ENS M1 General Relativity - Lecture 4 - Space-time curvature

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1 Space-time curvature and gravity

An essential ingredient of Einstein's theory of gravity is that there is no *gravitational force* per se, as in Newtonian gravity, rather space-time is *locally flat* (i.e. Minkowskian) but curved on large scales. and freely falling particles follow *geodesic* paths (curves of extremal proper distance or proper time) on a *curved manifold*.

Their trajectories are rather like rubber fibres stretched across the surface of an apple. The geometry of the surface is locally flat – i.e. appears Euclidean if a small patch is viewed with a magnifying glass. So we can have two fibres that are locally parallel, for instance, but the distance between them will change over scales comparable to the radius of curvature. For particle geodesics in the vicinity of the earth that 'radius of curvature' is about $ct \simeq 15$ light-minutes (the distance light travels in the time a near-earth satellite advances along its orbit by 1 radian) or about 3×10^{11}m. That is much larger than the radius of the Earth, which accounts for the fact that the effects of spatial curvature are not readily perceived. The curvature of *time*, however, is very much in evidence.

The other essential ingredient is that the matter in the universe and the geometry are linked by Einstein's field equations.

1.1 The manifold: the arena for gravity

A *manifold* is pretty much anything that is, or can be visualised as, a N-dimensional sub-space in some higher $M > N$ dimensional space and is endowed with a coordinate system.

The manifold of interest in GR is the space-time continuum, which we know has $N = 4$ (though it may have more dimensions that are 'curled up' on some invisibly small scale).

We do not know if our universe is, in reality, 'embedded' in a large higher dimensional space. Some older texts (like Dirac's) invoke a fictitious 5-dimensional space to discuss e.g. the curvature of the 4D manifold. This allows one to construct a 4D 'tangent space-time' that is flat, rather like a 2D plane that is flat and tangent to the surface of an apple, and the terminology of 'tangent vector' stems from that. The mathematical machinery of GR may, however, be formulated in a way that only concerns itself with the *intrinsic* properties of the 4-dimensional manifold that is space-time. In GR, the events and world-lines of particles and all fields, such as EM or matter fields, exist only in the 4D manifold.

1.2 Coordinates, foliation and distances

The *coordinate system* is a labelling of each event in space-time with 4 numbers. This is fairly arbitrary, though we require nearby that nearby events have coordinates – denoted by x^α – that are also close and we require that different events have different coordinates.

The set of events with the same value for one of the 4 coordinates defines a 3-surface (or hyper-surface), and a set of such surfaces, for a sequence of values of the coordinate, 'foliate' space-time.

On such a, rather vaguely defined, manifold it is possible to define vectors and 1-forms and their contractions. Local flatness implies that for two neighbouring events on the manifold there are observers who can measure the physical spatial and temporal intervals between these events (with rulers and clocks – see below) and form the proper separation $(\Delta s)^2$. Empirically these can be positive, zero or negative and we assume that this is true everywhere on the manifold. With a collection of such measurements for a set of pairs of events in some small region, as illustrated in figure 1, together with coordinate separations Δx^α that they obtain from the labels on the events, these squared intervals will be bi-linear, at lowest order for small separations: $(\Delta s)^2 = g_{\alpha\beta}\Delta x^\alpha \Delta x^\beta$. So one can thus empirically determine the metric $g_{\alpha\beta}(\vec{x})$, which will be some function of the coordinates. We will discuss this in a little more detail shortly.

1.3 Locally flat coordinates on a 2D manifold

Consider some points $\vec{x}_i$, for $i = 1, 2, \ldots$ on a 2-dimensional manifold, as illustrated in figure 1. Let's take one of the points, $\vec{x}_0$ (marked by the '+' symbol), call it the origin, and subtract its $x^\alpha = (\xi, \eta)$ coordinates from those of the other points.

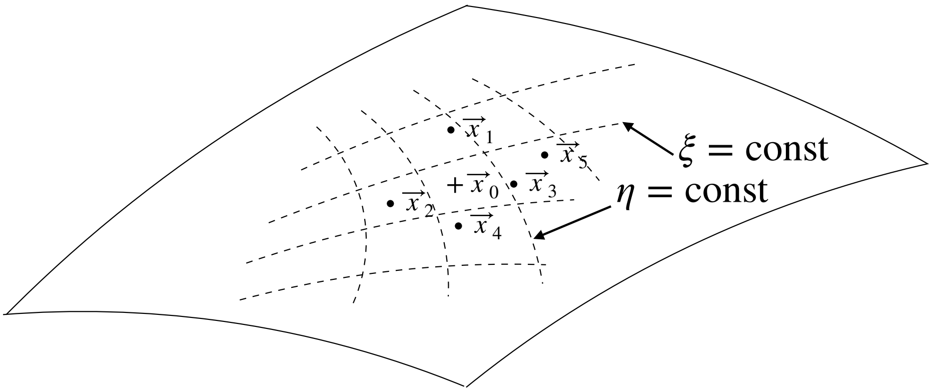


Figure 1: A portion of a 2-dimensional manifold – shown here embedded in a 3-dimensional space – on which lines of (unprimed) coordinates $x^\alpha = (\xi, \eta)$ have been drawn. Also shown are a set of points on the manifold. We assume that the points – events if we were dealing with a 4D locally Minkowskian manifold – are labelled with their (ξ, η) coordinates. Distances – in the manifold itself, not taking a 'short-cut' through the fictitious embedding space – can be measured using pieces of string or rulers. This allows the determination of the metric; which we can think of as a distillation of such distance measurements.

Let's now construct another (primed) coordinate system $x^{\alpha'} = x^{\alpha'}(x^\alpha)$ as a first order Taylor series expansion and require that $x^{\alpha'}(x_0^\alpha) = 0$:

$$x^{\alpha'} = 0 + x^{\alpha'}{}_{,\alpha} x^\alpha. \tag{1}$$

We thus have a 2×2 matrix of coefficients $\Lambda^{\alpha'}{}_\alpha = x^{\alpha'}{}_{,\alpha}$ defining the mapping.

The physical distance squared from the origin to a point with (ξ, η) coordinates x^α is $s^2 = g_{\alpha\beta} x^\alpha x^\beta$ so, using the above mapping, this is $s^2 = g_{\alpha\beta} \Lambda^\alpha{}_{\alpha'} x^{\alpha'} \Lambda^\beta{}_{\beta'} x^{\beta'}$, where $\Lambda^\alpha{}_{\alpha'}$ is the matrix inverse of $\Lambda^{\alpha'}{}_\alpha$. Thus the metric in the primed frame, defined such that $s^2 = g_{\alpha'\beta'} x^{\alpha'} x^{\beta'}$, is[1]

$$g_{\alpha'\beta'} = \Lambda^\alpha{}_{\alpha'} \Lambda^\beta{}_{\beta'} g_{\alpha\beta}. \tag{2}$$

The metric $g_{\alpha\beta}$ is 2×2 and symmetric and is therefore determined by 3 numbers. The matrix $\Lambda^\alpha{}_{\alpha'}$ is also 2×2, but not, in general, symmetric, so we have more than enough freedom to adjust the coefficients of $\Lambda^\alpha{}_{\alpha'}$ to render the primed frame metric diagonal and equal to the identity matrix: $g_{\alpha'\beta'} = \mathrm{diag}(1, 1)$. That is to say equal to the Euclidean metric in 2D. That makes sense. The transformation should not be fully determined by the requirement $g_{\alpha'\beta'} = \mathrm{diag}(1, 1)$, since any other coordinate system $x^{\alpha''}$ that is a rotated version of the $x^{\alpha'}$ system would have the same metric.

[1] Note that this differs from the usual matrix transformation law in matrix algebra. There, if a vector $\mathbf{v}$ is transformed with a transformation matrix $\mathbf{\Lambda}$ as $\mathbf{v}' = \mathbf{\Lambda} \cdot \mathbf{v}$ then a matrix $\mathbf{M}$ gets transformed as $\mathbf{M}' = \mathbf{\Lambda} \cdot \mathbf{M} \cdot \mathbf{\Lambda}^{-1}$. Here, instead, we have $\mathbf{g}' = (\mathbf{\Lambda}^{\mathrm{T}})^{-1} \cdot \mathbf{g} \cdot \mathbf{\Lambda}^{-1}$ with inverses of $\mathbf{\Lambda}$ on both sides, and with one transposed. The reason for the difference is that in matrix algebra we demand that multiplying a vector by a matrix gives another vector, e.g. $\mathbf{u} = \mathbf{M} \cdot \mathbf{v}$, and the transformation law ensures that $\mathbf{M}' \cdot \mathbf{v}' = (\mathbf{\Lambda} \cdot \mathbf{M} \cdot \mathbf{\Lambda}^{-1}) \cdot (\mathbf{\Lambda} \cdot \mathbf{v}) = \mathbf{\Lambda} \cdot (\mathbf{M} \cdot \mathbf{v}) = \mathbf{\Lambda} \cdot \mathbf{u} = \mathbf{u}'$. Here the transformation of $\mathbf{g}$ is designed instead to ensure that the quadratic form $\mathbf{v}^{\mathrm{T}} \cdot \mathbf{g} \cdot \mathbf{v}$ be invariant. In matrix algebra, in contrast, $\mathbf{v}'^{\mathrm{T}} \cdot \mathbf{M}' \cdot \mathbf{v}' = (\mathbf{v}^{\mathrm{T}} \cdot \mathbf{\Lambda}^{\mathrm{T}}) \cdot \mathbf{M}' \cdot (\mathbf{\Lambda} \cdot \mathbf{v}) = \mathbf{v}^{\mathrm{T}} \cdot (\mathbf{\Lambda}^{\mathrm{T}} \cdot \mathbf{\Lambda} \cdot \mathbf{M}) \cdot \mathbf{v} \neq \mathbf{v}^{\mathrm{T}} \cdot \mathbf{M} \cdot \mathbf{v}$, with this inequality readily confirmed for the simple case that $\mathbf{\Lambda}$ is simply a non-unity multiple of the identity matrix (i.e. just a scaling transformation) so the quadratic form is not invariant.

This allows us to set $g_{\alpha'\beta'}(\vec{x}_0) = \mathrm{diag}(1,1)$, but only at that point. If we move away from $\vec{x}_0$ we will have $g_{\alpha\beta} = g_{\alpha\beta}(\vec{x}_0) + g_{\alpha\beta,\gamma}x^\gamma$ at linear order. These will feed through into $g_{\alpha'\beta'}$ which will have correction terms that will also, at leading order, be linear in the displacement from the origin.

But now consider what happens if we make a 2nd order Taylor series expansion:

$$x^{\alpha'} = x^{\alpha'}{}_{,\beta}x^\beta + \tfrac{1}{2!}x^{\alpha'}{}_{,\beta\gamma}x^\beta x^\gamma \tag{3}$$

with the same 1st order coefficients $x^{\alpha'}{}_{,\beta} = \Lambda^{\alpha'}{}_\beta$ and with an extra set of coefficients $x^{\alpha'}{}_{,\beta\gamma}$. To lowest order we can put $x^\beta = \Lambda^\beta{}_{\beta'}x^{\beta'}$ and $x^\gamma = \Lambda^\gamma{}_{\gamma'}x^{\gamma'}$ in the (small) second term on the right hand side. Taking this to the other side, and multiplying both sides by $\Lambda^\alpha{}_{\beta'}$ we see that this is equivalent to a second order expansion of $x^\alpha(x^{\alpha'})$:

$$x^\alpha = \Gamma^\alpha{}_{\beta'}x^{\beta'} + T^\alpha{}_{\beta'\gamma'}x^{\beta'}x^{\gamma'} + \ldots \tag{4}$$

with 2nd order coefficients $T^\alpha{}_{\beta'\gamma'} = -\frac{1}{2}\Gamma^\alpha{}_{\alpha'}\Gamma^\beta{}_{\beta'}\Gamma^\gamma{}_{\gamma'}x^{\alpha'}{}_{,\beta\gamma}$.

