = (H/c, \mathbf{P}) \rightarrow -i\hbar\partial^\mu$ in the energy-momentum relation $p^\mu p_\mu = (P^\mu - qA^\mu)(P_\mu - qA_\mu) = -m^2c^2$ gives the relativistic Schrödinger-equation alluded to above,

$$\boxed{(\hbar\partial^\mu - iqA^\mu)(\hbar\partial_\mu - iqA_\mu)\psi = m^2c^2\psi} \tag{30}$$

Equation (30) is gauge invariant in the sense that if $\psi(\vec{x})$ is a solution for some $A_\nu(\vec{x})$, then $\psi'(\vec{x}) = \psi(\vec{x})e^{i(q/\hbar)\xi(\vec{x})}$ is a solution for a different potential $A'_\nu(\vec{x}) = A_\nu(\vec{x}) + \xi_{,\nu}(\vec{x})$. This can be seen most readily by noticing that $(\hbar\partial_\mu - iqA'_\mu)\psi' = e^{i(q/\hbar)\xi}(\hbar\partial_\mu - iqA_\mu)\psi$, so the phase factor $e^{i(q/\hbar)\xi}$ in ψ' 'passes through' the operator $\hbar\partial_\mu - iqA'_\mu$, converting it, in the process, back to $\hbar\partial_\mu - iqA_\mu$, with the result that, under a gauge transformation, (30) is just multiplied by the phase factor and remains valid.

6.2 The gauge-covariant derivative

It is conventional to define the *gauge-covariant derivative operator:*

$$\boxed{D_\mu \equiv \partial_\mu - i(q/\hbar)A} \tag{31}$$

in terms of which the Schrödinger equation is

$$\boxed{D_\mu D^\mu\psi = (m^2c^2/\hbar^2)\psi} \tag{32}$$

As we will see, something rather similar appears in GR where there is a also a gauge invariance – arising from the freedom to choose one's coordinates – and we can profitably define a covariant derivative that looks superficially like that above.

6.3 Classical wave electrodynamics

As discussed in the last lecture, cosmologists are very keen on relativistic scalar fields. For the most part, they use real scalar fields, with Lagrangian densities like $\mathcal{L} = -\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}\mu^2\phi^2$ where is μ is a constant with units of spatial frequency.

Electric charge can be introduced by having a 2-component field, which can be represented as a complex field, with $\mathcal{L} = -\frac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi)^\star - \frac{1}{2}\mu^2\phi\phi^\star$. As it stands, that is rather sterile as the two field components are decoupled. But this field can, if we like, be coupled to Maxwell's electromagnetic field by replacing the ordinary derivatives ∂_μ by their gauge covariant counterparts D_μ. The field ϕ then has the same equation

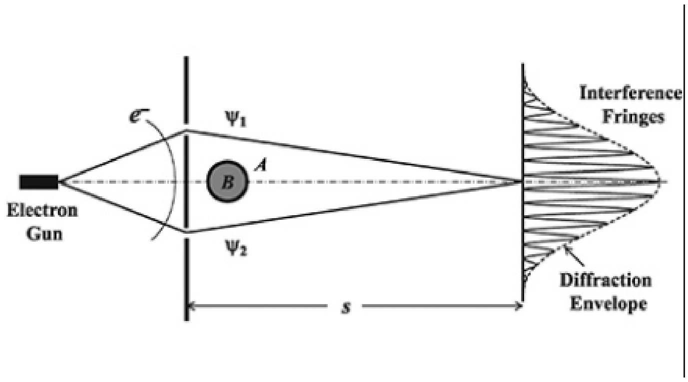

Figure 4: The Aharonov-Bohm (1959) effect is that the interference fringes in the above experimental setup will shift depending on the magnetic flux though the solenoid even though the actual **E** and **B** fields in the region that electrons explore are negligible. This shows how the magnetic potential **A** has real effects and is not simply a calculational device. Q: Does the shift of the fringes change if one makes a change of gauge? The title of the Aharonov and Bohm paper was *"Significance of Electromagnetic Potentials in the Quantum Theory".* Unknown to them, the effect had been previously analysed in 1949 by Ehrenberg and Siday, who were at pains to point out that this was a purely classical effect in electron-optics (they were interested in electron microscopes) despite the appearance of Planck's constant in the formula. Q: So what is it? Classical or quantum? Finally, our galaxy has large-scale (of order 10 kpc scale) currents generating fields of strengths on the order of $|\mathbf{B}| \sim 10^{-9}$ Tesla. Q: How rapidly does a field of that strength and scale cause the phase of an electron's wave function to rotate? How does the distance over which the phase changes by 1 radian compare to the Compton wavelength of an electron?

of motion as above, usually called the 'Klein-Gordon equation', with μ in place of $mc/\hbar$. And there is a charge and current density 4-vector which which looks superficially like the probability density and current for a QM wave function (but with $\vec{D}$ in place of $\vec{\nabla}$). But it must be kept in mind that the interpretation is very different; the field ϕ not a QM wave function. A full description of such fields does involve quantum mechanics, of course. There is a wave function, but it is not the field ϕ, rather it is a function – or rather functional – of ϕ. What the classical equations describe is the evolution of the expectation value of the field.

7 Covariant vs. non-covariant formulation of particle electrodynamics

7.1 The components of the Faraday tensor

The scalar and vector potentials were defined in terms of $\mathbf{E} = -\boldsymbol{\nabla}\varphi - \dot{\mathbf{A}}$ and $\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}$. From the former, and recalling that $A_0 = -\varphi/c$ and that $X_{,0} \equiv c^{-1}\partial X/\partial t$ (so $X_{,0}$ has the same units as $X_{,i}$), we have $E_i = c(A_{0,i} - A_{i,0}) = cF_{0i}$ which, together with $F_{00} = 0$ (as **F** is anti-symmetric), determines the top row and left column of $F_{\mu\nu}$ (remember we adopt the – standard – convention that the first index labels the rows and the second the columns). From the second we have $\mathbf{B} = (A_{[z,y]}, A_{[x,z]}, A_{[y,x]})$ which determines the spatial 3 by 3 sub-matrix of the Faraday tensor:

$$F_{\mu\nu} \equiv A_{[\mu,\nu]} = \begin{bmatrix} 0 & E_x/c & E_y/c & E_z/c \\ -E_x/c & 0 & -B_z & B_y \\ -E_y/c & B_z & 0 & -B_x \\ -E_z/c & -B_y & B_x & 0 \end{bmatrix}. \tag{33}$$

Note that in SI, the electric and magnetic fields have different units, hence the appearance of c to make all of the components of Faraday have the same units.

7.2 The Lorentz force law and the work equation

One can readily verify from (33) that the spatial components of (21) $dp_i/dt = q\dot{x}^\mu A_{[\mu,i,]}$ are the *Lorentz force law*:

$$\boxed{d\mathbf{p}/dt = q(\mathbf{E} + \dot{\mathbf{x}} \times \mathbf{B})} \tag{34}$$

while the time component $dp^0/dt = q\dot{x}^\nu F_\nu{}^0 = -q\dot{x}^\nu F_{\nu 0}$ implies the *work equation*

$$\boxed{dE/dt = q\dot{\mathbf{x}} \cdot \mathbf{E}} \tag{35}$$

where, as before, $E \equiv \gamma mc^2 = H - q\varphi$ is the mechanical (i.e. rest-mass plus kinetic) energy, and which states that it is only the **E** field that does work on a charged particle, the magnetic force being perpendicular to the particle's trajectory.

7.3 Maxwell's equations in terms of the Faraday tensor

Maxwell's equations relate **E**, **B** and the charge and current density. How are these equations expressed in terms of the Faraday tensor?

One can readily verify, directly from the definition $F_{\mu\nu} \equiv A_{[\mu,\nu]}$, the 'cyclic' identity

$$\boxed{F_{\mu\nu,\gamma} + F_{\gamma\mu,\nu} + F_{\nu\gamma,\mu} = 0} \tag{36}$$

This is also known as the *Bianchi identity*, and we will see that the analogue of this features prominently in the construction of the Einstein field equations.

Equation (36) has $4^3 = 64$ components, but the only non-trivial combinations are when all indices are different, of which there are four, depending on whether the non-appearing index is time or one of the 3 spatial components. The former yields $\boldsymbol{\nabla} \cdot \mathbf{B} = 0$ while the latter give the 3 equations $\boldsymbol{\nabla} \times \mathbf{E} + \dot{\mathbf{B}} = 0$, so (36) usefully encodes the 'homogeneous' pair of Maxwell's equations.

The 'inhomogeneous' (or 'sourced') pair of Maxwell's equations

$$\nabla \cdot \mathbf{E} = \rho/\epsilon_0 \quad \text{and} \quad \boldsymbol{\nabla} \times \mathbf{B} - c^{-2}\dot{\mathbf{E}} = \mu_0 \mathbf{j}, \tag{37}$$

where ρ is the electric charge density and $\mathbf{j}$ is the electric current density, are encoded in

$$\boxed{F^{\mu\nu}{}_{,\mu} = \mu_0 j^\nu} \tag{38}$$

where the 4-current density is

$$\boxed{j^\nu \equiv (c\rho, \mathbf{j})} \tag{39}$$

So Maxwell's equations, of which there are 8 in total, are, in terms of $F_{\mu\nu}$, given by the two 4-vector equations (36) and (38).

7.4 Maxwell's equations in the Lorentz gauge

- gauge invariance of electromagnetism is both a complication, conceptually at least, but also a convenience as, by a judicious 'choice of gauge', one can simplify Maxwell's equations.
- one particularly useful choice is the so-called 'Lorentz gauge', where we assert that
  - $\boxed{A^\mu{}_{,\mu} = 0}$
  - so we are demanding that the 4-divergence of $\vec{A}$ vanishes
- to see that one is free to impose such a restriction on the potential, imagine a solution to Maxwell's equations for which $A^\mu{}_{,\mu}(\vec{x}) = f(\vec{x}) \neq 0$
  - under a gauge transformation, $A^\mu{}_{,\mu} \to A'^\mu{}_{,\mu} = A^\mu{}_{,\mu} + \xi^{,\mu}{}_{,\mu}$
  - so if we can find a gauge function $\xi(\vec{x})$ that solves $\xi^{,\mu}{}_{,\mu} = -f$ the desired condition is satisfied
  - but such a solution can be found by, for example, writing $\xi(\vec{x})$ and $f(\vec{x})$ as Fourier synthesis
- in the Lorentz gauge, Maxwell's equations take the particularly simple form
  - $\boxed{\Box\vec{A} = -\mu_0\vec{j}}$
  - where $\Box$ is the d'Alembertian operator defined such that $\Box X \equiv \ddot{X}/c^2 - \nabla^2 X$

  - and these are convenient for solving many problems – particularly those involving radiation from accelerated charges
  - it also reveals that $\vec{A}$ is a genuine Lorentz 4-vector

- we will see that a very similar trick can be applied in GR to simplify the equations of so-called 'weak field' gravity

  - and, as in EM, we do not try to solve for the gauge transformation function. It is sufficient to know that a solution exists

- There are many other useful choices of gauge. For example, one is the so-called 'Hamiltonian gauge' in which $A^0 = 0$. Jackson and Okun have written a very nice review of the history of the EM gauge transformation.