If we plug this expression into the right hand side of $s^2 = g_{\alpha\beta}(\vec{x})x^\alpha x^\beta$ then we will still have $s^2 = g_{\alpha'\beta'}(\vec{x}_0)x^{\alpha'}x^{\beta'}$ with $g_{\alpha'\beta'}(\vec{x}_0) = \mathrm{diag}(1,1)$ and with 1st order corrections $g_{\alpha\beta,\gamma}x^\gamma$ – if we move away from $\vec{x} = \vec{x}_0$ – coming from the variation of $g_{\alpha\beta}$ as before, but now with extra 1st order corrections coming from non-linear terms in the expansion of $x^\alpha(x^{\alpha'})$ above. Now since $g_{\alpha\beta}$ is symmetric under $\alpha \Leftrightarrow \beta$ it has 3 independent components and hence $g_{\alpha\beta,\gamma}$ has $2 \times 3 = 6$ 'degrees of freedom'. But the 3-dimensional matrix of coefficients $x^{\alpha'}{}_{,\beta\gamma}$ the we have at our disposal is symmetric under $\beta \Leftrightarrow \gamma$, so this has $2 \times 3 = 6$ degrees of freedom also. Thus by judicious choice of these coefficients we can make $g_{\alpha'\beta'}(\vec{x}_0) = \mathrm{diag}(1,1)$ and make the 1st order corrections to this vanish. The problem §6.4 has you show that, in fact, the coefficients of the transformation are just the Christoffel symbols.

This is what is meant by 'local flatness'. Close to some point $\vec{x}_0$ – which was arbitrary, we could have chosen any point as the origin – we can find primed frame coordinates – in fact a family of such frames, as they can be rotated with respect to each other – such that the metric is exactly Euclidean at $\vec{x}_0$ and where any corrections are at most of second order in distance from $\vec{x}_0$. This has the implication that, if we find the equations of motion of a curve of extremal length – i.e. a curve for which $\delta \int d\lambda L(x^{\alpha'}, \dot{x}^{\alpha'}) = 0$ with $L(x^{\alpha'}, \dot{x}^{\alpha'}) = \sqrt{-g_{\alpha'\beta'}(x^{\alpha'})\dot{x}^{\alpha'}\dot{x}^{\beta'}}$ and where $\dot{x}^{\alpha'} = dx^{\alpha'}/d\lambda$ and the parameter λ is chosen to be 'affine', so it measures distance along the path – then close to $\vec{x}_0$ the 'generalised force' $\partial L/\partial x^{\alpha'}$ vanishes, and we get $\ddot{x}^{\alpha'} = 0$; the equation of a straight path.

1.4 Inertial observers and locally inertial coordinates

All of the above carries over to higher dimensional manifolds, and to non-Euclidean geometry. For a $N = 4$ dimensional manifold that is locally Minkowskian (this being an inherent property of the manifold – not something we can influence in any way) we can find a linear transformation that renders the the metric Minkowskian at any point $\vec{x}_0$ that we choose for the origin. The linear transformation has $N \times N = 16$ degrees of freedom we can adjust, allowing us to fix the $N + N(N-1)/2 = N(N+1)/2 = 10$ independent components of the symmetric matrix. And if we add a 2nd order term to the mapping we get an extra $N^2(N+1)/2 = 40$ extra coefficients that allow us to make the derivatives of the primed-frame metric vanish at $\vec{x}_0$ also. In these primed coordinates, the equation of motion for a particle – obtained either as the Euler-Lagrange equations obtained by extremising the proper length of the trajectory or by using the Christoffel symbols computed from the metric (they vanish in the primed frame) in the geodesic equation – is the same as in the absence of gravity.

Einstein's insight – encapsulated in the 'Einstein equivalence principle' (EEP) – is that such coordinates –which are called *locally inertial coordinates* – can be realised physically by having a freely falling massive observer – a so-called *'inertial observer'* – who is not spinning – as measured with respect to gyroscopes carried by the observer or by means of stress measurements – carry an orthogonal set of rulers (or a lattice of such rulers) populated with other observers with clocks (synchronised by exchanging light signals). As illustrated in figure 2, these observers can measure and record the *physical* space-time coordinates $x^{\alpha'}$ of events in the vicinity of the massive observer: these simply being the clock-time and 3-D coordinates of an observer on the grid at the same location as the event in question.

The physical content of the EEP is that by going to free fall – e.g. by jumping out of a window – effects of gravity such as objects we release being accelerated downwards and photons suffering gravitational redshifts as measured by Pound and Rebka are locally 'nulled out'. We say that gravity is 'transformed away' by going to a freely falling (and non-spinning) frame. The EEP says that – locally at least – there is nothing more to such phenomena as experienced standing on the surface of the Earth than what we would experience in empty space being accelerated by a rocket motor.

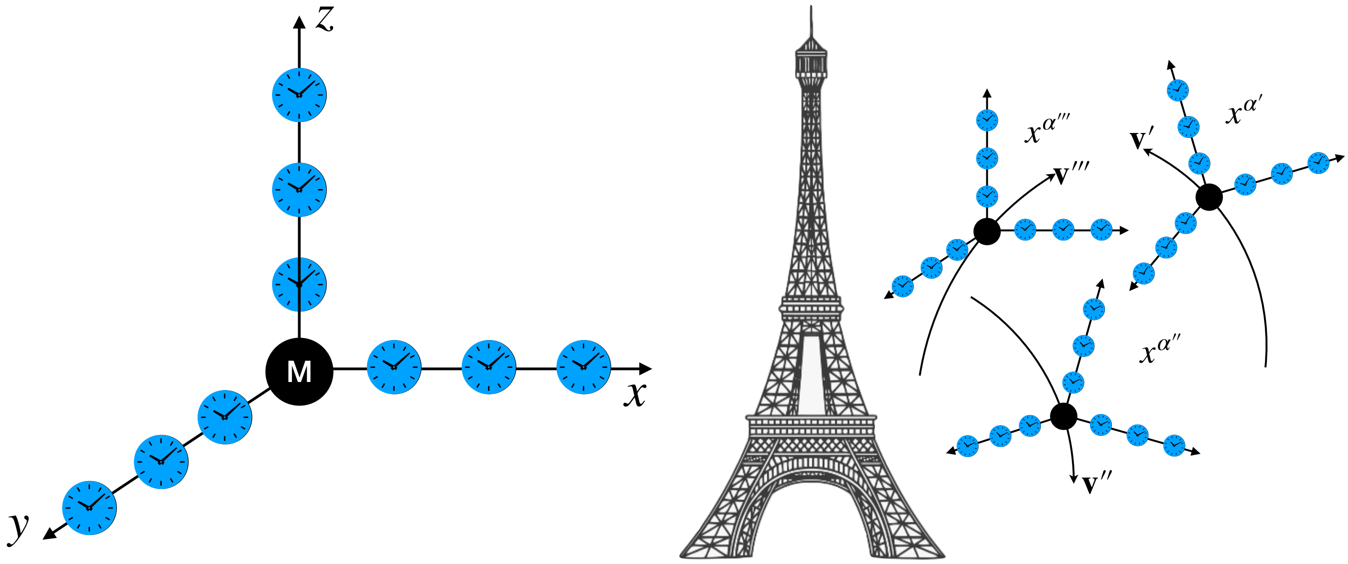


Figure 2: Inertial coordinates $x^{\alpha'}$ can be realised by having a massive freely falling non-spinning observer carry a set of rigid rulers on which there are other (massless) observers with clocks. This is shown at the left. The clocks are assumed to have been synchronised by exchanging light signals. These observers can record the $x^{\alpha'}$ coordinates of events that are labelled with the general coordinates x^{α}. From this they can establish the transformation matrix $\Lambda^{\alpha}{}_{\alpha'} = \partial x^{\alpha}/\partial x^{\alpha'}$ of the mapping $\vec{x} = \vec{x}(\vec{x}')$. As illustrated at the right – with the tower there to remind us that we are in the presence of a gravitational field – we can have multiple inertial frames that are moving and/or rotated with respect to each other. They form a 6-parameter family.

With the exception of the massive observer who anchors the grid of rulers these observers are *not* inertial; rather they are accelerating to maintain themselves at fixed location with respect to the massive observer. This is similar to the situation in Newtonian gravity where spatial coordinates are defined with respect to observers on a rigid lattice; these observers not being allowed to move in free-fall in the gravitational field. This acceleration will cause the clocks to drift out of synch. But the acceleration here is zero for the massive observer, and increases, in general, linearly with distance from the massive observer, and any drifting out of synch is negligible sufficiently close to the massive observer. It is for this reason that we say that the reference frame thus defined is *locally* inertial

Just as in SR there is some freedom in the choice of inertial coordinates. We could use physical coordinates as measured by another massive observer whose spatial coordinate system is rotated with respect the one we have considered, or we could use those measured by a boosted observer. So there is a 6-parameter family of reference frames at each tangent point (i.e. for each possible location for the massive observer). This meshes perfectly with what we infer from counting of degrees of freedom for the metric (10) and the transformation matrix (16). These different observers will ascribe different coordinate separations $\Delta x^{\alpha'}$ to the same pair of events, but local flatness implies that they will agree on the proper separation squared $(\Delta s)^2 = \eta_{\alpha'\beta'}\Delta x^{\alpha'}\Delta x^{\beta'}$.

These observers can, in principle, establish, locally, the transformation matrix $\Lambda^{\alpha}{}_{\alpha'} = \partial x^{\alpha}/\partial x^{\alpha'}$ of the mapping $\vec{x} = \vec{x}(\vec{x}')$ between the x^{α} values they read off – or assign to – the events and their physical (or inertial) coordinates $x^{\alpha'}$. This gives the physical components $(\vec{e}_{\alpha})^{\beta'} = \partial x^{\beta'}/\partial x^{\alpha}$of a set of 4 basis vectors $\vec{e}_{\alpha}$ (the α^{th} basis vector being that which points along the line along which all coordinates except for the α^{th} are constant and which connects a pair of events with unit separation in x^{α}). This relies on the assumption that the coordinate labelling is such that these mappings are differentiable, which we assume to be valid.

With a dense enough set of events, they can determine not only the local physical components of the basis vectors but how they are varying with physical location $(\vec{e}_{\alpha})^{\beta'}{}_{,\gamma'}$. This gives $\vec{e}_{\alpha,\gamma'} = (\vec{e}_{\alpha})^{\beta'}{}_{,\gamma'}\vec{e}_{\beta'}$ (the basis vectors in the physical frame being constant) and hence $\vec{e}_{\alpha,\gamma} = \Lambda^{\gamma'}{}_{\gamma}\vec{e}_{\alpha,\gamma'}$. Finally they can then solve for the coefficients $\Gamma^{\mu}{}_{\alpha\gamma}$ of the decomposition $\vec{e}_{\alpha,\gamma} = \Gamma^{\mu}{}_{\alpha\gamma}\vec{e}_{\mu}$. So they can determine the connection.

1.5 Measurement of the metric

In somewhat more detail, the way the metric can be operationally defined is as follows. Let there be a collection of n events (labelled by their general coordinates x_i^α where $i = 1 \ldots n$ is a label and as illustrated in figure 1) in some small region of the manifold that have had their inertial coordinates measured by one of our inertial observers. This gives a collection of a pairs of events labelled by ij for which the observers determine $(\Delta s)^2_{ij}$. They then proceed to adjust the 10 parameters $g_{\alpha\beta}$ of a model $(\Delta s)^2_{ij} = g_{\alpha\beta}\Delta x^\alpha_{ij}\Delta x^\beta_{ij}$. There are 10 parameters rather than $4 \times 4 = 16$ because the $\Delta x^\alpha_{ij}\Delta x^\beta_{ij}$ values are symmetric under $\alpha \Leftrightarrow \beta$ and so $g_{\alpha\beta}$ is taken to be symmetric. This model fitting might be done by least squares minimisation; the details are not important here. The essential thing is that the metric is really just a distillation of some measured data (the $(\Delta s)^2_{ij}$) and numbers Δx^α_{ij} obtained by subtracting the general coordinate labels on the events. This provides the metric at that location. Repeating at other locations on the manifold gives $g_{\alpha\beta}(\vec{x})$.

The metric components $g_{\alpha\beta}(\vec{x})$ as a function of position encode information about the manifold *and* about the (arbitrary) coordinate system x^α that we have adopted to label the events.

1.6 Light-cone structure and its orientability

The local flatness (Minkowskianity) assumption says that separations between pairs of events have invariant squared proper separation $\vec{\Delta x}^2 = \eta_{\alpha'\beta'}\Delta x^{\alpha'}\Delta x^{\beta'}$ that may be positive, zero or negative. This means that the manifold of universe has a light-cone structure that is an *absolute* property of the manifold. At each and every point on the manifold there is a light-cone that distinguishes the 3 types of vectors in an unambiguous manner This is an *intrinsic* property of the manifold and is entirely independent of the coordinate system or the frame of reference of the observer.

Interestingly, though, there is nothing intrinsic to the manifold that tells us which direction along the axis of these cones an observer would consider to be the future. Experience tells us that my future is the same as your future; we've never met anyone who we perceive to be getting younger as we age. And we've never seen a mixture of gases or liquids spontaneously 'un-mix'. There seems to be a 'thermodynamic arrow of time'. But there does not seem to be anything in the metric that distinguishes past from future. And when we talk about fundamental particles, or perhaps matter wave-packets, there is nothing in the basic physics that indicates direction of time. And the geodesic equation does not care if we change $\tau \Rightarrow -\tau$. In special relativity one can draw 'future' directed arrows everywhere in a consistent way; e.g. neighbouring light-cones centred on space-like separated points have the arrow pointing in the same direction. That gives an *orientable* manifold with the seemingly sensible property that two time-like observers starting at different locations who come together will both see the other one to be ageing rather than getting younger. It is an interesting question whether this is generally true for a curved manifold.

At a more mundane level, given a metric expressed in some coordinate system, a very useful starting point for understanding the physical meaning of the metric, and of the coordinate system, is to sketch the form of the light-cones.

2 Curvature: the breakdown of local flatness

So if the manifold of GR is locally flat – as our intuition suggests – we can always find primed coordinates in which not only does the metric locally take the Minkowskian form but that it should be locally constant. And a corollary of this is that the effect of curvature – or gravity – will appear, at lowest non-vanishing order, in the *second derivatives* of the metric. That meshes well with a) what we know from Newtonian gravity, where the tidal acceleration for a pair of test particles is proportional to the second derivative of the potential $\phi_{\rm N}$, and b) gravitational time dilation, where the time-time component of the metric is $g_{00} = -(1 + 2\phi_{\rm N}/c^2)$.

In the following we will extend the Taylor expansion of the mapping $\vec{x}' = \vec{x}'(\vec{x})$ used above to 3rd order. The goal here is to determine, first of all, how many numbers are required to describe the curvature of space-time.

To motivate this, consider blind ants living on an egg making measurements on a small region of the the surface with strings and a protractor. They can also measure changes in the the separation between pairs of ants that march along straight, initially parallel lines. They can determine the surface is curved, and not flat, right? What else can they deduce about the curvature? Can they determine the orientation of the egg? Can they tell, for instance, that they are not living on a sphere?

2.1 2D locally Euclidean manifold

Let's start in 2-dimensions. Assuming differentiability, we can Taylor expand the metric as

$$g_{\alpha\beta}(x^\alpha) = g_{\alpha\beta} + g_{\alpha\beta,\gamma} x^\gamma + \tfrac{1}{2!} g_{\alpha\beta,\gamma\delta} x^\gamma x^\delta \ldots \tag{5}$$

and the transformation as

$$x^{\alpha'} = 0 + x^{\alpha'}{}_{,\beta} x^\beta + \tfrac{1}{2!} x^{\alpha'}{}_{,\beta\gamma} x^\beta x^\gamma + \tfrac{1}{3!} x^{\alpha'}{}_{,\beta\gamma\delta} x^\beta x^\gamma x^\delta \ldots \tag{6}$$

so the transformation matrix is

$$\Lambda^{\alpha'}{}_\beta = \partial x^{\alpha'} / \partial x^\beta = x^{\alpha'}{}_{,\beta} + x^{\alpha'}{}_{,\beta\gamma} x^\gamma + \tfrac{1}{2!} x^{\alpha'}{}_{,\beta\gamma\delta} x^\gamma x^\delta \ldots \tag{7}$$

We've already shown how the 2nd order expansion allows us to nullify deviations of the metric – in primed coordinates – from Euclidean form. But at cubic order we cannot, in general, adjust the coefficients $x^{\alpha'}{}_{,\beta\gamma\delta}$ to nullify 2nd derivatives of the metric $g_{\alpha'\beta',\gamma'\delta'}$. The latter – being symmetric under $\alpha \Leftrightarrow \beta$ and $\gamma \Leftrightarrow \delta$ – has $3 \times 3 = 9$ d.f. – while in $x^{\alpha'}{}_{,\beta\gamma\delta}$ the index α' can take values $0, 1$, while the independent combinations of the other indices are $(\beta\gamma\delta) = (000), (001), (011), (111)$, for a total of $2 \times 4 = 8$ d.f.. So we do not have enough freedom at our disposal to make $g_{\alpha'\beta',\gamma'\delta'}$ vanish.

That this be so is not at all surprising. Our intuition tells us that for a curved, but locally Eucidean, 2D manifold the effect of curvature appears at lowest order in the 2nd derivative of the metric – and so in non-vanishing derivatives of $\Gamma^\mu{}_{\alpha\beta}$ What may be surprising is that we are left with a single 'shortfall' in the number of degrees of freedom. The curvature – in 2D – *is evidently described by a single number* (at each point in space). There is no measurable 'anisotropy' or 'directionality' of the curvature.

This might seem a little odd, if one thinks about the curvature of a 2D *surface* $z(x, y)$ where the curvature is characterised by the 2×2 symmetric matrix of 2nd derivatives of $z(x, y)$. I.e. by 3 numbers at each point. But that matrix describes the *extrinsic curvature* of the surface embedded in the 3-space (x, y, z). In GR the 'embedding' space does not exist, and we would like, if at all possible to avoid any reference to it. It is the *intrinsic curvature* in 2D that has only a single degree of freedom. The distinction between intrinsic and extrinsic curvature is nicely illuminated by a curled up sheet of paper; that is intrinsically flat, but the extrinsic curvature is non-zero.

Coming back to the question posed at the outset, evidently the blind ants cannot sense the orientation of the egg! They can send out a pair of lines from a point with a certain opening angle and measure the deficit in the length of the arc joining their ends as compared to the Euclidean expectation. Or they can send a pair of ants marching along initially parallel paths and sense the change of separation. But the result of any such experiment is independent of the direction they do this measurement. Only if they were to do *non-local* experiments such as circumnavigating the egg can they learn, for example, that they are not living on a sphere.

2.2 3D Locally Euclidean manifold

Things are very similar for a locally Euclidean 3D manifold. Here the 3×3 metric has 6 independent components so the constant transformation matrix (not, in general, being symmetric and thus having 9 components) allows us to simultaneously make the primed frame metric be the identity matrix and set the three 'Euler angles' defining the orientation of the primed coordinate axes.

Here $g_{\alpha\beta,\gamma\delta}$ (being the symmetric 3×3 derivative operator $\partial_\gamma \partial_\delta$ acting on the symmetric 3×3 metric $g_{\alpha\beta}$) has $6 \times 6 = 36$ d.f. While $x^{\alpha'}{}_{,\beta\gamma\delta}$ has 30: There being 3 possible values for α' while, for the β, γ & δ indices, there are 3 like 000, another six like 011 and a final possibility 012 where they are all different (this does not occur in 2-dimensions). This give $3 \times (3 + 6 + 1) = 30$ in all. So there are $6 = 36 - 30$ numbers that describe the intrinsic curvature in 3 dimensions.

2.3 General N-dimensional manifold

- here the metric has $N + N(N-1)/2$ d.f. (or 10 for $N = 4$)
- while the transformation matrix $x^{\alpha'}{}_{,\beta}$ has $N \times N$ d.f. (or 16 for $N = 4$)

- allowing one (for $N = 4$) to specify the 10 degrees of freedom of the metric (a 4×4 symmetric matrix) and leaving over 6 additional degrees of freedom to specify the three components of the boost velocity and the three Euler angles for the specific Lorentz frame
- At next order, $g_{\alpha\beta,\gamma}$ and $x^{\alpha'}{}_{,\beta\gamma}$ both have $N \times (N + N(N-1)/2)$ d.f.
- (for $N = 4$ this is $4 \times 10 = 40$) d.f. giving us just enough freedom to make space-time locally inertial
  - i.e. to make all of the $g_{\alpha'\beta',\gamma'}$ vanish, and hence also nullify the connection $\Gamma^{\mu'}{}_{\alpha'\beta'}$, at $\mathcal{P}$
- The number of distinct combinations of β, γ & δ indices in $x^{\alpha'}{}_{,\beta\gamma\delta}$ is, in general, N where all indices are equal plus $N(N-1)$ where two are the same and the other is different and $N!/(N-3)!3!$ where they are all different (that is if $N > 2$; for $N = 2$ there is no way to have 3 indices all different). Multiplying by N, for the possible values of α', gives $N(N^2 + N!/(N-3)!3!)$ (or 80 for $N = 4$) as the number of independent combinations of indices of $x^{\alpha'}{}_{,\beta\gamma\delta}$
- while $g_{\alpha\beta,\gamma\delta}$ has, in general, $(N(N+1)/2)^2$ independent components, so for $N = 4$ there are 100 d.f.
- so the intrinsic curvature in 4D is characterised by 100 - 80 = 20 numbers
- in general the curvature in N dimensions is characterised by $(N(N+1)/2)^2 - N(N^2 + N!/(N-3)!3!) = N^2(N^2-1)/12$ numbers
  - this works for $N = 2$ if we define $(-1)! = 0$

In 4D then, the curvature – which we will see shortly is a tensor – has 20 independent components. This is more than the 6 independent components of the Newtonian tidal field tensor $\phi_{\mathrm{N},ij}$ that appears in the Newtonian geodesic deviation equation $d^2\Delta x_i/dt^2 = -\phi_{\mathrm{N},ij}\Delta x_j$. That's not unreasonable as we would expect the relativistic version of this to involve a 4-vector rather that a 3-vector displacement.

3 Parallel transport and covariant differentiation on a manifold

In the previous lecture we constructed – in flat space or space-time – the covariant derivative of a vector or tensor field in arbitrary curvilinear coordinates. To recapitulate:

The covariant derivative of a vector field $\vec{V}(\vec{x})$, for instance, is the $\binom{1}{1}$ tensor field $\nabla\vec{V}$, and this, when contracted with the tangent vector of a curve $\vec{U} = d\vec{x}/d\lambda$, gives $\nabla_{\vec{U}}\vec{V}$ which is the vector $d\vec{V}/d\lambda$.

This is simplest, of course, in rectilinear coordinates, where the components of $\nabla\vec{V}$ are simply $(\nabla\vec{V})^{\alpha'}{}_{\beta'} = V^{\alpha'}{}_{,\beta'}$, whereas, in curvilinear coordinates, the components are $(\nabla\vec{V})^{\alpha}{}_{\beta} = V^{\alpha}{}_{,\beta} + \Gamma^{\alpha}{}_{\mu\beta}V^{\mu}$ where the connection is defined such that the α^{th} component of rate of change of the μ^{th} basis with respect to the β^{th} coordinate is $((\vec{e}_\mu)_{,\beta})^{\alpha} = \Gamma^{\alpha}{}_{\mu\beta}$.

And we showed how the connection (the Christoffel symbols) $\Gamma^{\mu}{}_{\alpha\beta}$ can be determined from the metric components $g_{\alpha\beta}$ in the curvilinear coordinate system, without any reference to any rectilinear coordinate system, as

$$\Gamma^{\alpha}{}_{\mu\beta} = \tfrac{1}{2}g^{\alpha\gamma}(g_{\gamma\beta,\mu} + g_{\gamma\mu,\beta} - g_{\beta\mu,\gamma}). \tag{8}$$

We also discussed how this allows one to calculate the rate of change of the components of a vector – like the 4-momentum of a particle – that is 'parallel transported' along the path: $\nabla_{\vec{p}}\vec{p} = 0$. This gives the geodesic equation, which can be integrated to give the path $\vec{x}(\lambda)$ and the 4-momentum.

We now want to do all of the above on a curved manifold. At the *calculational* level this is rather straightforward, as all of the above still holds. At a *conceptual* level things are different; essentially because it is now more difficult to compare vectors (or tensors) at different points on a manifold to say whether in fact they are changing.

To overcome this conceptual hurdle, it is best to start with the physical concept of parallel transport of a vector. The parallel transport of a vector along a curve on a 2D manifold is illustrated in the left hand side of figure 3 by means of a geometrical construction known as 'Schild's ladder'. There are other ways to physically realise parallel transport on a 2D manifold. If you ride a tricycle across such a surface and cross a line of wet paint then the marks that your wheels will make have separations that are parallel.