7.5 Conservation of electric charge

The inhomogeneous Maxwell's equations tell us that the divergence of the current is $\mu_0 j^\nu{}_{,\nu} = F^{\mu\nu}{}_{,\mu\nu}$. But this is invariant if we interchange the dummy indices $\mu \leftrightarrow \nu$, i.e. $F^{\mu\nu}{}_{,\mu\nu} = F^{\nu\mu}{}_{,\nu\mu}$. And $F^{\mu\nu}$ is anti-symmetric under $\mu \leftrightarrow \nu$ so applying this to the right hand side gives $F^{\mu\nu}{}_{,\mu\nu} = -F^{\mu\nu}{}_{,\nu\mu}$. But the partial derivatives in the last term commute so we have $F^{\mu\nu}{}_{,\mu\nu} = -F^{\mu\nu}{}_{,\mu\nu}$, implying that $F^{\mu\nu}{}_{,\mu\nu} = 0$ and therefore that

$$\boxed{j^\nu{}_{,\nu} = 0} \tag{40}$$

In 3+1 form this equation says that the rate of change of the charge density ρ is minus the divergence of the current $\mathbf{j}$, so the form of Maxwell's equations *require*, in some sense, that electric charge be conserved.

This is remarkable. Maxwell's equations, along with the Lorentz force and work equations, encapsulate empirical facts (the laws of Gauss, Ampère and Faraday and the dipole nature of magnets), based on measurements of forces. Another empirical fact of electromagnetism – one that pre-dates Maxwell – is that charge does not seem to spontaneously appear or disappear, and this, together with the constancy of charge to mass ratio for elementary particles, encouraged the idea that charge is carried by particles and is an intrinsic and fixed attribute. The above result shows that this does not have to be added to the equations as an additional assumption; it is already 'built in'. In a universe where charge were not precisely conserved, the equations of EM would have to be different from Maxwell's.

Writing (40) as $\partial_t \rho + \boldsymbol{\nabla} \cdot \mathbf{j} = 0$ and defining $Q \equiv \int d^3x \rho$ we have

$$\frac{dQ}{dt} = \frac{d}{dt} \int d^3x\rho = \int d^3x \partial_t \rho = -\int d^3x \boldsymbol{\nabla} \cdot \mathbf{j} = -\int dxdydz(\partial_x j_x + \partial_y j_y + \partial_z j_z) \tag{41}$$

but $\int_{x_1}^{x_2} dx \partial_x j_x = [j_x]_{x_1}^{x_2}$ and similarly for the equivalent integrals involving $\partial_y j_y$ and $\partial_z j_z$, so the right hand side vanishes if we let the limits of integration tend to $\pm\infty$ and we have that the total charge $Q \equiv \int d^3x\rho$ is conserved. If we have a bounded charge distribution, so we can draw some surface around it where $\mathbf{j} = 0$, then the charge within the surface is constant.

7.6 Useful ways to express the 4-current density

There are various useful ways to express the 4-current for a collection of particles, as described in detail in appendix (A). One is in terms of the spatial density of particles, which we can write as a sum of 3-dimensional Dirac δ-functions: $n(\mathbf{x}, t) = \sum_P \delta^{(3)}(\mathbf{x} - \mathbf{x}_P(t))$, where the index P labels the particles. In terms of these, the 4-current is

$$j^\nu(\mathbf{x}, t) = \sum_P q_P \dot{x}^\nu_P(t) \delta^{(3)}(\mathbf{x} - \mathbf{x}_P(t)). \tag{42}$$

Note that it is *not* the 4-velocity $u^\nu_P \equiv dx^\nu_P / d\tau$ that appears here, but the non-4-vector $\dot{x}^\nu_P \equiv dx^\nu_P / dt$.

Another is in terms of the *space-time* density of particles: a set of filaments $n(\vec{x}) = \sum_P \int d\tau \delta^{(4)}(\vec{x} - \vec{x}_P(\tau))$, where $\vec{x}_P(\tau)$ is the world-line, parameterised by proper time, and in relation to which

$$j^\nu(\vec{x}) = \sum_P q_P \int d\tau u^\nu_P(\tau)_P \delta^{(4)}(\vec{x} - \vec{x}_P(\tau)). \tag{43}$$

A third is in terms of the density in 6-dimensional phase space $f(\mathbf{x},\mathbf{p},t)=\sum_P \delta^{(3)}(\mathbf{x}-\mathbf{x}_P(t))\delta^{(3)}(\mathbf{p}-\mathbf{p}_P(t))$ for which, for particles of identical charge,

$$j^\nu(\vec{x}) = q\int d^3p\,\dot{x}^\nu f(\mathbf{x},\mathbf{p},t), \tag{44}$$

with the total current being, in general, the sum of this over species of particles with different charges.

All the above are equivalent. In the appendix (A) we show how the continuity equation (40) is implicit in each of the above definitions.

7.7 Transformation of the 4-current under a Lorentz boost

The charge current-density j^ν and particle current-density n^ν are both 4-vectors (they transform like x^ν under boosts), the number of particles in a spatial volume δV is $\delta N = n^0\delta V/c$, for example, being a scalar even though both n^0 and δV are frame dependent. This is relatively easily understood; if we have a cubical volume of side L in the 'lab-frame' then, in the frame of particles moving in the x-direction relative to the lab, the x-separation between two events that occur at $x=\pm L/2$ – i.e. on opposite the ends of the cube – and at the same time – in the lab-frame – will have particle-frame x-separation enhanced by a factor γ and the particle-frame volume is γL^3. The number of particles in the box in the particle frame will be (γL^3) times the proper number density. The number of particles being Lorentz invariant it must be that the number density of particles with this velocity will be enhanced in the lab-frame by a factor γ relative to the proper number density, just like the enhancement of the time component of the particles' 4-momentum, and the mean volume per particle δV is therefore decreased by a factor $1/\gamma$ as compared to the proper value. The is consistent with the statement that the particle current density $\vec{n}$ is a 4-vector, having a proper – i.e. particle-frame – value $\vec{n}\to n^\nu = (cn_{\rm proper},0,0,0)$, so its value in the lab-frame has time component $n^0 = c\gamma n_{\rm proper}$, and so $n^0\delta V$ is frame independent.

It is very common to refer to the 4-current-density as the '4-current'. This is a little dangerous, as one might get confused with $q\dot{x}^\mu$ that appears in the Lagrangian (3), or with δJ^ν in (82), whose spatial components are an electrical current (i.e. electrical charge times velocity), but which do not transform as 4-vectors.

8 Liouville's theorem for charged particles

In terms of the canonical phase space density $f(\mathbf{x},\mathbf{P})$, conservation of particles is expressed in the *Vlasov equation:*

$$\partial_t f + \boldsymbol{\nabla}_\mathbf{x}\cdot(\dot{\mathbf{x}}f) + \boldsymbol{\nabla}_\mathbf{P}\cdot(\dot{\mathbf{P}}f) = 0 \tag{45}$$

I.e. the rate of change of density at some point in phase-space is the 6-divergence of the particle 6-current $(\dot{\mathbf{x}}f,\dot{\mathbf{P}}f)$. Equivalently, the convective derivative of f is

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \dot{\mathbf{x}}\cdot\boldsymbol{\nabla}_\mathbf{x}f + \dot{\mathbf{P}}\cdot\boldsymbol{\nabla}_\mathbf{P}f = -f[\boldsymbol{\nabla}_\mathbf{x}\cdot\dot{\mathbf{x}} + \boldsymbol{\nabla}_\mathbf{P}\cdot\dot{\mathbf{P}}]. \tag{46}$$

But Hamilton's equations ($\dot{\mathbf{P}} = -\boldsymbol{\nabla}_\mathbf{x}H$ and $\dot{\mathbf{x}} = \boldsymbol{\nabla}_\mathbf{P}H$) say the right hand side is $-f\times(\boldsymbol{\nabla}_\mathbf{x}\cdot\boldsymbol{\nabla}_\mathbf{P} - \boldsymbol{\nabla}_\mathbf{P}\cdot\boldsymbol{\nabla}_\mathbf{x})H$ which, because partial derivatives commute, vanishes and we have

$$\boxed{df/dt = 0} \tag{47}$$

which says that the 6-D density of particles in the vicinity of any chosen particle is constant.

What is generally more interesting is the mechanical phase space density $f(\mathbf{x},\mathbf{p})$, for which we have $\partial_t f + \boldsymbol{\nabla}_\mathbf{x}\cdot(\dot{\mathbf{x}}f) + \boldsymbol{\nabla}_\mathbf{p}\cdot(\dot{\mathbf{p}}f) = 0$ and

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \dot{\mathbf{x}}\cdot\boldsymbol{\nabla}_\mathbf{x}f + \dot{\mathbf{p}}\cdot\boldsymbol{\nabla}_\mathbf{p}f = -f[\boldsymbol{\nabla}_\mathbf{x}\cdot\dot{\mathbf{x}} + \boldsymbol{\nabla}_\mathbf{p}\cdot\dot{\mathbf{p}}] \tag{48}$$

We cannot now simply appeal to Hamilton's equations to show it, but this also vanishes: On the right hand side, $\boldsymbol{\nabla}_\mathbf{x}\cdot\dot{\mathbf{x}}$ is, by definition, the rate of change of $\dot{\mathbf{x}}$ with respect to position $\mathbf{x}$ at fixed $\mathbf{p}$. But $\mathbf{p} = m\dot{\mathbf{x}}/\sqrt{1-|\dot{\mathbf{x}}|^2/c^2}$ implies $\dot{\mathbf{x}} = \mathbf{p}/\sqrt{m^2+|\mathbf{p}|^2/c^2}$ which is solely a function of $\mathbf{p}$, so the rate of change of $\dot{\mathbf{x}}$ at fixed $\mathbf{p}$ is the same as rate of change of $\dot{\mathbf{x}}$ at fixed $\dot{\mathbf{x}}$, i.e. zero, so $\boldsymbol{\nabla}_\mathbf{x}\cdot\dot{\mathbf{x}} = 0$. The equation of motion is $\dot{p}_i = qcF_{0i}(\mathbf{x}) + q\dot{x}_j F_{ji}(\mathbf{x})$ so the second term in brackets on the right hand side of (48) (being, by definition, the rate of change of $\dot{\mathbf{p}}$ with respect to $\mathbf{p}$ at fixed $\mathbf{x}$), is $\boldsymbol{\nabla}_\mathbf{p} \cdot \dot{\mathbf{p}} = qF_{ji}\partial \dot{x}_j / \partial p_i$. But from $\dot{\mathbf{x}} = \mathbf{p}/\sqrt{m^2 + |\mathbf{p}|^2/c^2}$ we have $\partial \dot{x}_j / \partial p_i = (\gamma m)^{-1}(\delta_{ij} - \dot{x}_i \dot{x}_j / c^2)$ which is symmetric. But F_{ji} is anti-symmetric, so (using an exactly analogous argument to that used above, flipping dummy indices etc.) we have $\boldsymbol{\nabla}_\mathbf{p} \cdot \dot{\mathbf{p}} = 0$.

Thus, even though the Lorentz force on a particle is velocity, and hence momentum, dependent, so, in general, $\partial \dot{p}_i / \partial p_j \neq 0$, the trace of this, which is the momentum-space divergence $\boldsymbol{\nabla}_\mathbf{p} \cdot \dot{\mathbf{p}} = 0$. Therefore both terms on the right of (48) vanish, and we have, as a consequence, *Liouville's theorem:*

$$\boxed{df/dt = 0} \tag{49}$$

i.e. the phase space density along any particle trajectory for charged particles moving under the influence of an external electromagnetic field $F_{\mu\nu}(\mathbf{x}, t)$ (but ignoring their collisions with one another) is constant.

9 The stress-energy tensor in electromagnetism

9.1 The stress-energy tensor for charged particles

As well as electric charge, particles can transport other attributes. In particular they transport their 4-momentum. That means that just as there is a current-density 4-vector j^ν there is a current for the 4-momentum. But as the quantity being transported is itself a 4-vector, its current is a rank two tensor $T^{\mu\nu}$, being defined as the rate at which the ν^{th} component of 4-momentum is being transported along the μ^{th} coordinate axis. Another distinction is that the 4-momentum, unlike charge, is not, in general, constant along the particle trajectory.

9.1.1 Stress-energy in terms of the 3, 4 or 6 dimensional particle densities

This motivates one to define the 4-momentum current for particles, also known as the stress-energy tensor, simply by replacing the charge q in the formulae for the 4-current-density j^μ by the particles' 4-momenta $\vec{p} = m\vec{u}$.

The components of $T^{\mu\nu}$ give the flux density of the ν^{th} component of the 4-momentum in the μ^{th} direction, and, as with the charge 4-current density, this can be expressed in terms of the 3, 4 or 6-dimensional particle densities:

$$\begin{aligned} T^{\mu\nu} &= \sum_P \dot{x}_P^\mu(t) p_P^\nu(t) \delta^{(3)}(\mathbf{x} - \mathbf{x}_P(t)) \\ &= \sum_P \int d\tau u_P^\mu(\tau) p_P^\nu(\tau) \delta^{(4)}(\vec{x} - \vec{x}_P(t)) \\ &= \int d^3p f(\mathbf{x}, \mathbf{p}, t) \dot{x}^\mu(\mathbf{p}) p^\nu(\mathbf{p}) = \int \frac{d^3p}{p^0} f(\mathbf{x}, \mathbf{p}, t) p^\mu p^\nu. \end{aligned} \tag{50}$$

For more details see the appendix.

9.1.2 Continuity of energy and momentum

If the Faraday tensor vanishes, the 4 components of the 4-momentum of each particle are independent of time, so the four 4-current densities $T^{\mu 0}$, $T^{\mu x}$, $T^{\mu y}$ and $T^{\mu z}$, each analogous to the charge 4-current density j^μ, are all conserved, or

$$T^{\mu\nu}{}_{,\mu} = 0. \tag{51}$$

Each of the $\nu = t, x, y, z$ components of this equation are saying that the rate of change of density $T^{t\nu}$ of ν-momentum in a volume is the (integral over the volume of the) 3-divergence of the ν-momentum flux density $T^{i\nu}$.

If $F_{\mu\nu} \neq 0$, the EM field will be transferring 4-momentum to the particles at a rate, per particle, given by $\dot{p}^\nu = F_\mu{}^\nu \dot{x}_P^\mu$, so there will be an additional rate of change in the amount of ν-momentum in a volume δV, over and above that being convected in or out, given by the sum over the particles of $\sum_{P \in \delta V} \dot{p}^\nu = \sum_{P \in \delta V} \dot{x}^\mu F_\mu{}^\nu$.

So there will be an addition rate of change in the *density* of ν-momentum equal to this divided by the volume: ${F_\mu}^\nu \delta V^{-1} \sum_{P\in\delta V} \dot{x}^\mu = {F_\mu}^\nu j^\mu$ and we therefore have

$$\boxed{{T^{\mu\nu}}_{,\mu} = j^\mu {F_\mu}^\nu} \tag{52}$$

This may also be derived, more arduously, directly from the various expressions for $T^{\mu\nu}$ in (50) as shown in appendix (B). These 4 equations are essentially the work equation and the 3-components of the Lorentz force equation in density form.

If $F_{\mu\nu} = 0$, the four continuity equations ${T^{\mu\nu}}_{,\mu} = 0$ imply four globally conserved quantities $p^\nu_{\rm tot} = \int d^3x T^{t\nu}$ each analogous to the globally conserved charge Q. While if $F_{\mu\nu} \neq 0$ there will be a transfer of energy and momentum to the particles given, per unit volume, in (52) by $j^\mu {F_\mu}^\nu$, the time and space components of which are the rate-of-work density and the Lorentz force density respectively.

We could have also computed the *canonical stress-energy tensor* $T^{\mu\nu}_{\rm c}$ defined as the flux density of canonical 4-momentum P^ν. This is gauge dependent and has a divergence ${T^{\mu\nu}_{\rm c}}_{,\mu} = j^\mu {A_\mu}^{,\nu}$ with a source which is gauge dependent also. Both sides of (52), on the other hand, are gauge invariant.

9.2 The stress-energy tensor for EM radiation

9.2.1 Energy density of the electromagnetic field

Consider the work done pulling two capacitor plates – with separation along the x-axis – apart as illustrated on the left side of in figure 5. Assuming, for simplicity, a separation much less than their size of the plates, the field between the plates will be $\mathbf{E} = \hat{\mathbf{x}} E_x$ with, by Gauss's law, $E_x = \Sigma/\epsilon_0$, where Σ is the charge density. The force is $F = AE_x\Sigma/2$ with A the area of the plates – the factor $1/2$ coming from the fact that the field ramps from zero to the inter-plate value passing through the plate so the mean field is $E_x/2$ – and so the work done in increasing the separation by δx is $\delta W = AF\delta x = \epsilon_0 A\delta x E_x^2/2 = \epsilon_0 \delta V E_x^2/2$. Equating this to the change in the volume times the energy density $\mathcal{E}$ of the field gives $\mathcal{E} = \epsilon_0|\mathbf{E}|^2/2$.

It is crucial to recognise that we are here attributing the energy entirely to the field. In this process there was no change in the mechanical energy of the plates as they started and ended at rest. If we release the plates, they will gain mechanical energy at the expense of the field and field plus mechanical energy will be conserved. The *canonical* energy H of the particles in the plates, on the other hand, contains, in addition to the mechanical energy, contributions from the term $q\varphi$, so canonical particle energy plus field energy (as defined above) would not be conserved. It is somewhat arbitrary how we assign the energy; whether we say it resides in the field or whether it resides in the particles.

Similar considerations can be applied to the magnetic field. Consider a long solenoid of length L with radius r and with N turns as illustrated on the right side of in figure 5. Ampère's law says the field is $B = \mu_0 NJ/L$, where J is the current. If we ramp the current up to increase the field there will be an induced EMF (according to Faraday's law) such that $E \times 2\pi r = AdB/dt$ where $A = \pi r^2$ is the area. The power required to drive the current J against the induced electric field is $dW/dt = EJ \times 2\pi r N = \mu_0^{-1} AL \times BdB/dt$. It follows that the work done is $dW = \mu_0^{-1} V dB^2/2$, with $V = AL$ the volume, so, attributing this to the the energy density of the magnetic field gives $\mathcal{E} = \mu_0^{-1}|\mathbf{B}|^2/2$. The sum of the electric and magnetic field densities is therefore

$$\boxed{\mathcal{E} = (\epsilon_0|\mathbf{E}|^2 + \mu_0^{-1}|\mathbf{B}|^2)/2} \tag{53}$$

9.2.2 Poynting's theorem

Dotting Faraday's law $\dot{\mathbf{B}} = -\boldsymbol{\nabla} \times \mathbf{E}$ with $\mathbf{B}$ and dotting the Ampère-Maxwell equation $\dot{\mathbf{E}}/c^2 = \boldsymbol{\nabla} \times \mathbf{B} - \mu_0 \mathbf{j}$ – which we can also write as $\epsilon_0 \dot{\mathbf{E}} = \mu_0^{-1} \boldsymbol{\nabla} \times \mathbf{B} - \mathbf{j}$, since $c^2 = 1/(\epsilon_0\mu_0)$ – with $\mathbf{E}$ and adding gives

$$\partial_t(\epsilon_0|\mathbf{E}|^2 + \mu_0^{-1}|\mathbf{B}|^2)/2 = \mu_0^{-1}[\mathbf{E} \cdot (\boldsymbol{\nabla} \times \mathbf{B}) - \mathbf{B} \cdot (\boldsymbol{\nabla} \times \mathbf{E})] - \mathbf{j} \cdot \mathbf{E} \tag{54}$$

Using the identity $\boldsymbol{\nabla} \cdot (\mathbf{a} \times \mathbf{b}) = \mathbf{b} \cdot (\boldsymbol{\nabla} \times \mathbf{a}) - \mathbf{a} \cdot (\boldsymbol{\nabla} \times \mathbf{b})$ gives *Poynting's theorem:*

$$\boxed{\partial_t \mathcal{E} + \boldsymbol{\nabla} \cdot \mathbf{S} = -\mathbf{j} \cdot \mathbf{E}} \tag{55}$$

where the *Poynting vector* (or *Poynting flux*) is defined as

$$\boxed{\mathbf{S} \equiv \mu_0^{-1} \mathbf{E} \times \mathbf{B}} \tag{56}$$

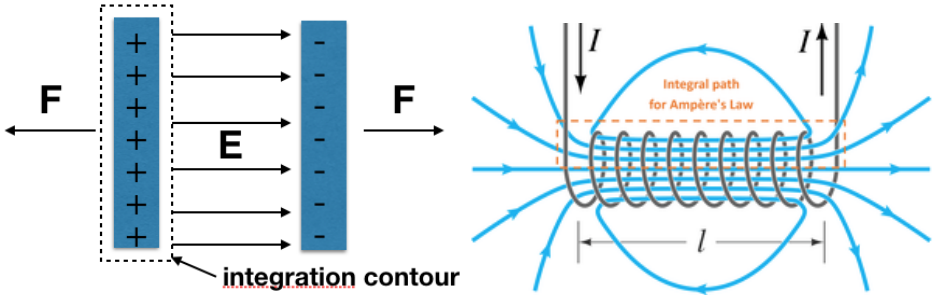

Figure 5: The energy density of an electric field can be determined by considering the work done when pulling two capacitor plates apart (see text for details) which, in the process, creates new field. The work done is proportional to the field times the charge density, but the field and charge density are linearly related – by Gauss's law – so the work done is proportional to the volume increase times the square of the field strength. The energy density of a magnetic field can be obtained by considering the work done in increasing the current flowing in a solenoid; here the increase in current causes an increase in the **B**-field, and hence in the magnetic flux through the solenoid. This induces an EMF opposing the change and the energy going into the field is the work we need to do to keep the current flowing in opposition to the induced **E**-field.

The right hand side of (55) is minus the rate at which the particles are gaining energy from the field. So Poynting's theorem expresses conservation of total energy in which we identify the Poynting flux with the energy flux density.

9.2.3 Maxwell's electromagnetic stress tensor

Considering again a capacitor, now static with its plates kept apart by being pulled by springs as illustrated in figure 6. It is evident that since the flux of momentum must be continuous then, as there is a flux of momentum in the stretched springs, there must be a flux associated with the **E**-field between the plates also. If we let the separation of the plates $\mathbf{d}$ (and therefore also the field $\mathbf{E}$) be parallel to $\hat{\mathbf{x}}$ and their area be A, then the momentum flux is $-\epsilon_0 A E_x^2/2$. This is negative since the springs (and the field also) are in tension, so the spring at negative $\mathbf{x}$ is delivering negative p_x to the plate at $-\mathbf{d}/2$; this is transferred within the plate to the field (the plates being static so there being no change of their momentum) and this is delivered to the plate at $+\mathbf{d}/2$, which it, in turn, transfers to the spring at positive $\mathbf{x}$. The rate at which x-momentum is being transported along the x-axis per unit area (i.e. the xx-component of the stress – or momentum flux density – tensor) is, in this situation, $T_{xx} = -\epsilon_0 E_x^2/2$.