This is unambiguous and conceptually straightforward and allows one to generate a parallel transported copy of a vector $\vec{V}(\vec{x}_0)$ living at $\vec{x}_0$ at another location $\vec{x}_1$. If we have a vector *field* $\vec{V}(\vec{x})$ we can subtract

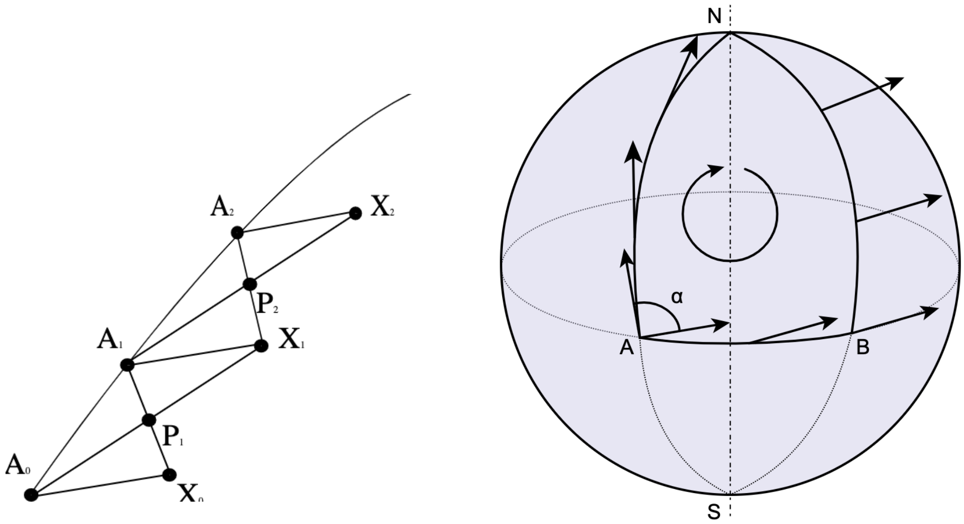


Figure 3: The left panel shows 'Schild's ladder'. The idea here is that if we have a vector (here $\vec{X}_0$) at some point $\vec{x} = \vec{A}_0$, and a path $\vec{x}(\lambda)$ passing through that point, we can obtain an approximation to a parallel transported copy $\vec{X}_1$ of $\vec{X}_0$ at a point $\vec{A}_1$ by erecting a vector $\vec{A}_1 - (\vec{A}_0 + \vec{X}_0)$. We then construct a vector from $\vec{A}_0$ to the mid point of that vector and extend it the same distance. This gives the end point of the vector $\vec{X}_1$. If we make the steps of the ladder smaller we obtain in the limit the (scaled down) parallel transported vector along the path. The right panel shows, using a sphere as an example, that the result of parallel transporting a vector on a curved manifold depends not just on the end points of the path but on the path taken. Here, the vector transported directly from the N-pole to A is rotated with respect to the result of transporting it to A via B. The difference – here just a rotation – is, quite generally, proportional to the area enclosed between the paths.

the parallel transported copy from $\vec{V}(\vec{x}_1)$ and, dividing by the length of the displacement we and taking the limit we get the derivative of the field. *This is the definition of the covariant derivative on a curved manifold.*

The only problem with this, as illustrated in the right hand side of figure 3, is that the result of parallel transporting a vector depends not just on the end points, but on the path taken. This is an interesting fact, but it is important to realise *that this does not affect the derivative defined as the limit.* The reason is that the difference in parallel transporting along different paths scales with the *area* of the region enclosed by the paths. It is therefore of second order in $\Delta\vec{x}$.

We are more interested in parallel transport on a 4D locally Minkowskian manifold. This is much the same; the result of transporting a vector over a finite distance will depend on the path taken, but in the limit of a small displacement it becomes unambiguous. We can, if we like, appeal to our inertial observers; for them space-time is locally flat and parallel transported vectors (or tensors) simply maintain constant physical components up to first order in the displacement.

One can formalise this as follows: Given the value of a vector field $\vec{V}$ at some position $\vec{x}$, we can decompose this as a sum of basis vectors in the locally inertial frame (LIF) $\vec{V}(\vec{x}) = V^{\alpha'}(\vec{x})\vec{e}_{\alpha'}$. Since the LIF basis vectors are constant, at linear order in the displacement, we do not need to indicate where the basis vector $\vec{e}_{\alpha'}$ is evaluated. The parallel transported version of this at $\vec{x}+\Delta\vec{x}$, which we denote by $\vec{V}_p(\vec{x}+\Delta\vec{x})$ is just $V_p(\vec{x}+\Delta\vec{x}) = V^{\alpha'}(\vec{x})\vec{e}_{\alpha'}$. I.e. the same components as at $\vec{x}$, because the basis vectors have not changed. We'd like to express this in terms of the components in the general frame and in terms of general frame basis vectors. To do so we use $V^{\alpha'}(\vec{x}) = \Lambda^{\alpha'}{}_{\alpha}(\vec{x})V^{\alpha}(\vec{x})$ and $\vec{e}_{\alpha'} = \Lambda^{\mu}{}_{\alpha'}(\vec{x}+\Delta\vec{x})\vec{e}_{\mu}(\vec{x}+\Delta\vec{x})$ so the parallel transported vector is

$$\vec{V}_p(\vec{x}+\Delta\vec{x}) = V^{\alpha'}(\vec{x}) \times \vec{e}_{\alpha'} = \Lambda^{\alpha'}{}_{\alpha}(\vec{x})V^{\alpha}(\vec{x}) \times \Lambda^{\mu}{}_{\alpha'}(\vec{x}+\Delta x)\vec{e}_{\mu}(\vec{x}+\Delta\vec{x}) \tag{9}$$

Or, Taylor expanding $\Lambda^{\alpha'}{}_{\alpha}(\vec{x}) = \Lambda^{\alpha'}{}_{\alpha}(\vec{x}+\Delta\vec{x}) - \Lambda^{\alpha'}{}_{\alpha,\beta}\Delta x^{\beta}$ and $V^{\alpha}(\vec{x}) = V^{\alpha}(\vec{x}+\Delta\vec{x}) - V^{\alpha}{}_{,\beta}\Delta x^{\beta}$, and using the fact that $\Lambda^{\alpha'}{}_{\alpha}$ and $\Lambda^{\alpha}{}_{\alpha'}$ are inverses of one another when evaluated at the same position, we get

$$\vec{V}_p = V^{\mu}\vec{e}_{\mu} - \Delta x^{\beta}(\Lambda^{\mu}{}_{\alpha'}\Lambda^{\alpha'}{}_{\alpha,\beta}V^{\alpha} + V^{\mu}{}_{,\beta})\vec{e}_{\mu} \tag{10}$$

where now everything is evaluated at $\vec{x}+\Delta\vec{x}$. But $V^{\mu}\vec{e}_{\mu} = \vec{V}$, so we have

$$\vec{V}(\vec{x}+\Delta\vec{x}) - \vec{V}_p(\vec{x}+\Delta\vec{x}) = (V^{\mu}{}_{,\beta} + \Gamma^{\mu}{}_{\alpha\beta}V^{\alpha})\Delta x^{\beta}\vec{e}_{\mu} \tag{11}$$

where we have used $\Gamma^\mu{}_{\alpha\beta} = \Lambda^\mu{}_{\alpha'}\Lambda^{\alpha'}{}_{\alpha,\beta}$.

This is valid for arbitrary (but small) $\Delta\vec{x}$. Taking $\Delta\vec{x} = \Delta\lambda\vec{e}_\beta$ and dividing the above by $\Delta\lambda$, taking the limit, and calling the result $(\nabla\vec{V})_\beta$ we have

$$(\nabla\vec{V})_\beta \equiv \lim_{\Delta\lambda\to 0} \frac{\vec{V}(\vec{x}+\Delta\lambda\vec{e}_\beta) - \vec{V}_p(\vec{x}+\Delta\lambda\vec{e}_\beta)}{\Delta\lambda} = (V^\mu{}_{,\beta} + \Gamma^\mu{}_{\alpha\beta}V^\alpha)\vec{e}_\mu \tag{12}$$

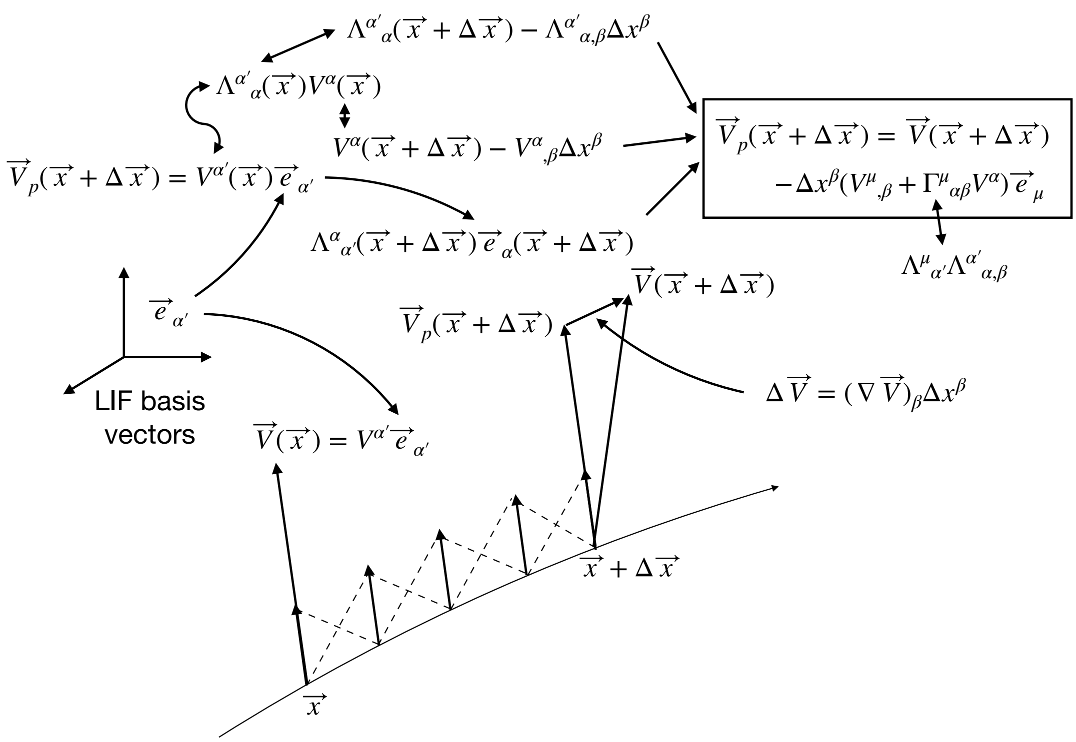


Figure 4: The steps leading to (13) for the components of the covariant derivative. For small $\Delta\vec{x}$ we can take the local inertial frame (LIF) basis vectors to be constant at linear order. So the LIF components of the parallel transported copy $V^{\alpha'}(\vec{x}+\Delta\vec{x})$ are the same as the original components at $\vec{x}$ (again to 1st order precision). We convert these to the general frame: $V^{\alpha'}(\vec{x}) = \Lambda^{\alpha'}{}_\alpha(\vec{x})V^\alpha(\vec{x})$ and then do a Taylor expansion of both factors to express each one as their value at $\vec{x}+\Delta\vec{x}$ minus a 'delta'. Together with $\vec{e}_{\alpha'} = \Lambda^\alpha{}_{\alpha'}(\vec{x}+\Delta\vec{x})\vec{e}_\alpha(\vec{x}+\Delta\vec{x})$ this gives an expression for $\vec{V}_p(\vec{x}+\Delta\vec{x})$ as $\vec{V}(\vec{x}+\Delta\vec{x})$ minus a 'delta' in which we recognise the familiar expression for the (flat space-time) covariant derivative. The bottom line is that the rate at which a vector field $\vec{V}(\vec{x})$ is changing *relative to a parallel transported copy of itself* is the same as the flat space-time covariant derivative formula, in which the connection coefficients may be computed from the metric also using the flat space-time formula.

For the limit to exist requires that both the vector field $\vec{V}(\vec{x})$ and the manifold itself be differentiable, which we assume to be the case. The above equation says that the μ^{th} component of $(\nabla\vec{V})_\beta$, which we will denote by $(\nabla\vec{V})^\mu{}_\beta = V^\mu{}_{;\beta}$, is

$$\boxed{V^\mu{}_{;\beta} = V^\mu{}_{,\beta} + \Gamma^\mu{}_{\alpha\beta}V^\alpha} \tag{13}$$

which is identical to the flat space-time formula.

But this should come as no real surprise. As we have discussed, inertial observers measuring locally inertial coordinates (i.e. physical coordinates) $x^{\alpha'}$ of events with known general coordinates x^α can determine the differential mapping $\Lambda^{\alpha'}{}_\alpha = \partial x^{\alpha'}/\partial x^\alpha = (\vec{e}_\alpha)^{\alpha'}$. And they can determine how this varies, so they can compute the connection $\Gamma^\mu{}_{\alpha\beta} = \Lambda^\mu{}_{\alpha'}\Lambda^{\alpha'}{}_{\alpha,\beta} = (\partial_\beta\vec{e}_\alpha)^\mu$. Or, equivalently, they can use the same data to compute the components of the metric $g_{\alpha\beta}(\vec{x})$ and how it varies, and and compute the connection using (8). All of this is valid on a curved manifold, just as in flat space-time, because it only involves first derivatives, and these are unaffected by curvature.

Armed with the covariant derivative we can define the rate of change of $\vec{V}(\vec{x})$ along a path with tangent vector $\vec{U} = U^\beta\vec{e}_\beta$ to be the sum of the $(\nabla\vec{V})_\beta$ – being the rate at which $\vec{V}$ is changing with respect to x^β –

weighted by U^β – the rate at which x^β is changing with path length λ – or

$$d\vec{V}/d\lambda = \nabla_{\vec{U}}\vec{V} = (\nabla\vec{V})_\beta U^\beta. \tag{14}$$

This is sometimes denoted by $D\vec{V}/d\lambda$. Either way, the physical meaning is that this is how the vector field is varying with respect to a locally parallel transported copy of itself.

And, again just as in flat space time, if we have a particle that is freely moving, and therefore conserving – i.e. parallel transporting – its momentum $\vec{p} = m\vec{U}$ the we can use $\nabla_{\vec{U}}\vec{U} = 0$ to give us the geodesic equation that we can integrate to find the particle trajectory.

If you need further convincing of the legitimacy of all of this consider a curved manifold with metric $\mathbf{g}(\vec{x})$ in some general coordinate system x^α, and consider a curve $x^\alpha(\lambda)$ parameterised by distance along the curve λ. This forces the constraint on the tangent vector $\vec{U} \to dx^\alpha/d\lambda$ (which we will also denote by $\dot{x}^\alpha$) that, since $d\lambda = \sqrt{g_{\alpha\beta}dx^\alpha dx^\beta}$, we have $\sqrt{\vec{U}\cdot\vec{U}} = \sqrt{g_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta} = d\lambda/d\lambda = 1$. Now require that the path be of *extremal length*: $\delta\int d\lambda\sqrt{g_{\alpha\beta}(\vec{x})\dot{x}^\alpha\dot{x}^\beta} = 0$. This looks like a problem in variational mechanics for a particle with velocity $\dot{x}^\alpha$ and a Lagrangian $L(x^\alpha, \dot{x}^\alpha) = \sqrt{g_{\alpha\beta}(\vec{x})\dot{x}^\alpha\dot{x}^\beta}$. Write down the Euler-Lagrange equations. Show that these are equivalent to $d^2x^\alpha/d\lambda^2 = -\Gamma^\gamma{}_{\mu\beta}(dx^\mu/d\lambda)(dx^\beta/d\lambda)$ which is the *geodesic equation.*

A key point here is that we are *defining* the covariant derivative of a vector field $\vec{V}(\vec{x})$ in terms of parallel transport:

- *The covariant derivative of a vector field is defined to be the rate of change with position of the field with respect to a parallel transported copy of itself.*

Many text books take the opposite path: they first *construct* the covariant derivative operator $\nabla_{\vec{U}}$ by requiring that it have certain pleasant properties such as linearity and that it obey the usual rules when applied to a product etc. and that it carry over to the flat-space, but curvilinear, derivative, and then define parallel transport of a vector $\vec{V}$ to be the condition that $\nabla_{\vec{U}}\vec{V} = 0$. But that seems backwards.

We have then shown that the formulae we developed for flat space-time are still valid. So far we considered only the covariant derivative of vectors, but (e.g. considering the derivative of a scalar $\phi = \vec{V}(\tilde{p}) = p_\alpha V^\alpha$) we can readily get the covariant derivatives of 1-forms and/or tensors of arbitrary rank.

Another key point is that the validity of (8) derives, as in flat space-time, from the fact that the metric – considered as a geometric object $\mathbf{g}$ – is covariantly (locally) constant: $\boldsymbol{\nabla}\mathbf{g} = 0$. But that only holds up to first derivatives; if we look at 2nd derivatives we expect to see the effect of the gravitational (i.e. tidal) field.

4 Space-time curvature: the gravitational field

4.1 Introduction

4.1.1 Homogeneous 2-dimensional spaces

Consider these 2-dimensional line elements:

$$\begin{aligned} dl^2 &= dr^2 + r^2 d\theta^2 \\ dl^2 &= d\theta^2 + \sin(\theta)^2 d\phi^2 \\ dl^2 &= d\chi^2 + \sinh(\chi)^2 d\phi^2 \end{aligned} \tag{15}$$

We immediately recognise that the first as the metric on a 2D flat sheet of paper in polar coordinates while the second is the metric on the unit sphere: a curved 2D surface - which we can visualise as embedded in 3D Euclidian space – expressed in terms of the conventional polar coordinates θ, ϕ. They are illustrated – embedded in Euclidean 3-space – in figure 5. They have the property that they are spatially homogeneous; all points are equivalent – though this is not obvious from the line elements (or the components of the metric tensors that we can read off from the line elements). The third is also homogeneous – though that is not immediately obvious even from the diagram, which shows it to be ‘saddle-like’ or ‘hyperbolic’.

Figure 5 illustrates two ways that we might measure the local curvature of these surfaces, which we know from §2.1 to be characterised by a single number.

One is to measure the sum of the angles of a triangle made from 3 geodesics. For the plane the result is π while for the spherical and hyperbolic cases these is respectively an excess and deficit, which grows in proportion to the area of the triangle (being a large – order unity – effect for triangles of linear size of order the radius of curvature (unity in the coordinates we have adopted)). Legend has it that Gauss tried to

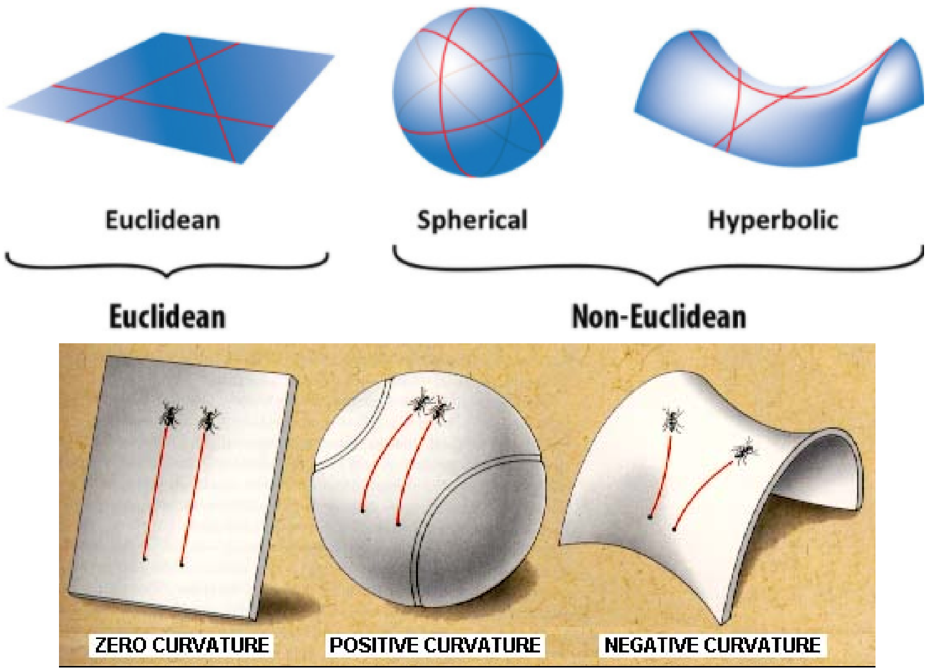


Figure 5: Geometry of the three possible homogeneous locally Euclidean spaces in 2D. The space can be flat (left) or be spherical (positive curvature) or hyperbolic (negatively curved). The curvature in 2D is, quite generally, characterised by a single number at each point on the manifold. Here it is a single global number for each geometry, which could be taken to be the radius of curvature. It can be measured by summing the angles of triangles; by measuring deviation of initially parallel geodesics; or by comparing circumferences to radii of circles.

measure this experimentally in the Alps. Q: Assuming he had baselines of order 10km in length, and that the orbital period for near-earth satellites is about 90 minutes, how precise a measurement of angles would he have needed to be successful?

Another approach is to use 'geodesic deviation'. Here we extend two neighbouring, initially parallel, geodesics and measure how their separation changes. Or equivalently measure how their deviation from parallelism changes.

All of these implicitly involve parallel transport as the geodesics are curves that parallel transport their tangent vectors. Another way to measure curvature in 2D would be to measure the circumference of circles made by extending geodesics of fixed length from a common starting point and comparing with the radius length.

In 2D curvature is simple as it is characterised by 1 number (at each point in space). In 3D we can measure 6 quantities, and in 4D there are 20. What is the best way to quantify the curvature in such spaces?

4.1.2 Definition of the Riemann curvature tensor

Figure 6 illustrates, now in 3D, the curvature measurement concept used in GR: It is an expression of the result of taking a vector $\vec{V}$ and parallel transporting it around a closed loop and measuring how much it has changed relative to its initial value. That is a vector valued function of 3 other vectors; one being the initial vector $\vec{V}$ and the other two being the two vectors lying along the edges of the parallelogram defining the path. It is evidently a linear function of the initial vector. Our intuition – based on experience with spheres etc. – tells us that the result will scale with the *area* of the parallelogram, and would therefore have to be linear in the length of each leg. Were we to let this act on a 1-form (at the starting point) we would have a scalar function of three vectors and a 1-form that would be linear in each of its arguments. It is therefore a $\binom{1}{3}$ tensor, and it is called the *Riemann curvature tensor* defined[2] such that

$$\Delta\vec{V} = -\mathbf{R}(\ ,\vec{V},\vec{a},\vec{b}). \tag{16}$$

[2]If this notation, with the missing first argument to $\mathbf{R}()$ makes you nervous, don't be. Simply read $\mathbf{R}(\ ,\vec{V},\vec{a},\vec{b})$ as saying: if I were to provide the basis 1-form $\tilde{\omega}^\alpha$ here, the result would be (minus) the $\alpha^{\rm th}$ component of the vector $\Delta\vec{V}$.

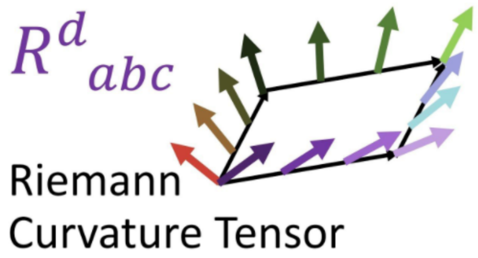


Figure 6: The Riemann curvature tensor can be defined in terms of parallel transport of a vector around a closed loop. It is illustrated here in 3D where we take the purple vector and parallel transport it around a parallelogram to obtain the red vector. The difference of these is given by the Riemann tensor acting on 3 vectors; the initial purple vector and the 2 vectors parallel to the sides of the parallelogram.

At first sight this might seem problematic as such a tensor has, in general, N^4 components, or 256 in 4D, not 20. In 3D it has 81 components, not 6. But there is much redundant information in the full complement of components since, as we will see, there are symmetries that reduce the number of degrees of freedom. One set of symmetries is the dependence on the choice of parallelogram; the result only depends on the area (1 number) and the direction normal to the plane defined by the 2 vectors (2 numbers in 3D), for a total of 3 degrees of freedom, rather than the full 6 d.f. for 2 vectors in 3D.