If we consider a charged sphere, then, at the North pole, there is a field in the z-direction, so there is a (negative) flux density of z-momentum. But that flux density falls off as the square of the field strength, or as $1/r^4$. This may seem strange as this does not look like momentum is being conserved since the flux density times an element of area dA subtending a certain solid angle $d\Omega$ at the centre of the sphere is falling off like $1/r^2$. What this is telling us is that there is, in fact, not just a flux of z-momentum along the field lines, there must be z-momentum flowing in the direction perpendicular to the field.

To elucidate this, consider a skinny cylinder of radius r_p and height δh sitting atop the North pole. From the foregoing, more z-momentum is flowing out of this volume through its bottom than is flowing in through its top. If that were all the momentum transport, the amount of z-momentum in the volume would be changing at a rate $\dot{p}_z \equiv dp_z/dt = \delta(\pi r_p^2 \times \epsilon_0 E_z^2(r)/2)$. The field is $E_z(r) = E_z \times r_s^2/r^2$ (where $E_z = E_z(r_s)$) so $\dot{p}_z = \pi \epsilon_0 r_s^4 r_p^2 E_z^2 \delta(1/2r^4)$. Assuming $\delta h \ll r_s$ this is $\dot{p}_z = -2\pi\epsilon_0 r_s^{-1} r_p^2 \delta h E_z^2$. For momentum to be conserved, there must evidently be a negative flux of z-momentum flowing out the sides of the cylinder (i.e. z-momentum flowing in).

The area of the side of the plug is $2\pi r_p \delta h$ so in modulus the flux density of z-momentum inward at the wall must be rate of change of momentum divided by area, or $|\dot{p}_z|/(2\pi r_p \delta h) = (r_p/r_s)E_z^2$. This must point towards the polar axis, or (assuming positive E_z) anti-parallel to the vector $\mathbf{E}_\perp \equiv (E_x, E_y, 0) = (x_p, y_p, 0) \times E_z/r_s$. It follows that the three components of the flux density of z-momentum are

$$T_{iz} = -\epsilon_0 E_z (E_x, E_y, E_z/2). \qquad (57)$$

That is in the vicinity of the N-pole, and it tells us, to a precision that is 1st order in E_x and E_y, what is the z-momentum flux density at locations where $\mathbf{E}$ is not exactly along the z-axis. To obtain the general

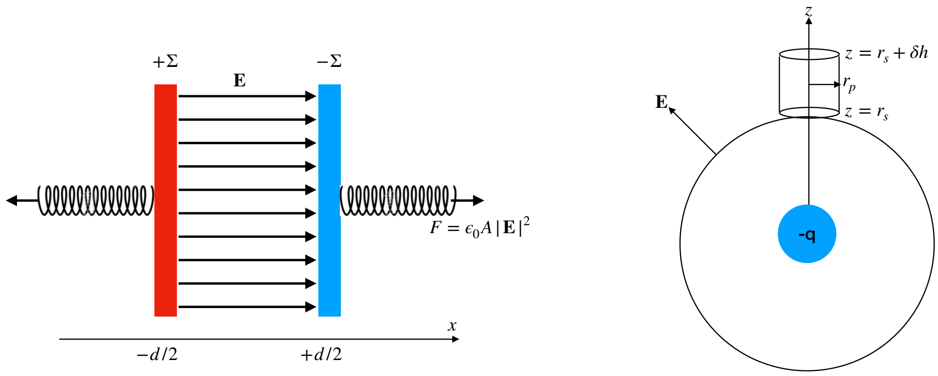

Figure 6: On the left is shown a capacitor with the plates being held apart by springs. The right-hand spring is delivering positive x-momentum to the right-hand plate. So positive x-momentum is flowing towards *negative* x in that spring. I.e. there is a negative flux of x-momentum in the right spring. There is also a negative flux of x-momentum in the left spring, as it is delivering negative x-momentum to the left plate. That is to say negative x-momentum is flowing towards positive x in the left spring. For momentum to be conserved there must also be a negative flux (in the x-direction) of x-momentum in the field between the plates. The density of that x-directed x-momentum flux is $T_{xx} = -\epsilon_0|\mathbf{E}|^2/2$. On the right we have a negative charge producing a radial electric field that falls of as $1/r^2$. Along the N-polar axis (the z-axis) there is a negative flux of z-momentum in the z-direction (just as for the field between the capacitor plates on the left). So z-momentum is flowing out of the bottom of the 'plug' and in to the top of the plug. But as the flux density falls off as $1/r^4$ there is more momentum flowing out of the bottom than in at the top. But the situation is static; there can be no 'build-up' of momentum anywhere. This enables us to compute the fluz of z-momentum in through the walls of the plug and hence determine the stress tensor.

expression for the electric field stress we may ask: what *tensor*, that is quadratic in the field (i.e. whose components are products of components of the field) agrees with the above expression? The answer is:

$$\boxed{T_{ij} = -\epsilon_0(E_i E_j - \tfrac{1}{2}\delta_{ij}|\mathbf{E}|^2)} \tag{58}$$

which we can confirm by inspection.

An alternative route to (58) is simply to argue that a) it agrees with $T_{zz} = -\epsilon_0 E_z^2/2$ on the polar axis and b) it is divergence free.

For the case of an E-field along the x-axis, equation (58) tells us the stress is $T_{ij} = \frac{1}{2} E_x^2 \text{diag}(-1, 1, 1)$ so there is tension along the field line and pressure in the perpendicular direction (i.e. positive flux of y-momentum in the y-direction and similarly for z). This can be understood qualitatively in terms of the work needed (or released) if we change the field configuration. The tension along the field lines follows from the fact that pulling the capacitor plates apart requires energy whereas if we were to increase the area of the plates for a fixed amount of charge the field strength times the area would be constant so the total field energy (being the field energy density times the area) would go down and this would release energy.

It is interesting to take the divergence of (58). Assuming $\dot{\mathbf{A}} = 0$ (so the magnetic field, if any, is static) we have $E_i = -\varphi_{,i}$, so the divergence of the $j^{\rm th}$ component of (58) is $T_{ij,i} = \epsilon_0(-\varphi_{,ii}\varphi_{,j} - \varphi_{,i}\varphi_{,ji} + \delta_{ij}\varphi_{,k}\varphi_{,ki})$. The last two terms cancel, so we have $T_{ij,i} = -\epsilon_0\varphi_{,ii}\varphi_{,j} = \epsilon_0 E_j \nabla^2\varphi = \rho E_j$, where in the last step we invoked Poisson's equation. This vanishes if there are no charges present, and the momentum flux density is then divergence-free. Otherwise, if integrated over a region containing charges, this is equal to the rate at which those charges are losing j-momentum. Invoking the divergence theorem, this says that the inward flux of field j-momentum across a surface (being minus the integral of the divergence of the j-momentum flux density over the enclosed volume) is equal to the rate at which charges in the volume are gaining j-momentum as a result of the presence of the E-field.

Very similar considerations apply to magnetic fields. On the left in figure 7 is illustrated how there is an outward force on the coils of a solenoid. There is radial momentum flowing into the coils from the interior. In the interior there must therefore be a positive transverse stress – a *pressure* – perpendicular to the field lines. As with an electric fields there is *tension* along the field lines.

Q: As this is static, you might want to ask yourself where does the radial momentum flowing into the coil go?

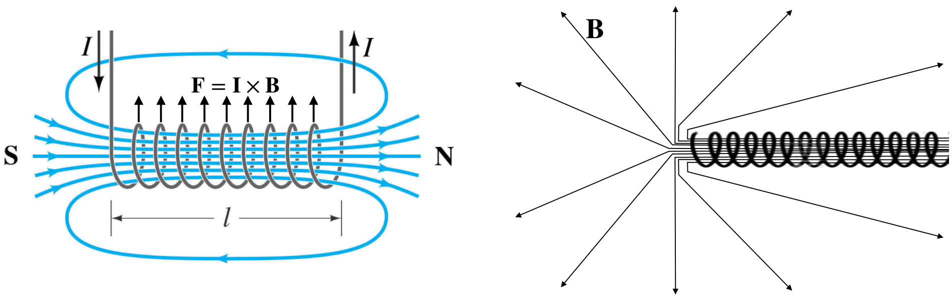

Figure 7: For the solenoid on the left, there is a strong field inside the solenoid, which ramps down to close to zero on the outside. It follows that there is a $\mathbf{j} \times \mathbf{B}$ force acting on the wire carrying the current. This force is outward directed. This is known to people who make high-power solenoids; they need to have straps around them so stop the solenoid exploding. Thus there is momentum being delivered to the coils in a direction perpendicular to the field. By continuity, there is a flux of radial momentum in the outward radial direction within the solenoid. Unlike the momentum transport along the field lines, which like that of a spring in *tension*, is negative, this is positive; i.e. it is a *pressure*. On the right is one end of a very long skinny solenoid, the field outside of which is identical, in the limit, to that of a monopole. The argument used to calculate the stress tensor for an electric field can be carried over directly to this situation.

One might also, for instance, consider the stresses in the monopole-like field around one end of a long skinny solenoid as illustrated in the right hand side of figure 7. This can be analysed exactly as for the radial electric field

Such considerations led Maxwell to his electromagnetic stress tensor $\boldsymbol{\sigma}$, which is, aside from a minus sign, the sum of (58) and a identical expression for the magnetic field with $\mathbf{E}$ replaced by $\mathbf{B}$ so the combined stress tensor is

$$\boxed{T_{ij} = -\sigma_{ij} = -\epsilon_0(E_i E_j - \tfrac{1}{2}\delta_{ij}|\mathbf{E}|^2) - \mu_0^{-1}(B_i B_j - \tfrac{1}{2}\delta_{ij}|\mathbf{B}|^2)} \tag{59}$$

9.2.4 The momentum density of the electromagnetic field

Poynting's theorem suggests that the left (i.e. energy) column of the stress energy tensor for radiation has components $T_{\rm r}^{\mu t} = (\mathcal{E}, \mathbf{S}/c)$ since then $T_{\rm r}^{\mu 0}{}_{,\mu} = \frac{1}{c}T_{\rm r}^{00}{}_{,0} + T_{\rm r}^{i0}{}_{,i} = \frac{1}{c}(\dot{\mathcal{E}} + \boldsymbol{\nabla}\cdot\mathbf{S})$ so Poynting's theorem, with the left hand side expressed in terms of the stress-tensor, is $T_{\rm r}^{\mu 0}{}_{,\mu} = -\mathbf{j}\cdot\mathbf{E}/c$.

But note that $\mathbf{j}\cdot\mathbf{E}/c = j^\mu F_\mu{}^0 = T_{\rm m}^{\mu 0}{}_{,\mu}$ where $T_{\rm m}^{\mu 0}$ is the energy column of the mechanical stress energy tensor. So the sum of the 4-divergences of the energy columns for the particles and the field vanishes:

$$T_{\rm r}^{\mu 0}{}_{,\mu} + T_{\rm m}^{\mu 0}{}_{,\mu} = 0 \tag{60}$$

and consequently the total (mechanical plus field) energy $E = \int d^3r\ (T_{\rm r}^{00} + T_{\rm m}^{00})$ is conserved. So that all makes sense.