There is another, essentially equivalent, ways to define the curvature tensor, which is as the commutator of two second (covariant) derivatives of a vector field. The definition in terms of parallel transport around a loop is mathematically as simple as any other approach – and arguably conceptually clearer – and it is also the definition that is most closely linked to how we actually *use* the Riemann tensor in practice. So we will develop that approach in the following section and then afterwards show the relation to the commutator of derivatives. Following that we will show that the symmetries of the tensor do indeed result in the expected number of independent components. We end this section with the *geodesic deviation equation* (GDE). We show how the covariant derivative gives us an equation for the evolution of the separation between a pair of neighbouring geodesics that is very similar in form to the Newtonian equation $d^2\mathbf{x}/dt^2 = -\mathbf{x}\cdot\boldsymbol{\nabla\nabla}\phi$ but with the right hand side containing the curvature tensor in place of the Newtonian tidal field tensor $\boldsymbol{\nabla\nabla}\phi$. This establishes a rather direct connecting between the relativistic curvature and the Newtonian tidal field and suggests that a certain contraction of the curvature is equal to the matter density, in the Newtonian limit, and, by extension this indicates the path to Einstein's field equations.

4.2 The Riemann curvature tensor from parallel transport

Here we obtain an expression for the curvature tensor in terms of the connection.

A common approach is to argue, reasonably, that the above definition is equivalent to asking what is the difference between the results of transporting a vector from one corner of a parallelogram to the opposite one; i.e. first along a vector $\vec{a}$ followed by $\vec{b}$ as compared to the path $\vec{b}$ followed by $\vec{a}$. And then to argue this must somehow be essentially equivalent to $\nabla_{\vec{b}}\nabla_{\vec{a}}\vec{V} - \nabla_{\vec{a}}\nabla_{\vec{b}}\vec{V}$; i.e. the difference between applying the covariant derivative to a vector *field* $\vec{V}(\vec{x})$, first in direction $\vec{a}$ followed by direction $\vec{b}$ and vice versa, and that this must be the same as what we would get by applying the Riemann tensor to $\vec{a}$, $\vec{b}$ and $\vec{V}$.

But there is a serious problem with this. In the definition we have set out there is no vector field $\vec{V}(\vec{x})$. There is just a vector at a point that is transported along a line or a series of lines. And we are not differentiating anything. Rather we are asking what is the difference between transporting this vector along two paths *while holding it fixed.*

So while the commutator of second derivatives is one route to the desired answer, the reason this should be equivalent is, *a priori,* not at all clear. As the curvature tensor is such an essential tool in GR we will instead tackle the problem head on. The route we will follow is summarised in figure 7.

The equation of parallel transport is $dV^\alpha/dx^\beta = -\Gamma^\alpha{}_{\gamma\beta}V^\gamma$, so transporting $\vec{V}$ around a loop gives

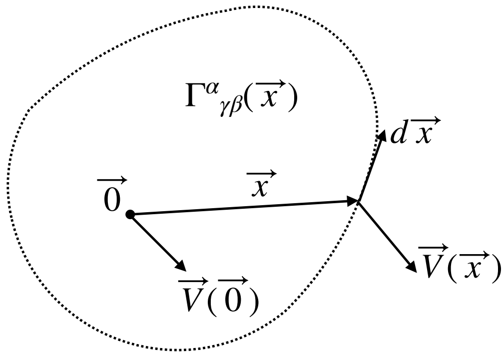
$$\Delta V^\alpha = \oint dV^\alpha = -\oint dx^\beta \Gamma^\alpha{}_{\gamma\beta}(\vec{x})V^\gamma(\vec{x}) \tag{17}$$

which, on quite general grounds will, for a small loop, scale as the area of the loop.

Evaluating this appears difficult because $V^\gamma(\vec{x})$ here is itself the result of transporting the vector we started with. But we can proceed by replacing $V^\gamma(\vec{x})$ by the *field* created by parallel transporting the initial

Riemann curvature tensor $R^{\alpha}{}_{\nu\beta\mu}$

- Parallel transport: $dV^{\alpha}/dx^{\beta} = -\Gamma^{\alpha}{}_{\gamma\beta}V^{\gamma}$
- So transporting $\vec{V}$ around a loop gives
  - $\Delta V^{\alpha} = -\oint dx^{\beta}\Gamma^{\alpha}{}_{\gamma\beta}(\vec{x})V^{\gamma}(\vec{x})$
- **key feature:** ΔV^{α} scales as the loop area
  - i.e. second order in $|\vec{x}|$, the size of the loop



- so we can calculate ΔV^{α} (and find its dependence on $\Gamma^{\alpha}{}_{\gamma\beta}(\vec{x})$) using a linear approximation $V^{\gamma}(\vec{x}) = V^{\gamma}(\vec{0}) - \Gamma^{\gamma}{}_{\nu\mu}(\vec{0})V^{\nu}(\vec{0})x^{\mu} + \ldots$ i.e. a linear (single-step) approximation to the result of parallel transporting $\vec{V}$ from $\vec{0}$ to $\vec{x}$
- along with $\Gamma^{\alpha}{}_{\gamma\beta}(\vec{x}) = \Gamma^{\alpha}{}_{\gamma\beta}(\vec{0}) + \Gamma^{\alpha}{}_{\gamma\beta,\mu}(\vec{0})x^{\mu} + \ldots$ this gives, in the integral above,
  - $\Delta V^{\alpha} = -\Gamma^{\alpha}{}_{\gamma\beta}V^{\gamma}\oint dx^{\beta} - (\Gamma^{\alpha}{}_{\nu\beta,\mu} - \Gamma^{\alpha}{}_{\gamma\beta}\Gamma^{\gamma}{}_{\nu\mu})V^{\nu}\oint dx^{\beta}x^{\mu} + \ldots$
  - but $\oint dx^{\beta} = 0$ and, for loop in $x-y$ plane, $\oint dx^{\beta}x^{\mu} = \delta A(\delta^{\beta}_{y}\delta^{\mu}_{x} - \delta^{\beta}_{x}\delta^{\mu}_{y})$ so

$\Delta V^{\alpha} = \delta A R^{\alpha}{}_{\nu xy}V^{\nu}$ where $R^{\alpha}{}_{\nu xy} \equiv \Gamma^{\alpha}{}_{\nu x,y} - \Gamma^{\alpha}{}_{\gamma x}\Gamma^{\gamma}{}_{\nu y} - \Gamma^{\alpha}{}_{\nu y,x} + \Gamma^{\alpha}{}_{\gamma y}\Gamma^{\gamma}{}_{\nu x}$

Figure 7: The essence of the approach used here, and described fully in the text, to obtain the Riemann curvature tensor in terms of the connection is as follows: We take some point in the vicinity of the loop to be the origin $\vec{0}$, and we construct a vector field $\vec{V}(\vec{x})$ around this by parallel transporting the vector $\vec{V}(\vec{0})$ along a straight line to $\vec{x}$. This is accurate to only first order in the size of the loop, but it is sufficient, if we plug it into the integral giving $\Delta\vec{V}$, to give this at second order which is all that we need as the result scales, for small loops, in proportion to the area. As the field $\vec{V}(\vec{x})$ has a zeroth order component $\vec{V}(\vec{0})$ we need to use a 1st order Taylor series expansion of $\Gamma^{\alpha}{}_{\gamma\beta}(\vec{x})$ also, and this involves, of course, the derivative of the connection. The result is – for the case of a loop lying in the $x-y$ plane – the formula at the bottom for the curvature tensor.

vector along a straight line from the point in question – which we can take, without loss of generality, to be the origin $\vec{0}$ – to $\vec{x}$:

$$V^{\gamma}(\vec{x}) = V^{\gamma}(\vec{0}) - \Gamma^{\gamma}{}_{\nu\mu}(\vec{0})V^{\nu}(\vec{0})x^{\mu} + \ldots \tag{18}$$

where we can use a first-order (in loop size) approximation as the value of any loop integral $\oint dx^{\beta}\ldots$ is at most on the order of the length of the path times the integrand. So first order precision in the integrand is sufficient to obtain second order precision in $\Delta\vec{V}$.

Replacing the Christoffel symbol appearing in the integral by a 1st order Taylor expansion

$$\Gamma^{\alpha}{}_{\gamma\beta}(\vec{x}) = \Gamma^{\alpha}{}_{\gamma\beta}(\vec{0}) + \Gamma^{\alpha}{}_{\gamma\beta,\mu}(\vec{0})x^{\mu} + \ldots \tag{19}$$

we obtain

$$\Delta V^{\alpha} = -\Gamma^{\alpha}{}_{\gamma\beta}V^{\gamma}\oint dx^{\beta} - (\Gamma^{\alpha}{}_{\nu\beta,\mu} - \Gamma^{\alpha}{}_{\gamma\beta}\Gamma^{\gamma}{}_{\nu\mu})V^{\nu}\oint dx^{\beta}x^{\mu} + \ldots \tag{20}$$

where all quantities outside the integral are their values at the origin. But $\oint dx^{\beta} = 0$ so the first term vanishes, and we are almost done, we just need to evaluate the matrix $\oint dx^{\beta}x^{\mu}$ in the second term.

Evidently, this vanishes if $\beta = \mu$, since e.g. $\oint xdx = \oint dx^2/2 = 0$. For the special case of a loop lying in the $x-y$ plane, and assuming an anti-clockwise path, this is equal to the area δA of the loop if $\beta = y$ and $\mu = x$ and is $-\delta A$ if $\beta = x$ and $\mu = y$, so $\oint dx^{\beta}x^{\mu} = \delta A(\delta^{\beta}_{y}\delta^{\mu}_{x} - \delta^{\beta}_{x}\delta^{\mu}_{y})$.

Specialising further to a path that is a rectangle with sides Δx and Δy we have

$$\Delta V^{\alpha} = \Delta x \Delta y (\Gamma^{\alpha}{}_{\beta x,y} - \Gamma^{\alpha}{}_{\gamma x}\Gamma^{\gamma}{}_{\beta y} - \{x \Leftrightarrow y\}) V^{\beta} \tag{21}$$

where $\{x \Leftrightarrow y\}$) indicates the result of exchanging the indices.

Writing the two rectangle edge vectors as $\vec{a} \to a^{\mu} = \delta^{\mu}{}_{x}\Delta x$ and $\vec{b} \to b^{\nu} = \delta^{\nu}{}_{y}\Delta y$ this gives

$$\boxed{\Delta V^{\alpha} = -R^{\alpha}{}_{\beta\mu\nu} V^{\beta} a^{\mu} b^{\nu}} \tag{22}$$

where the minus sign is purely conventional[3], and which is evidently linear in the vectors $\vec{a}$, $\vec{b}$ and $\vec{V}$, and where we are defining the *Riemann tensor* as

$$\boxed{R^{\alpha}{}_{\beta\mu\nu} \equiv \Gamma^{\alpha}{}_{\beta\nu,\mu} - \Gamma^{\alpha}{}_{\gamma\nu}\Gamma^{\gamma}{}_{\beta\mu} - \{\nu \Leftrightarrow \mu\}} \tag{23}$$

which evidently contains both derivatives and products of the Christoffel symbols.

Considering a rectangular path is not a serious restriction, as one can consider a more general path to be the exterior of a region composed of multiple rectangles.

An illustration of this machinery is provided by the exercise at the end of this section. It asks you to compute the connection on a sphere and to use this to compute how a vector changes if it is transported around a closed loop. In the process it will introduce you to an example of a *non-coordinate basis*.

4.3 Riemann tensor as the commutator of second covariant derivatives

As already mentioned, the curvature tensor is also, it turns out, expressible as the commutator of the second covariant derivatives of a vector field:

$$\mathbf{R}(\ \ , \vec{V}, \vec{a}, \vec{b}) = (\nabla_{\vec{a}}\nabla_{\vec{b}} - \nabla_{\vec{b}}\nabla_{\vec{a}})\vec{V} = [\nabla_{\vec{a}}, \nabla_{\vec{b}}]\vec{V} \tag{24}$$

and many books take this to be the definition of the curvature.

At first sight, this seems unreasonable. The thing on the left is a vector with α^{th} component $R^{\alpha}{}_{\beta\mu\nu}V^{\beta}a^{\mu}b^{\nu}$, which depends only on the value of the components of $\vec{V}$ at the point in question, whereas the thing on the right hand side would appear to also depend on how $\vec{V}$ is varying as well.

To show that this is not the case and that (24) is actually valid, consider first $\nabla_{\vec{b}}\vec{V} = \nabla_{\vec{b}}(V^{\beta}\vec{e}_{\beta}) = b^{\nu}(V^{\beta}\vec{e}_{\beta})_{;\nu} = b^{\nu}(V^{\beta}{}_{,\nu}\vec{e}_{\beta} + V^{\beta}\vec{e}_{\beta;\nu})$, where we have used the usual rule for the derivative of a product and, since the components V^{β} are just ordinary functions of the coordinates, have used $V^{\beta}{}_{;\nu} = V^{\beta}{}_{,\nu}$.

Taking the derivative of this along the tangent vector $\vec{a}$ gives

$$\nabla_{\vec{a}}\nabla_{\vec{b}}\vec{V} = a^{\mu}b^{\nu}(V^{\beta}{}_{,\nu}\vec{e}_{\beta} + V^{\beta}\vec{e}_{\beta;\nu})_{;\mu} = a^{\mu}b^{\nu}(V^{\beta}{}_{,\nu\mu}\vec{e}_{\beta} + V^{\beta}{}_{,\nu}\vec{e}_{\beta;\mu} + V^{\beta}{}_{,\mu}\vec{e}_{\beta;\nu} + V^{\beta}\vec{e}_{\beta;\nu\mu}) \tag{25}$$

where we are assuming b^{ν} to be constant: $b^{\nu}{}_{,\mu} = 0$. So, as expected, this contains first and second derivatives of the components of $\vec{V}$, in addition to the components of $\vec{V}$ itself.

But the first term is symmetric under exchange of the order of derivatives $\vec{a} \Leftrightarrow \vec{b}$ – or equivalently if we exchange $\mu \Leftrightarrow \nu$ – as are the 2nd and 3rd terms taken together. So these drop out when we commute the derivatives and we find

$$[\nabla_{\vec{a}}, \nabla_{\vec{b}}]\vec{V} = V^{\beta}a^{\mu}b^{\nu}(\vec{e}_{\beta;\nu\mu} - \vec{e}_{\beta;\mu\nu}) = V^{\beta}[\nabla_{\vec{a}}, \nabla_{\vec{b}}]\vec{e}_{\beta} \tag{26}$$

So there is indeed no dependence on the way that the components of $\vec{V}$ are varying. The vector $\vec{V}$ only enters through the value of its components V^{β} at the point where $[\nabla_{\vec{a}}, \nabla_{\vec{b}}]$ is being applied as it has 'passed through' the derivative operator. The derivatives of V^{β} having dropped out, all that remains is the commutator of derivatives of the basis vectors. And these, of course, are only a property of the manifold and coordinate system and are independent of $\vec{V}$.

The various commutators $\vec{e}_{\beta;\nu\mu} - \vec{e}_{\beta;\mu\nu}$, for the various possible values of β, μ and ν, do not, in general, vanish. One can verify this from the example of $\vec{e}_{\theta}$ on a sphere. This vector field is like a set of compass needles on the Earth. These are parallel transported in the latitude direction: $\vec{e}_{\theta;\theta} = 0$, so $\vec{e}_{\theta;\theta\phi} = 0$ also. But they do vary with longitude: $\vec{e}_{\theta;\phi} \neq 0$ (recall that what we mean by this physically is that $\vec{e}_{\theta}$ is changing

[3] As with many other vital ingredients such as the signature of the metric, there is no general agreement on the sign convention. Here we follow Schtuz, who follows MTW, the preface of which provides a useful table.

with respect to a parallel transported copy of itself as one moves along a line of constant latitude[4]). This variation *is* a function of latitude (it is zero on the equator and increases linearly as one moves off the equator) so $\vec{e}_{\theta;\phi\theta} \neq 0$. Hence the commutator $\vec{e}_{\theta;\phi\theta} - \vec{e}_{\theta;\theta\phi}$ is non-zero.

But it is not difficult to see that, if one is working in arbitrary curvilinear coordinates (ξ, η) on a flat surface with physical Cartesian coordinates (x, y), all the commutators *will* vanish. This is because the components of say the ξ basis vector $\vec{e}_\xi$ in the (x, y) frame are just the rate at which x and y are varying with respect to ξ: i.e. $\vec{e}_\xi = \vec{e}_x \partial x/\partial\xi + \vec{e}_y \partial y/\partial\xi$. But $\vec{e}_x$ and $\vec{e}_y$ are independent of position, so the commutator $\vec{e}_{\xi;\xi\eta} - \vec{e}_{\xi;\eta\xi}$, for example, has x-component $\partial^3 x/\partial\xi\partial\xi\partial\eta - \partial^3 x/\partial\xi\partial\eta\partial\xi$ which vanishes, as does the y-component.

The commutator $\vec{e}_{\beta;\nu\mu} - \vec{e}_{\beta;\mu\nu}$ (for some choice of the indices β, μ and ν) is a vector. What are its components?

Let's start with one of the 1st derivatives, say $\vec{e}_{\beta;\nu}$. The β^{th} basis vector is some vector field $\vec{e}_\beta(\vec{x})$. So, like any field, we can write it as $V^\alpha \vec{e}_\alpha$, and its covariant derivative $(\nabla\vec{e}_\beta)_\nu = (\nabla\vec{V})_\nu$ – being the rate at which it changes with respect to the ν^{th} coordinate relative to a parallel transported copy of itself – has components given by $V^\alpha{}_{;\nu} = V^\alpha{}_{,\nu} + \Gamma^\alpha{}_{\gamma\nu} V^\gamma$. But here the components of the basis vectors are constant: $(\vec{e}_\beta)^\alpha = \delta^\alpha_\beta$, so $V^\alpha = \delta^\alpha_\beta$, and consequently the components $\vec{e}_{\beta;\nu}$ are just the Christoffel symbols: $(\vec{e}_{\beta;\nu})^\alpha = \Gamma^\alpha{}_{\beta\nu}$

Taking a further derivative gives $\vec{e}_{\beta;\nu\mu} = (\Gamma^\alpha{}_{\beta\nu}\vec{e}_\alpha)_{;\mu} = \Gamma^\alpha{}_{\beta\nu,\mu}\vec{e}_\alpha + \Gamma^\alpha{}_{\beta\nu}\Gamma^\gamma{}_{\alpha\mu}\vec{e}_\gamma$, and flipping the dummy indices $\alpha \Leftrightarrow \gamma$ in the last term gives $\vec{e}_{\beta;\nu\mu} = (\Gamma^\alpha{}_{\beta\nu,\mu} + \Gamma^\gamma{}_{\beta\nu}\Gamma^\alpha{}_{\gamma\mu})\vec{e}_\alpha$. Finally, forming the commutator, we have

$$\boxed{\vec{e}_{\beta;\nu\mu} - \vec{e}_{\beta;\mu\nu} = (\Gamma^\alpha{}_{\beta\nu,\mu} - \Gamma^\alpha{}_{\gamma\nu}\Gamma^\gamma{}_{\beta\mu} - \{\nu \Leftrightarrow \mu\})\vec{e}_\alpha} \tag{27}$$

But the quantity in parentheses is identical to the definition of the Riemann tensor in terms of parallel transport, so the α^{th} components of the commutator of second derivatives of a basis vector $\vec{e}_\beta$ – i.e. what we get if we let $\vec{e}_{\beta;\nu\mu} - \vec{e}_{\beta;\mu\nu}$ act on the basis 1-form $\tilde{\omega}^\alpha$ – is $R^\alpha{}_{\beta\mu\nu}$.

So just as the Christoffel symbols are the components of the 1st derivatives of the basis vectors, the curvature tensor provides the components of the commutator of 2nd derivatives of the basis vectors.

Equation (27) provides a very nice alternative way to conceptualise the curvature. On a 2D manifold, for example, you have the freedom to choose the mapping $\xi(x, y)$ and $\eta(x, y)$ from locally inertial to general coordinates arbitrarily. It is up to you how to label the events. There is thus also great freedom in the basis vectors $\vec{e}_\xi$ and $\vec{e}_\eta$ and how they change with position on the manifold. But not *complete* freedom. Equation (27) says that the curvature of the manifold constrains the way that the basis vectors can vary. It says that, regardless of how you choose to label the events, if you take one of your basis vectors, let's say $\vec{e}_\xi$, and measure its rate of change with respect to ξ and then ask how that thing is changing with respect to η you will, in general get a different answer to how $\vec{e}_\xi$'s rate of change with η is varying with ξ – *unless you happen to be on a flat manifold*. The difference between these operations tells you about the intrinsic geometry of the manifold. It is a rank-4 tensor, with units of inverse length squared. While the numerical value of its *components* depend on the coordinate system (i.e. how you chose to label the events), when viewed as a geometric entity is absolute and invariant. And this, as we will shortly see, is the relativistic analogue of the Newtonian gravitational field.

Equation (27) allows us to write, rather abstractly,

$$\mathbf{R}(\ \ ,\ \ ,\vec{a},\vec{b}) = [\nabla_{\vec{a}}, \nabla_{\vec{b}}]. \tag{28}$$

To my mind, this is rather *too* abstract as it elides the crucial fact that while this operator only acts *algebraically* in regards to the *components* of its second argument, it acts as a differential operator on the basis vectors therein, so I prefer to write

$$\boxed{\mathbf{R}(\ \ ,\vec{e}_\alpha,\vec{a},\vec{b}) = [\nabla_{\vec{a}}, \nabla_{\vec{b}}]\vec{e}_\alpha} \tag{29}$$

to keep this explicit.

Finally, we should mention a technicality: We have assumed in all of the above that a^μ and b^ν are constant. If they are not then one has

$$\mathbf{R}(\ \ ,\vec{e}_\alpha\vec{a},\vec{b}) = ([\nabla_{\vec{a}}, \nabla_{\vec{b}}] - \nabla_{[\vec{a},\vec{b}]})\vec{e}_\alpha \tag{30}$$

[4]If it is not obvious that $\vec{e}_{\theta;\phi} \neq 0$ think about what happens if you ride your tricycle on a sphere along a line of constant latitude and cross a line of wet paint at the meridian $\phi = 0$. Your path is not a geodesic – unless you're at the equator – and the back wheel closer to the pole will turn less fast than the other. That means that the vectors joining the marks of paint that your wheels leave – which *are* being parallel transported – will rotate with respect to the lines of constant longitude, or with respect to compass needles, which are parallel to $\vec{e}_\theta$.

where the vector $[\vec{a}, \vec{b}] \equiv \nabla_{\vec{a}}\vec{b} - \nabla_{\vec{b}}\vec{a}$ has α^{th} component $a^\beta b^\alpha{}_{,\beta} - b^\beta a^\alpha{}_{,\beta}$. Or, more – perhaps too – abstractly, this is

$$\boxed{\mathbf{R}(\ \ ,\ \ , \vec{a}, \vec{b}) = [\nabla_{\vec{a}}, \nabla_{\vec{b}}] - \nabla_{[\vec{a},\vec{b}]}.} \tag{31}$$

The definition of the curvature tensor in terms of parallel transport is very useful as it leads, as we will show shortly, rather directly to the geodesic deviation equation and thus to directly observable quantities. One use of the definition here is that it shows there is a danger in naively applying the comma ⇒ semicolon rule. To see this, consider Maxwell's equations in flat space-time: $F^{\mu\nu}{}_{,\mu} = j^\nu$. From the definition of $F^{\mu\nu} = A^{\mu,\nu} - A^{\nu,\mu}$ we see that the LHS of Maxwell's equations will involve a term $g^{\nu\alpha}A^\mu{}_{,\alpha,\mu}$. The second derivative operator is symmetric, so this is the same as $g^{\nu\alpha}A^\mu{}_{,\mu,\alpha}$ but if we change to covariant derivatives the results will differ by the commutator. See MTW for further discussion.

4.4 Symmetries of the Riemann tensor

The Riemann tensor has $4^4 = 256$ components, but we have seen that the deviation of the metric from its flat space form should be described by 20 numbers. We now show that the symmetries of $\mathbf{R}$ do indeed result in only 20 independent components.

First of all, it is clear from the definition that the curvature tensor is anti-symmetric (changes sign) under exchange of its last two indices. This reduces the number of independent components.

To go further it is convenient to consider a *locally inertial* coordinate system ($g_{\alpha\beta} = \eta_{\alpha\beta}$ and $\Gamma^\gamma{}_{\epsilon\beta} = 0$ at point $\mathcal{P}$).