Maxwell's stress tensor $-\boldsymbol{\sigma}$ provides the spatial parts of $T_{\rm r}^{\mu\nu}$; the momentum flux density. Note that σ has the same units as energy density (i.e. T^{00}) so there are no factors of c here. All that remains to be determined is the spatial part of the top row (i.e. the momentum density). One might guess that, like $T_{\rm m}^{\mu\nu}$, the radiation stress tensor $T_{\rm r}^{\mu\nu}$ should be symmetric, so $T_{\rm r}^{t\mu} = (\mathcal{E}, \mathbf{S}/c)$ and hence the momentum density (times c) be simply the Poynting (energy) flux density $\mathbf{S}$ (divided by c).

That is correct. With $T_{\rm r}^{0j} = S_j/c$ and $T_{\rm r}^{ij} = -\sigma_{ij}$, the 4-divergence of the $j^{\rm th}$ column turns out to be $T_{\rm r}^{\mu j}{}_{,\mu} = -(\rho\mathbf{E} + \mathbf{j}\times\mathbf{B})_j$, which is minus the Lorentz force density. And this is $-j^\mu F_\mu{}^j$ which is $-T_{\rm m}^{\mu j}{}_{,\mu}$. So the total of the $j^{\rm th}$ component of spatial momentum is conserved.

To prove this, consider first the time-time component $T_{\rm r}^{0j}{}_{,0} = \frac{1}{c}\partial_t T_{\rm r}^{0j} = \dot{S}_j/c^2$. But

$$\dot{\mathbf{S}} = \partial_t(\mathbf{E}\times\mathbf{B})/\mu_0 = \mu_0^{-1}(\dot{\mathbf{E}}\times\mathbf{B} + \mathbf{E}\times\dot{\mathbf{B}}) \tag{61}$$

but, from Maxwell's equations (1), $\dot{\mathbf{B}} = -\boldsymbol{\nabla}\times\mathbf{E}$ and $\dot{\mathbf{E}} = c^2(\boldsymbol{\nabla}\times\mathbf{B} - \mu_0\mathbf{j})$, so this is

$$\dot{\mathbf{S}} = -c^2\mathbf{j}\times\mathbf{B} - \mu_0^{-1}(\mathbf{E}\times(\boldsymbol{\nabla}\times\mathbf{E}) - c^2\mathbf{B}\times(\boldsymbol{\nabla}\times\mathbf{B})) \tag{62}$$

Next, using the identity $\mathbf{a}\times(\boldsymbol{\nabla}\times\mathbf{a}) = \frac{1}{2}\boldsymbol{\nabla}|\mathbf{a}|^2 - (\mathbf{a}\cdot\boldsymbol{\nabla})\mathbf{a}$ we find, in component form,

$$T_{\rm r}^{0j}{}_{,0} = c^{-2}\dot{S}_j = -(\mathbf{j}\times\mathbf{B})_j + \epsilon_0(E_i E_{j,i} - E_i E_{i,j}) + \mu_0^{-1}(B_i B_{j,i} - B_i B_{i,j}). \tag{63}$$

The rest of the 4-divergence $T_{\rm r}^{\mu j}{}_{,\mu}$ is the spatial divergence part. This is

$$T_{\rm r}^{ij}{}_{,i} = -\sigma_{ij,i} = \epsilon_0(E_i E_{i,j} - E_j E_{i,i} - E_i E_{j,i}) + \mu_0^{-1}(B_i B_{i,j} - B_j B_{i,i} - B_i B_{j,i}). \tag{64}$$

Adding (63) and (64), four of the terms in common cancel, while $B_{i,i} = \nabla \cdot \mathbf{B} = 0$ and $E_{i,i} = \nabla \cdot \mathbf{E} = \rho$, so we have, finally, for $T_{\rm r}^{\mu j}{}_{,\mu} = T_{\rm r}^{tj}{}_{,t} + T_{\rm r}^{ij}{}_{,i}$:

$$T_{\rm r}^{\mu j}{}_{,\mu} = -(\rho \mathbf{E} + \mathbf{j} \times \mathbf{B})_j = -j^\mu F_\mu{}^j. \tag{65}$$

The above equation is the analogue for the 3-momentum of Poynting's theorem for energy conservation.

We have thus determined that the stress-energy tensor for the radiation is

$$\boxed{T_{\rm r}^{\mu\nu} = \begin{bmatrix} \mathcal{E} & \mathbf{S}/c \\ \mathbf{S}/c & -\boldsymbol{\sigma} \end{bmatrix}} \tag{66}$$

the left column coming from Poynting's theorem and consisting of the field energy density $\mathcal{E} = (|\mathbf{E}|^2 + |\mathbf{B}|^2)/2$ and the energy flux density $\mathbf{S}$ divided by c. The upper right is the 3-momentum density (times c to get units of energy density), and is also equal to $\mathbf{S}/c$, and the bottom right is the flux density of 3-momentum: (minus) the Maxwell stress. We will return to this presently. Next, however, we will see how this emerges from a Lagrangian field theory approach.

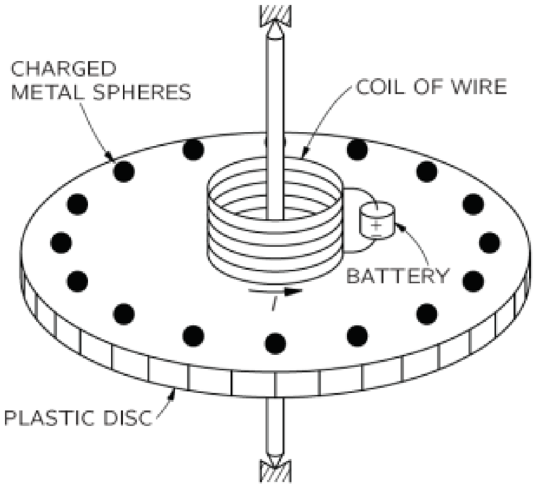

Figure 8: The Feynman disk paradox. In Feynman's *Lectures on Physics* (vol 2 17-4) he presents the above diagram and gives two conflicting arguments as to what will happen when the battery runs down. The first is that there is a magnetic flux threading the ring of charges and so if the current stops there will be an induced electromotive force that will act on the charges and the disk will spin up. The second is that angular momentum is conserved, so the (initially non-spinning) disk will remain at rest. He adds *"We should also warn you that the solution is not easy, nor is it a trick. When you figure it out, you will have discovered an important principle of electromagnetism"*

9.2.5 The Lagrangian for electromagnetism in the presence of charges

The Lagrangian (3) is that of a free particle $L = -m/\gamma$ plus an interaction term $L_{\rm int} = q\dot{x}^\mu A_\mu$. Generalising to a collection of particles, or a continuous distribution of charge, we have $L_{\rm int} = \sum q\dot{x}^\mu A_\mu = \int d^3x A_\mu q \int d^3p f(\mathbf{x}, \mathbf{p})\dot{x}^\mu = \int d^3x A_\mu j^\mu$. So $L_{\rm int} = \int d^3x \mathcal{L}_{\rm int}$ with interaction Hamiltonian density

$$\mathcal{L}_{\rm int} = j^\mu A_\mu. \tag{67}$$

To this we need to add the free-field Lagrangian density for radiation which, it turns out, is

$$\mathcal{L}(A_{\alpha,\beta}) = -\frac{1}{4\mu_0} F^{\mu\nu} F_{\mu\nu}. \tag{68}$$

which, as indicated, only depends on derivatives of the potential and not on the potential itself, and which, in terms of the EM fields is

$$\mathcal{L}(A_{\alpha,\beta}) = \tfrac{1}{2}(\epsilon_0|\mathbf{E}|^2 - \mu_0^{-1}|\mathbf{B}|^2) \tag{69}$$

The Lagrangian density for the electromagnetic field in the presence of a current $j^\mu(\vec{x})$

$$\mathcal{L}(A_\alpha, A_{\alpha,\beta}, x^\gamma) = -\tfrac{1}{4\mu_0} F^{\mu\nu} F_{\mu\nu} + j^\mu A_\mu. \tag{70}$$

with the dependence on x^γ because we are considering this to be the Lagrangian density for the EM field in the presence of some given current density $\vec{j}(\vec{x})$.

To justify (70) we may note that from this and the definition $F_{\mu\nu} \equiv A_{\mu,\nu} - A_{\nu,\mu}$ we find $\partial\mathcal{L}/\partial A_{\alpha,\beta} = F^{\beta\alpha}$ while $\partial\mathcal{L}/\partial A_\alpha = j^\alpha$ so the Euler-Lagrange equations

$$\frac{\partial}{\partial x^\beta}\left(\frac{\partial\mathcal{L}}{\partial A_{\alpha,\beta}}\right) = \frac{\partial\mathcal{L}}{\partial A_\alpha} \tag{71}$$

become simply ${F^{\beta\alpha}}_{,\beta} = \mu_0 j^\alpha$, which are the inhomogeneous Maxwell's equations.

9.2.6 The canonical stress-energy tensor for the radiation

To obtain a continuity equation for energy and momentum of the radiation à la Noether we take the partial derivative of $\mathcal{L}(x^\gamma) = \mathcal{L}(A^\alpha(x^\gamma), {A^\alpha}_{,\beta}(x^\gamma), x^\gamma)$ with respect to x^ν. Using the chain rule gives

$$\frac{\partial\mathcal{L}(\vec{x})}{\partial x^\nu} = \frac{\partial\mathcal{L}}{\partial A_\alpha} A_{\alpha,\nu} + \frac{\partial\mathcal{L}}{\partial A_{\alpha,\mu}} A_{\alpha,\mu\nu} + \frac{\partial\mathcal{L}}{\partial x^\nu} \tag{72}$$

where, just to be clear, the last term represents the derivative of $\mathcal{L}(A_\alpha, A_{\alpha,\beta}, x^\gamma)$ with respect to its final argument keeping A_α and $A_{\alpha,\beta}$ fixed, and is equal to ${j^\alpha}_{,\nu} A_\alpha$ since the only *explicit* functional dependence of $\mathcal{L}$ on $\vec{x}$ is through the current $j^\alpha(\vec{x})$.

Eliminating $\partial\mathcal{L}/\partial A_\alpha$ from this using (71), and using $A_{\alpha,\mu\nu} = \partial_\mu A_{\alpha,\nu}$, the first two terms on the right combine, as usual, to become the derivative of a single product, so

$$\partial_\nu \mathcal{L}(x^\gamma) = \partial_\mu\left(A_{\alpha,\nu}\frac{\partial\mathcal{L}}{\partial A_{\alpha,\mu}}\right) + \frac{\partial\mathcal{L}}{\partial x^\nu} = \mu_0^{-1}\partial_\mu\left(A_{\alpha,\nu}F^{\mu\alpha}\right) + {j^\alpha}_{,\nu} A_\alpha. \tag{73}$$

Finally, using $\partial_\nu \mathcal{L}\vec{x} = \delta^\mu_\nu \partial_\mu \mathcal{L}(\vec{x}) = -\delta^\mu_\nu \partial_\mu(\frac{1}{4\mu_0} F^{\alpha\beta}F_{\alpha\beta} - j^\alpha A_\alpha)$, we obtain the continuity equation

$$T^{\mu}_{\mathrm{c}\ \nu,\mu} = -j^\mu A_{\mu,\nu} \tag{74}$$

where the *canonical stress tensor for radiation* is

$$\boxed{T^{\mu}_{\mathrm{c}\ \nu} \equiv \mu_0^{-1}(F^{\alpha\mu}A_{\alpha,\nu} - \tfrac{1}{4}\delta^\mu_\nu F^{\alpha\beta}F_{\alpha\beta})} \tag{75}$$

9.2.7 The symmetric stress-energy tensor for the radiation

The stress-energy tensor (75) is, like the canonical stress-energy tensor for particles, gauge-dependent. This might not seem surprising since we have obtained this from a Lagrangian density (70) containing a gauge-dependent interaction term. But even if we remove the interaction term, and start with the gauge invariant free-field electromagnetic Lagrangian density $\mathcal{L} = -\frac{1}{4\mu_0}F_{\mu\nu}F^{\mu\nu}$ alone, we still end up with the gauge dependent stress-energy tensor (75). Another unsatisfactory feature of (75) is that the source-term for its divergence in (74) is also gauge-dependent. It is in fact, however, minus the source term for the canonical stress tensor for the particles, so the sum of the canonical stress-energy tensors for the radiation and particles has a vanishing divergence; i.e. the total canonical 4-momentum is conserved. Also, it does not, at first sight, seem to agree with what one might expect from Poynting's theorem, but again that is perhaps not surprising as what appears in that theorem is $\mathbf{j}\cdot\mathbf{E}$ which is the rate at which the mechanical, rather than the canonical, energy of the particles is changing.

These unsatisfactory features are readily avoidable. This is because the continuity equation (74), while valid, does not uniquely specify the stress-energy tensor. It is possible to modify the radiation stress-energy tensor without affecting the continuity equation, and, in the process get rid of these problems.

Imagine we were to add to $T_{c}{}^{\mu}{}_{\nu}$ an additional term of the form $\partial_\alpha(C^{\mu\alpha}{}_\nu)$. Then, when we take the divergence, we get an extra term $\partial_\mu\partial_\alpha(C^{\mu\alpha}{}_\nu)$. If $C^{\mu\alpha}{}_\nu$ is anti-symmetric under $\alpha \leftrightarrow \mu$ then, since $\partial_\mu\partial_\alpha$ is symmetric, the extra divergence will vanish. Looking at the (gauge-dependent) first term in (75) suggests that we might want to try something like $C^{\mu\alpha}{}_\nu = \frac{1}{\mu_0}F^{\mu\alpha}A_\nu$. This is indeed anti-symmetric under $\alpha \leftrightarrow \mu$ and would add to $T_{c}{}^{\mu}{}_{\nu}$ a divergence-free contribution

$$\partial_\alpha C^{\mu\alpha}{}_\nu = \tfrac{1}{\mu_0}(F^{\mu\alpha}A_{\nu,\alpha} + F^{\mu\alpha}{}_{,\alpha}A_\nu) = \tfrac{1}{\mu_0}F^{\mu\alpha}A_{\nu,\alpha} - j^\mu A_\nu \tag{76}$$

where we have used the inhomogeneous Maxwell's equations: $F^{\alpha\mu}{}_{,\alpha} = \mu_0 j^\mu$. The first term here is $\frac{1}{\mu_0}F^{\mu\alpha}A_{\nu,\alpha} = -\frac{1}{\mu_0}F^{\alpha\mu}A_{\nu,\alpha}$ which, when combined with $\frac{1}{\mu_0}F^{\alpha\mu}A_{\alpha,\nu}$ in (75), gives the gauge invariant product $\frac{1}{\mu_0}F^{\alpha\mu}F_{\alpha\nu}$.

Adding $\partial_\alpha C^{\mu\alpha}{}_\nu + j^\mu A_\nu = \frac{1}{\mu_0}F^{\mu\alpha}A_{\nu,\alpha}$ to (75) gives the symmetric stress-tensor

$$\boxed{T_{r}{}^{\mu}{}_{\nu} \equiv \tfrac{1}{\mu_0}F^{\mu\alpha}F_{\nu\alpha} - \tfrac{1}{4\mu_0}\delta^\mu_\nu F^{\alpha\beta}F_{\alpha\beta}} \tag{77}$$

while changing its divergence, on the right hand side of (74), from $-j^\mu A_{\mu,\nu}$ to $-j^\mu A_{\mu,\nu} + \partial_\mu(j^\mu A_\nu) = -j^\mu F_{\mu\nu}$ (invoking charge conservation $j^\mu{}_{,\mu} = 0$) so, on raising the index ν,

$$\boxed{T_{r}^{\mu\nu}{}_{,\mu} = -j^\mu F_\mu{}^\nu} \tag{78}$$

So $T_{r}^{\mu\nu}$ is a symmetric, gauge-invariant tensor that depends only on the radiation fields in $F_{\mu\nu}$, and has a 4-divergence (78) with a source term which is gauge invariant also. Moreover, this source term is just the opposite to that which sources the divergence of the mechanical stress (52), so

$$\boxed{T_{r}^{\mu\nu}{}_{,\mu} + T_{m}^{\mu\nu}{}_{,\mu} = 0} \tag{79}$$

so whatever energy and momentum is given up by the radiation appears in the stress tensor for the matter and vice versa, the combination $T_{r}^{\mu\nu} + T_{m}^{\mu\nu}$ being conserved. The time component of the above equation expresses conservation of total (field plus mechanical) energy – in fact it is just Poynting's theorem – and the spatial components are the expression of Newton's law of action being equal and opposite to reaction.

If we compute the components of $T_{r}^{\mu\nu}$ in terms of $\mathbf{E}$ and $\mathbf{B}$ (see appendix C), we find that they are identical to what we found before (66) from Poynting's theorem and its analogue for momentum.

We arrived at (77) and (78) from Noether's theorem; i.e. by taking the derivative of the Lagrangian density with respect to the space-time coordinates. This actually led us to the canonical stress tensor, which we then had to massage to obtain the symmetric, gauge-invariant version. A much simpler alternative would have been to *postulate* (77) based on Poynting's theorem. Directly taking its derivative gives $T_{r}{}^{\mu}{}_{\nu,\mu} = \frac{1}{\mu_0}(F^{\mu\alpha}{}_{,\mu}F_{\nu\alpha} + F^{\mu\alpha}F_{\nu\alpha,\mu} - \frac{1}{2}F^{\alpha\beta}F_{\alpha\beta,\nu})$. The first term on the right is, from Maxwell's equations, equal to $-j^\alpha F_{\alpha\nu}$, so the other terms must vanish. To show this we replace the dummy index μ by α in the second term, so the last two terms become $-\frac{1}{2\mu_0}F^{\alpha\beta}[2F_{\nu\alpha,\beta} + F_{\alpha\beta,\nu}]$ and invoking the definition of $F_{\alpha\beta} \equiv A_{[\alpha,\beta]}$ this is $-\frac{1}{2\mu_0}F^{\alpha\beta}[2A_{\nu,\alpha\beta} - (A_{\alpha,\nu\beta} + A_{\beta,\nu\alpha})]$ where by inspection $[\ldots]$ is symmetric under $\alpha \leftrightarrow \beta$ so this, when contracted with the anti-symmetric $F^{\alpha\beta}$ vanishes and we obtain (78).

A The 4-current density in terms of 3, 4 and 6 dimensional particle densities

A.1 The 4-current density in terms of the density in 3D space

The space-density of a collection of point-like particles with label P and trajectories $x_P(t)$ is a sum of 3-dimensional Dirac δ-functions: $n = \sum_P \delta(\mathbf{x} - \mathbf{x}_P(t))$. It is indeed a density as it has the essential property that its integral over some volume is the number of particles contained therein.

For a *fluid* – i.e. a dense collection of particles whose velocities are some function of space $\mathbf{v}(\mathbf{x}, t)$ – the flux of particles across a surface $d\mathbf{A}$ is $n\mathbf{v} \cdot d\mathbf{A}$. The rate of change of the number of particles $\delta N = n\delta V$ in a volume δV with time is the sum of the inward fluxes across the surfaces. This gives $\partial_t n = -\boldsymbol{\nabla} \cdot (n\mathbf{v})$ or $n^\nu{}_{,\nu} = 0$ where $n^\nu \equiv n \times (c, \mathbf{v})$. For a *gas*, where there is, in general, a distribution of velocities at any position, the particle flux is still $n\mathbf{v} \cdot d\mathbf{A}$, but with $\mathbf{v}$ the mean velocity.

The 4-current n^ν (which we will see transforms as a 4-vector) can be defined as

$$n^\nu(\mathbf{x},t) \equiv \sum_P \dot{x}_P^\nu(t)\delta^{(3)}(\mathbf{x}-\mathbf{x}_P(t)). \tag{80}$$

If we integrate $\mathbf{n}$, the spatial part of this, over some volume δV then we get $\sum\limits_{P\in\delta V} \dot{\mathbf{x}}_P$, while, if we integrate n^0 over the same volume we get $\sum\limits_{P\in\delta V} c$, which is ($c$ times) the number of particles. Dividing these gives the mean velocity $\mathbf{v} = \langle\dot{\mathbf{x}}\rangle = \sum\limits_{P\in\delta V}\dot{\mathbf{x}}_P/\sum\limits_{P\in\delta V} 1 = c\mathbf{n}/n^0$, so the spatial components are indeed $\mathbf{n} = n\mathbf{v}$ and, since $\dot{x}^0 = d(ct)/dt = c$, the time component is $n^0 = cn$.

For a collection of particles with electric charges q_P, the charge 4-current-density can be defined similarly as the sum over particles weighted by their charge q_P:

$$j^\nu(\mathbf{x},t) = \sum_P q_P \dot{x}_P^\nu(t)\delta^{(3)}(\mathbf{x}-\mathbf{x}_P(t)). \tag{81}$$

We could, of course, replace the argument t of $\mathbf{x}_P$ and $\dot{x}_P^\nu$ by some other parameter along the path such as the proper time τ and replace $\dot{x}_P^\nu(t) \rightarrow \dot{x}_P^\nu(\tau) = \dot{x}_P^\nu(\tau(t))$ and $\mathbf{x}_P(t) \rightarrow \mathbf{x}_P(\tau) = \mathbf{x}_P(\tau(t))$.

If we integrate (81) over a spatial volume δV we obtain

$$\delta J^\nu = (c\delta Q, \delta\mathbf{J}) = \int\limits_{\delta V} d^3x j^\nu = \sum_{P\in\delta V} q_P\dot{x}_P^\nu = \sum_{P\in\delta V} q_P \times (c, \dot{\mathbf{x}}_P) \tag{82}$$

whose time component is the charge in the volume and whose spatial components are an element of current à la Biot and Savart. The charge/current element δJ^ν may look like a 4-vector, but it isn't. It doesn't transform properly under boosts.

We constructed n^ν so that it would obey ${n^\nu}_{,\nu} = 0$ by appealing to properties of fluids. Alternatively, we can show that ${n^\nu}_{,\nu} = 0$ is implicit in the definition (80) above as follows. Consider the particle with label P. Its contribution to $\vec{n}$ has time component $n_P^0 = c\delta^{(3)}(\mathbf{x}-\mathbf{x}_P(t))$ which is a function of $\mathbf{x}$ and $\mathbf{x}_P(t)$ so its partial derivative with respect to t at fixed $\mathbf{x}$ is, from the chain rule, $\partial_t n_P^0 = c\dot{\mathbf{x}}_P\cdot\boldsymbol{\nabla}_{\mathbf{x}_P}\delta^{(3)}(\mathbf{x}-\mathbf{x}_P(t))$. But $\boldsymbol{\nabla}_{\mathbf{x}_P}\delta^{(3)}(\mathbf{x}-\mathbf{x}_P(t)) = -\boldsymbol{\nabla}_{\mathbf{x}}\delta^{(3)}(\mathbf{x}-\mathbf{x}_P(t))$, so

$$\partial_t n_P^0 = -c\dot{\mathbf{x}}_P\cdot\boldsymbol{\nabla}_{\mathbf{x}}\delta^{(3)}(\mathbf{x}-\mathbf{x}_P(t)). \tag{83}$$

The spatial components of this particle's contribution to $\vec{n}$ are $\mathbf{n}_P = \dot{\mathbf{x}}_P(t)\delta^{(3)}(\mathbf{x}-\mathbf{x}_P(t))$. In the divergence of this, $\boldsymbol{\nabla}_{\mathbf{x}}\cdot\mathbf{n}_P$, the partial derivative operator – being the derivatives with respect to components of $\mathbf{x}$ at fixed t – acts only on the factor $\delta^{(3)}(\mathbf{x}-\mathbf{x}_P(t))$, so

$$\boldsymbol{\nabla}_{\mathbf{x}}\cdot\mathbf{n}_P = \dot{\mathbf{x}}_P(t)\cdot\boldsymbol{\nabla}_{\mathbf{x}}\delta^{(3)}(\mathbf{x}-\mathbf{x}_P(t)). \tag{84}$$

Comparing these shows that $\boldsymbol{\nabla}_{\mathbf{x}}\cdot\mathbf{n}_P = -\frac{1}{c}\partial_t n_P^0$, so, for a single particle,

$$\tfrac{1}{c}\partial_t n_P^0 + \boldsymbol{\nabla}\cdot\mathbf{n}_P = n_{P,\nu}^\nu = 0, \tag{85}$$

and summing the over particles gives ${n^\nu}_{,\nu} = 0$. It may seem a little strange to be considering partial derivatives of the charge and current density of a particle – this being a point-like entity – but it is legitimate nonetheless to manipulate the Dirac δ-function this way. Continuity of the charge current density ${j^\nu}_{,\nu} = 0$ follows in exactly the same way directly from (81).

A.2 The 4-current density and the density in 4D spacetime

A definition of the density in 4-dimensions of a set of particles with world-lines $\vec{x}_P(\tau)$, which is a set of filaments in spacetime that vanishes except on the world-lines, is

$$\sigma(\vec{x}) = \sum_P \int d\tau\delta^{(4)}(\vec{x}-\vec{x}_P(\tau)). \tag{86}$$

We can think of this as analogous to the density of beads on strings $\mathbf{x}_S(\lambda)$ in 3D space where the physical spacing $d\lambda$ between the beads is constant (and which we can usefully take to define a unit distance – which we will demand be much less than the radius of curvature of the strings). Here each 'bead' represents the existence of a particle for one unit of proper time.

The 4-D density $\sigma(\vec{x})$ is a Lorentz scalar. That the 4-dimensional δ-function is Lorentz invariant follows from the fact that it is a density – so when multiplied by a space-time volume element d^4x it gives a number (automatically Lorentz invariant) – while, as we showed previously, space-time volume elements like d^4x are invariant under boosts as the Jacobian of the Lorentz boost matrix is unity.

We define a particle 4-current-density $\vec{n}$ as

$$\boxed{\vec{n}(\vec{x}) = \sum_P \int d\tau \vec{u}_P(\tau)\delta^{(4)}(\vec{x} - \vec{x}_P(\tau))} \tag{87}$$

where we are effectively 'weighting' the 4-D density by the 4-velocity $\vec{u}_P = d\vec{x}_P/d\tau$, and in which all of the factors are Lorentz invariant.

Similarly, we define the charge 4-current-density to be

$$\boxed{j^\nu(\vec{x}) = \sum_P q_P \int d\tau u_P^\nu(\tau)_P \delta^{(4)}(\vec{x} - \vec{x}_P(\tau))} \tag{88}$$

Why are these reasonable definitions for the particle number or charge currents? To see, imagine we take a slice through the 4-D space-time at constant $x = x^1$ of the x-component of $\vec{n}(\vec{x})$ to get $n_x(ct, y, z; x)$. To make that precise, let's consider a single particle and take that to be the limit, as $\Delta x \to 0$ of the average of $n_x(\vec{x})$ through a slab of thickness Δx:

$$n_x(ct, y, z; x) = \lim_{\Delta x \to 0} \frac{1}{\Delta x} \int\limits_x^{x+\Delta x} dx\; n_x(\vec{x}) = \lim_{\Delta x \to 0} \frac{1}{\Delta x} \int d\tau u_x(\tau) \int\limits_x^{x+\Delta x} dx\; \delta^{(4)}(\vec{x} - \vec{x}(\tau)) \tag{89}$$

Now if τ in the last integral is such that $\vec{x}_P(\tau)$ lies within the slab, we have $\int dx\; \delta^{(4)}(\vec{x} - \vec{x}(\tau)) = \delta(ct - ct(x))\delta(y - y(x))\delta(z - z(x))$ where $t(x)$ etc. are the coordinates of the particle as it passes through the slab. The integral with respect to τ is then straightforward: its modulus is $|\int d\tau u_x(\tau)| = |\int d\tau (dx/d\tau)| = \Delta x$, and its sign is the same as that of u_x so we have, generalising to a collection of particles,

$$n_x(ct, y, z; x) = \sum_P \text{sign}(u_{Px})\delta(ct - ct_P(x))\delta(y - y_P(x))\delta(z - z_P(x)) \tag{90}$$

i.e. just a sum of 3-D delta functions – or the sum, if you prefer, of the crossing events – weighted by the direction of motion.

If we integrate $n_x(ct, y, z; x)$ over a 3-volume – let's say the cube $c\Delta t \times \Delta y \Delta z$ – we get the number of particles that went through the x=constant slice in the positive direction minus the number going in the opposite direction. So the x-component of $\vec{n}$ defined above really is the net rate at which particles are crossing the surface per unit time per unit area on the surface.

If, instead, we had taken a slice at constant $x_0 = ct$ we would have found

$$n_0(\mathbf{x}; x_0) = \sum_P \text{sign}(u_{P0})\delta^{(3)}(\mathbf{x} - \mathbf{x}_P(t)) \tag{91}$$

which, since all particles have positive u_{P0}, is just the space-density, and we would say that $n_0(\mathbf{x}; x_0)$ is the rate at which particles are moving into the future per unit spatial volume.

Introducing the particle charge q_P as in (88) we find similarly that integral of $j_x(ct, y, z; x)$ over the cube gives the net amount of charge crossing the surface per unit time per unit area. It is interesting to note that we can consider a negative charge to be a positive charge for which proper time is running in the opposite direction.

One can demonstrate somewhat more formally that (87) and (88) are equivalent to (80) and (81). The equivalence of (88) and (81) follows from the fact that $d\tau u_P^\nu = dx_P^\nu = \dot{x}_P^\nu dt$, while $\delta^{(4)}(\vec{x} - \vec{x}_P(\tau)) = \delta(t - t_P(\tau))\delta^{(3)}(\mathbf{x} - \mathbf{x}_P(\tau))$, so the integral is $\int dt \dot{x}_P^\nu(\tau)\delta^{(3)}(\mathbf{x} - \mathbf{x}_P(\tau))\delta(t - t_P(\tau)) = \dot{x}_P^\nu(t)\delta^{(3)}(\mathbf{x} - \mathbf{x}_P(t))$, where $\dot{x}_P^\nu(t)$ denotes the value of $\dot{x}_P^\nu(\tau)$ at the proper time τ which is the solution of $t_P(\tau) = t$, and similarly $\mathbf{x}_P(t) = \mathbf{x}_P(\tau)$. Multiplying by q_P and summing over particles gives (81).

Maxwell's equations, in which j^ν appears, are empirical laws based on observations involving quantities like δQ and $\delta \mathbf{J}$, as well as forces and hence measurements of fields. We can thus consider either (81) or (88) to be empirically based definitions of the 4-current-density.

While it is somewhat redundant, one can show that the continuity equation ${n^\nu}_{,\nu} = 0$ follows directly from the definition (87) as follows: The contribution to the particle 4-current density n^ν from an infinitesimal element of proper time $d\tau$ for one of the particles is $dn^\nu_P(\vec{x}) = d\tau u^\nu_P(\tau)\delta^{(4)}(\vec{x} - \vec{x}_P(\tau))$. Its 4-divergence is $dn^\nu_{P,\nu} = d\tau u^\nu_P(\tau)\partial/\partial x^\nu \delta^{(4)}(\vec{x} - \vec{x}_P(\tau)) = -d\tau u^\nu_P(\tau)\partial/\partial x^\nu_P(\delta^{(4)}(\vec{x} - \vec{x}_P(\tau))) = -dx^\nu_P \partial/\partial x^\nu_P(\delta^{(4)}(\vec{x} - \vec{x}_P(\tau)))$. The integral of this is just the boundary term: $n^\nu_{P,\nu} = \int_{\tau_1}^{\tau_2} dn^\nu_{P,\nu} = -[\delta^{(4)}(\vec{x} - \vec{x}_P(\tau))]^{\tau_2}_{\tau_1}$. But real world-lines do not end, or, if they do, it is at $t = \pm\infty$. Everywhere else ${n^\nu}_{,\nu} = 0$.

A.3 The 4-current density in terms of the density in 6D phase-space

The density $f(\mathbf{x}, \mathbf{p}, t)$ of particles in 6-dimensional phase space $(\mathbf{x}, \mathbf{p})$, defined such that the number of particles in a 6-dimensional volume element $d^3x d^3p$ at time t is $d^6N = f(\mathbf{x}, \mathbf{p}, t) d^3x d^3p$, is a sum of 6-dimensional Dirac δ-functions:

$$f(\mathbf{x}, \mathbf{p}, t) = \sum_P \delta^{(3)}(\mathbf{x} - \mathbf{x}_P(t))\delta^{(3)}(\mathbf{p} - \mathbf{p}_P(t)). \tag{92}$$

where $(\mathbf{x}_P(t), \mathbf{p}_P(t))$ is the trajectory of the P^{th} particle.

The space-density $n(\mathbf{x}, t) = \sum_P \delta(\mathbf{x} - \mathbf{x}_P(t))$ is simply $n(\mathbf{x}, t) = \int d^3p f(\mathbf{x}, \mathbf{p}, t)$. The mean velocity is $\mathbf{v} = \int d^3p \dot{\mathbf{x}} f(\mathbf{x}, \mathbf{p}, t) / \int d^3p f(\mathbf{x}, \mathbf{p}, t)$, from which if follows that the 4-current-density for a set of particles of equal charge q is

$$j^\nu(\vec{x}) = q n^\nu(\vec{x}) = q \int d^3p \, \dot{x}^\nu f(\mathbf{x}, \mathbf{p}, t) = q \int \frac{d^3p}{p^0} p^\nu f(\mathbf{x}, \mathbf{p}, t) \tag{93}$$

where, in the last form, d^3p/p^0 and $f(\mathbf{x}, \mathbf{p}, t)$ are both Lorentz scalars. This is for particles of equal charge and mass. For such particles, Hamilton's equations tell us that the 6-dimensional velocity $(\dot{\mathbf{x}}, \dot{\mathbf{p}})$ is only a function of 6-dimensional position $(\mathbf{x}, \mathbf{p})$. I.e. these particles are like a *fluid* in phase-space (unlike a gas in 3-dimensional space where, at any position $\mathbf{x}$, there is a range of velocities $\dot{\mathbf{x}}$). If we have different types of particles with different charges or charge-to-mass ratios, such as particles and their anti-particles, or electrons and ions in a plasma, then we need to sum over the different types as we then have a superposition of phase-space fluids.

B Continuity of 4-momentum in terms of 3, 4 and 6 dimensional particle densities

B.1 Continuity equation in terms of the 3D density

We now show how the continuity equation (52) follows from the expression $T^{\mu\nu} = m \sum_P \dot{x}^\mu_P(t) u_P(t) \delta^{(3)}(\mathbf{x} - \mathbf{x}_P(t))$ in terms of the 3-D density (50). Consider the P^{th} particle, using $\dot{x}^0 = c$ and $mu^\nu = p^\nu_P$ its contribution to the time-time part of the 4-divergence ${T^{0\nu}}_{,0}$ is ${T_P^{0\nu}}_{,0} = \frac{1}{c}\partial_t(p^\nu_P(t)\delta^{(3)}(\mathbf{x} - \mathbf{x}_P(t))) = \dot{p}^\nu_P \delta^{(3)}(\mathbf{x} - \mathbf{x}_P(t)) + p^\nu_P \partial_t \delta^{(3)}(\mathbf{x} - \mathbf{x}_P(t))$. Its contribution to the spatial divergence is ${T_P^{i\nu}}_{,i} = \boldsymbol{\nabla}_\mathbf{x} \cdot (\mathbf{x}_P(t) p^\nu_P(t) \delta^{(3)}(\mathbf{x} - \mathbf{x}_P(t))) = p^\nu_P(t)\mathbf{x}_P(t) \cdot \boldsymbol{\nabla}_\mathbf{x} \delta^{(3)}(\mathbf{x} - \mathbf{x}_P(t)) = -p^\nu_P(t)\mathbf{x}_P(t) \cdot \boldsymbol{\nabla}_{\mathbf{x}_P} \delta^{(3)}(\mathbf{x} - \mathbf{x}_P(t)) = -p^\nu_P \partial_t \delta^{(3)}(\mathbf{x} - \mathbf{x}_P(t))$. This is minus the second term in ${T_P^{t\nu}}_{,t}$ giving ${T_P^{\mu\nu}}_{,\mu} = {T_P^{0\nu}}_{,0} + {T_P^{i\nu}}_{,i} = \dot{p}^\nu_P \delta^{(3)}(\mathbf{x} - \mathbf{x}_P(t))$. Finally, using $\dot{p}^\nu_P = q_P \dot{x}^\mu_P {F_\mu}^\nu$ gives ${T_P^{\mu\nu}}_{,\mu} = q_P {F_\mu}^\nu \dot{x}^\mu_P \delta^{(3)}(\mathbf{x} - \mathbf{x}_P(t))$, and summing over particles gives (52).

B.2 Continuity equation in terms of the 4D density

The continuity equation (52) can also be obtained directly from the second expression in (50) as follows: The partial derivative ∂_μ acts only on the 4D δ-function so (again considering first the contribution from the P^{th} particle) ${T_P^{\mu\nu}}_{,\mu} = m \int d\tau u^\mu_P(\tau) u^\nu_P(\tau) \partial/\partial_{x^\mu} \delta^{(4)}(\vec{x} - \vec{x}_P(\tau)) = -\int dx^\mu_P p^\nu_P(\tau) \partial/\partial_{x^\mu_P} \delta^{(4)}(\vec{x} - \vec{x}_P(\tau))$ where we have used $d\tau u^\mu_P = dx^\mu_P$ and $mu^\nu_P = p^\nu_P$. Integrating by parts gives ${T_P^{\mu\nu}}_{,\mu} = \int dx^\mu_P \delta^{(4)}(\vec{x} - \vec{x}_P(\tau)) dp^\nu_P(\tau)/dx^\mu_P$ plus a boundary term at the end of the particle's world-line, which we may ignore, so ${T_P^{\mu\nu}}_{,\mu} = \int d\tau \delta^{(4)}(\vec{x} - \vec{x}_P(\tau)) dp^\nu_P(\tau)/d\tau$ and using $dp^\nu_P(\tau)/d\tau = q_P u^\mu_P {F_\mu}^\nu$ and summing gives ${T_P^{\mu\nu}}_{,\mu} = {F_\mu}^\nu \sum_P q_P \int d\tau u^\mu_P \delta^{(4)}(\vec{x} - \vec{x}_P(\tau)) = {F_\mu}^\nu j^\mu_P$.

B.3 Continuity equation in terms of the 6D density

Taking the divergence of the last expression for $T^{\mu\nu}$ in (50) gives

$$T^{\mu\nu}{}_{,\mu} = \partial_\mu \int d^3p f(\mathbf{x},\mathbf{p},t)\dot{x}^\mu(\mathbf{p})p^\nu(\mathbf{p}) = \int d^3p\, p^\nu \dot{x}^\mu \partial_\mu f(\mathbf{x},\mathbf{p},t) = -\int d^3p\, p^\nu \boldsymbol{\nabla}_\mathbf{p}\cdot(f\dot{\mathbf{p}}) \tag{94}$$

where we have used Liouville's theorem $df/dt = \partial_t f + \dot{\mathbf{x}}\cdot\boldsymbol{\nabla}_\mathbf{x} f + \dot{\mathbf{p}}\cdot\boldsymbol{\nabla}_\mathbf{p} f = \dot{x}^\mu\partial_\mu f + \dot{\mathbf{p}}\cdot\boldsymbol{\nabla}_\mathbf{p} f = 0$ and $\boldsymbol{\nabla}_\mathbf{p}\dot{\mathbf{p}} = 0$.

Integrating by parts, and assuming that f tends to zero at infinity, we have

$$T^{\mu\nu}{}_{,\mu} = \int d^3p f\dot{\mathbf{p}}\cdot\boldsymbol{\nabla}_\mathbf{p} p^\nu(\mathbf{p}) = qF_{\mu i}\int d^3p f\dot{x}^\mu \partial_{p_i} p^\nu(\mathbf{p}) \tag{95}$$

where we have used the Lorentz force law $\dot{p}_i = qF_{\mu i}\dot{x}^\mu$.

The $\nu = 0$ component of $\partial_{p_i} p^\nu(\mathbf{p})$ is, from $p^0 = \sqrt{m^2c^4 + |\mathbf{p}|^2c^2}$, given by $\partial_{p_i}p^0 = \dot{x}^i/c$, while $\partial_{p_i}p^j = \delta_{ij}$. Using the latter in (95) we obtain

$$T^{\mu i}{}_{,\mu} = F_{\mu i} q \int d^3p f\dot{x}^\mu = j^\mu F_\mu{}^i. \tag{96}$$

And using the former (in the penultimate expression for $T_\mathrm{m}^{\mu\nu}{}_{,\mu}$ in (95)) gives

$$T^{\mu 0}{}_{,\mu} = \int d^3p f\dot{p}_i\dot{x}_i = \int \frac{d^3p}{p^0} f\dot{p}_i p_i = -\int \frac{d^3p}{p^0} f\dot{p}^0 p_0 = \int d^3p f\dot{p}^0 = F_\mu{}^0 q \int d^3p f\dot{x}^\mu = j^\mu F_\mu{}^0 \tag{97}$$

where, in the third step, we used the fact that $p^\mu p_\mu = -m^2c^2$ is constant, so (half) its derivative $\dot{p}^\mu p_\mu = \dot{p}^0 p_0 + \dot{p}_i p_i = 0$, and in the fifth we used the work equation $\dot{p}^0 = qF_\mu{}^0\dot{x}^\mu$. Thus again we have $T^{\mu\nu}{}_{,\mu} = j^\mu F_\mu{}^\nu$ just as we found at the outset.

C The radiation Lagrangian and stress tensor in terms of E and B

In this section, for clarity, we will choose units of charge such that $\epsilon_0 = 1$ and units of length and time so that $c = 1$ (so $\mu_0 = 1$ also) and simply suppress ϵ_0, μ_0 and c.

The Lagrangian for the radiation is

$$\mathcal{L}_\mathrm{r} = -\tfrac{1}{4}F^{\mu\nu}F_{\mu\nu} \tag{98}$$

or equivalently

$$\mathcal{L}_\mathrm{r} = \tfrac{1}{4}\mathrm{Tr}(F^{\alpha\nu}F_{\nu\beta}). \tag{99}$$

Using

$$F^{\mu\nu} = \begin{bmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{bmatrix} \quad \text{and} \quad F_{\mu\nu} = \begin{bmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0 \end{bmatrix} \tag{100}$$

we obtain their product

$$F^{\alpha\nu}F_{\nu\beta} = \begin{bmatrix} |\mathbf{E}|^2 & E_zB_y - E_yB_z & E_xB_z - E_zB_x & E_yB_x - E_xB_y \\ E_yB_z - E_zB_y & E_x^2 - B_z^2 - B_y^2 & E_xE_y + B_xB_y & E_xE_z + B_xB_z \\ E_zB_x - E_xB_z & E_xE_y + B_xB_y & E_y^2 - B_z^2 - B_x^2 & E_yE_z + B_yB_z \\ E_xB_y - E_yB_x & E_xE_z + B_xB_z & E_yE_z + B_yB_z & E_z^2 - B_x^2 - B_y^2 \end{bmatrix} \tag{101}$$

Its trace is $2(|\mathbf{E}|^2 - |\mathbf{B}|^2)$ so

$$\mathcal{L}_\mathrm{r} = -\tfrac{1}{4}F^{\mu\nu}F_{\mu\nu} = \tfrac{1}{2}(|\mathbf{E}|^2 - |\mathbf{B}|^2). \tag{102}$$

We can write this product (101) more simply as

$$F^{\alpha\nu}F_{\nu\beta} = \begin{bmatrix} |\mathbf{E}|^2 & -\mathbf{S} \\ \mathbf{S} & \boldsymbol{\sigma} + \tfrac{1}{2}\mathbf{I}(|\mathbf{E}|^2 - |\mathbf{B}|^2) \end{bmatrix} \tag{103}$$

where $\mathbf{S} \equiv \mathbf{E} \times \mathbf{B}$ is the *Poynting energy flux density* and where $\mathbf{I} \to \delta_{ij} = \mathrm{diag}(1,1,1)$ is the 3-by-3 identity matrix and where $\boldsymbol{\sigma}$ is the 3 by 3 symmetric *Maxwell stress tensor* is defined by

$$\boldsymbol{\sigma} \equiv \mathbf{E}\mathbf{E} + \mathbf{B}\mathbf{B} - \tfrac{1}{2}\mathbf{I}(|\mathbf{E}|^2 + |\mathbf{B}|^2). \tag{104}$$

Raising the index β we obtain the symmetric matrix

$$F^{\alpha\nu} F_{\nu}{}^{\beta} = \begin{bmatrix} -|\mathbf{E}|^2 & -\mathbf{S} \\ -\mathbf{S} & \boldsymbol{\sigma} + \tfrac{1}{2}\mathbf{I}(|\mathbf{E}|^2 - |\mathbf{B}|^2) \end{bmatrix} \tag{105}$$

from which we obtain

$$T_{\mathrm{r}}^{\alpha\beta} = F^{\alpha\nu} F^{\beta}{}_{\nu} + \eta^{\alpha\beta} \mathcal{L}_{\mathrm{r}} = \begin{bmatrix} \tfrac{1}{2}(|\mathbf{E}|^2 + |\mathbf{B}|^2) & \mathbf{S} \\ \mathbf{S} & -\boldsymbol{\sigma} \end{bmatrix} \tag{106}$$

in which we recognise the usual expressions for the energy density in T_{r}^{tt} and the energy flux density and momentum density in T_{r}^{it} and T_{r}^{ti} respectively (these being equal).