At $\mathcal{P}$ the curvature is then $R^\alpha{}_{\beta\mu\nu} = \Gamma^\alpha{}_{\beta\nu,\mu} - \Gamma^\alpha{}_{\beta\mu,\nu}$. While $\Gamma^\alpha{}_{\mu\nu,\sigma} = \frac{1}{2}g^{\alpha\beta}(g_{\beta\mu,\nu\sigma} + g_{\beta\nu,\mu\sigma} - g_{\mu\nu,\beta\sigma})$ (since $g^{\alpha\beta}{}_{,\sigma} = 0$). giving an expression for $R^\alpha{}_{\beta\mu\nu}$ with six terms. Exploiting the symmetry of $g_{\alpha\beta}$ and the symmetry of $g_{\alpha\beta,\mu\nu}$ under exchange of μ and ν shows that two of these terms cancel, and lowering the first index give the fully covariant form of the Riemann tensor

$$R_{\alpha\beta\mu\nu} = g_{\alpha\lambda}R^\lambda{}_{\beta\mu\nu} = \tfrac{1}{2}(g_{\alpha\nu,\beta\mu} - g_{\alpha\mu,\beta\nu} + g_{\beta\mu,\alpha\nu} - g_{\beta\nu,\alpha\mu}). \tag{32}$$

This allows one to verify the following symmetries:

1. anti-symmetry under exchange of first or second pair of arguments:
   - $R_{\alpha\beta\gamma\delta} = -R_{\alpha\beta\delta\gamma} = -R_{\beta\alpha\gamma\delta}$
2. symmetry under interchange of first and second pairs:
   - $R_{\alpha\beta\gamma\delta} = R_{\gamma\delta\alpha\beta}$
3. and the 'first' or 'algebraic' Bianchi identities:
   - $R_{\gamma\delta\alpha\beta} + R_{\gamma\beta\delta\alpha} + R_{\gamma\alpha\beta\delta} = 0$
   - in which we cycle the last 3 indices and add the result, and which is sometimes written as $R_{\gamma[\alpha\beta\delta]} = 0$

These symmetries reduce the number of independent components from $4^4 = 256$,the value for a general rank-4 tensor in 4 dimensions, as follows:

- (1) implies that we can package the non-vanishing components of $R_{\alpha\beta\gamma\delta}$ into a 6×6 matrix R_{IJ} where each of the symbols I and J can take one of six possible values $\{01, 02, 03, 12, 13, 23\}$, with the understanding that if we want to retrieve a component with either $\alpha\beta$ or $\gamma\delta$ in non-increasing order we apply a minus sign.
- (2) tells us that R_{IJ} in fact a symmetric 6×6 matrix, and therefore has $6+5+4+3+2+1 = 21$ components
- the last identity (3) does not provide any additional information, save for the case where all the indices are distinct, for which it provides a single additional constraint, which we can take to be $R_{0123} + R_{0312} + R_{0231} = 0$

These reduce the number of independent components to 20, in agreement with what we inferred from the difference between the 100 independent components of $g_{\alpha\beta,\gamma\delta}$ and the 80 independent values of $x^{\alpha'}{}_{,\beta\gamma\delta}$.

4.5 Measuring curvature from geodesic deviation

The idea of locally inertial observers measuring physical coordinates of events of known general coordinates and determining the differential mapping from which they can extract the curvature tensor as the commutator of 2nd derivatives of the basis vectors is one way to conceptualise curvature.

The way that curvature is measured in practice, however, is from the tidal effect on the world-lines of neighbouring particles (including photons) following geodesics.

This is conceptually straightforward; we take the geodesic equation for a single particle $d^2x^\alpha/d\tau^2 = -\Gamma^\alpha{}_{\mu\beta}(dx^\mu/d\tau)(dx^\beta/d\tau)$, which is that of a particle subject to a force given by the Christoffel symbols contracted twice with the 4-velocity $\vec{U} = d\vec{x}/d\tau$, and subtract this from that of a neighbouring particle. The result is the *geodesic deviation equation* (GDE) that gives $d^2\xi^\alpha/d\tau^2$, the 2nd rate of change of the coordinate separation $\xi^\alpha \equiv x_2^\alpha - x_1^\alpha$, in which the differential (i.e. tidal) force involves the derivative of the connection 'dotted' with the separation vector $\vec{\xi}$. These derivatives of the connection are obviously closely related to the curvature tensor (recall that in a LIF only the derivatives of the connection appear in the Riemann tensor).

We develop that in the following section. The resulting equation (34) is of great practical utility. It is used extensively, for instance, in gravitational lensing where we apply it to photons (which we can consider to behave the same as massive particles in the limit that $m \to 0$) and it allows one to calculate, for example, the tidal 'shearing' of images of distant galaxies by the gravitational influence of the intervening matter distribution.

It has the aesthetic disadvantage, however, that it is not 'covariant'; it is not actually a 4-vector equation. It gives the 2nd rate of change of a component of the *coordinate* separation ξ^α, which is not a 4-vector. This obscures the connection with the curvature tensor. What we would prefer to have is an equation for the ($\alpha^{\rm th}$ component of) $d^2\vec{\xi}/d\tau^2$, which *is* a 4-vector. As we will see in the next-but-one section, this brings in another derivative of the connection that, combined with the non-covariant GDE, gives a formula for $d^2\vec{\xi}/d\tau^2$ where it is the Riemann tensor that appears in the 'force term' on the RHS. This establishes a direction connection between the Riemann tensor and observable effects such as the tidal focussing of particles and of photons.

4.5.1 The non-covariant geodesic deviation equation

The geodesic equation – for the path of a particle following a world-line of extremal proper time – is $\nabla_{\vec{U}}\vec{U} = 0$ or $U^\beta U^\alpha{}_{;\beta} = 0$ or $U^\beta U^\alpha{}_{,\beta} + U^\beta U^\mu \Gamma^\alpha{}_{\mu\beta} = 0$. With $\vec{U} \equiv d\vec{x}/d\tau$ we have $U^\beta U^\alpha{}_{,\beta} = (dx^\beta/d\tau)\partial_\beta(dx^\alpha/d\tau) = d^2x^\alpha/d\tau^2$ so

$$\frac{d^2x^\alpha}{d\tau^2} = -\Gamma^\alpha{}_{\mu\beta}\frac{dx^\mu}{d\tau}\frac{dx^\alpha}{d\tau}. \tag{33}$$

which, given the connection and the starting position and velocity, can be integrated[5] to give the trajectory $x^\alpha(\tau)$.

To obtain the (non-covariant) geodesic *deviation* equation, consider particle 1 that starts at $\vec{x}_1$. We adopt locally inertial coordinates, as we are free to do, so that $\Gamma^\alpha{}_{\mu\beta}(\vec{x}_1) = 0$ and so $\Gamma^\alpha{}_{\mu\beta}(\vec{x})$ will vary linearly with $\vec{x} - \vec{x}_1$, and we assume the particle starts at rest in these coordinates, so $\vec{U} \to (c, \mathbf{0})$. This particle will therefore remain at rest (or rather its 3-velocity will grow only quadratically with time so its displacement only grows cubically).

We consider a second particle (2), also initially at rest, but at $\vec{x}_2 = \vec{x}_1 + \vec{\xi}$ for which $\Gamma^\alpha{}_{\mu\beta} = 0 + \Gamma^\alpha{}_{\mu\beta,\gamma}\xi^\gamma + \ldots$, as illustrated in figure 8, and, by simply differencing the equations for the two particles, we obtain an equation of motion for $\xi^\alpha \equiv x_2^\alpha - x_1^\alpha$:

$$\boxed{d^2\xi^\alpha/d\tau^2 = -\Gamma^\alpha{}_{00,\gamma}\xi^\gamma.} \tag{34}$$

Note that in the figure we have drawn $\vec{\xi}$ perpendicular to $\vec{U}$, but this is not required.

[5]It is a non-linear equation, but it can always be integrated numerically; simply use it to update the velocity for a small step $d\tau$ and then use the velocity to update the position (you may need to use something like a time-centred leapfrog to do this stably, but that is an irrelevant detail for our purposes).

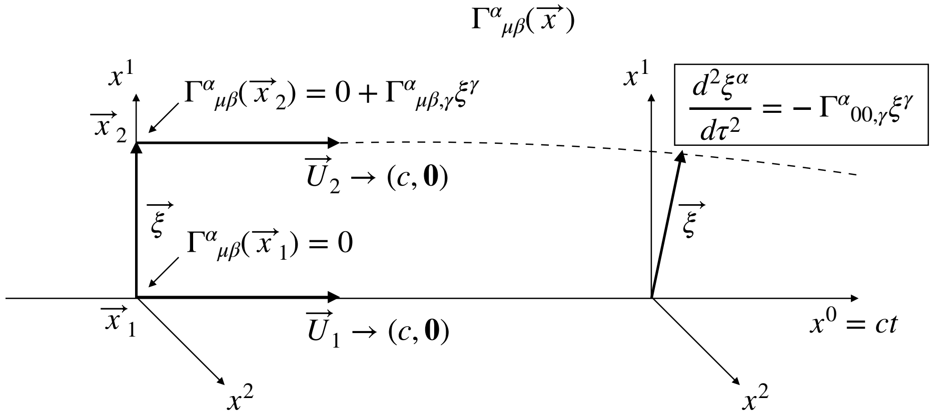


Figure 8: The non-covariant geodesic deviation equation. We have drawn this for the case where $\vec{\xi}$ is perpendicular to $\vec{U}$, and happens to be along the x^1 axis. But it could be arbitrary.

4.5.2 The covariant geodesic deviation equation

As mentioned, this is very useful, but is not covariant as $d^2\xi^\alpha/d\tau^2$ is not a 4-vector. You may well ask: why not? After all, $\vec{\xi}$ is a 4-vector and so is $d\vec{\xi}/d\tau$ and $d^2\vec{\xi}/d\tau^2$. Isn't $d^2\xi^\alpha/d\tau^2$ just the $\alpha^{\rm th}$ component of $d^2\vec{\xi}/d\tau^2$?

To see why not, consider the first derivative: we have $d\vec{\xi}/d\tau = d(\xi^\gamma \vec{e}_\gamma)/d\tau = \vec{e}_\gamma d\xi^\gamma/d\tau + \xi^\gamma d\vec{e}_\gamma/d\tau$. The second term involves the connection: $d\vec{e}_\gamma/d\tau = U^\beta \partial_\beta \vec{e}_\gamma = U^\beta \Gamma^\mu{}_{\gamma\beta}\vec{e}_\mu$, so

$$d\vec{\xi}/d\tau = \vec{e}_\gamma d\xi^\gamma/d\tau + \vec{e}_\mu \xi^\gamma U^\beta \Gamma^\mu{}_{\gamma\beta}. \tag{35}$$

Now if we work in a LIF, as here, the connection vanishes, so we don't need to worry about the 2nd term, and evidently $d\xi^\gamma/d\tau$ is then the $\gamma^{\rm th}$ component of $d\vec{\xi}/d\tau$.

But when we take a second time derivative we cannot, in general ignore this. When we differentiate (35) we get $\vec{e}_\gamma d^2\xi^\gamma/d\tau^2$ plus a number of terms involving Christoffel symbols, which we can drop as we're in locally inertial coordinates, plus a term $\vec{e}_\mu \xi^\gamma U^\beta U^\alpha \Gamma^\mu{}_{\gamma\beta,\alpha}$ involving the derivative of $\Gamma^\mu{}_{\gamma\beta}$ which will not, in general, vanish.

Using $\vec{U} \to (1, 0, 0, 0)$ we have

$$(\nabla_{\vec{U}} \nabla_{\vec{U}} \vec{\xi})^\alpha = (d^2\vec{\xi}/d\tau^2)^\alpha = d^2\xi^\alpha/d\tau^2 + \Gamma^\alpha{}_{\gamma 0,0}\xi^\gamma \tag{36}$$

and using $d^2\xi^\alpha/d\tau^2$ from (34) gives

$$(d^2\vec{\xi}/d\tau^2)^\alpha = (\Gamma^\alpha{}_{\gamma 0,0} - \Gamma^\alpha{}_{00,\gamma})\xi^\gamma \tag{37}$$

where $\Gamma^\alpha{}_{\gamma 0,0} - \Gamma^\alpha{}_{00,\gamma} = R^\alpha{}_{00\gamma}$ is a component of the curvature tensor (remember, we're working in a locally inertial coordinate system).

But $U^\alpha = \delta^\alpha{}_0$ so this is equivalent to

$$\boxed{(d^2\vec{\xi}/d\tau^2)^\alpha = R^\alpha{}_{\mu\nu\gamma} U^\mu U^\nu \xi^\gamma} \tag{38}$$

which is a tensor equation which is therefore valid in any frame.

Other ways of expressing this are

$$d^2\vec{\xi}/d\tau^2 = -\mathbf{R}(\ \ , \vec{U}, \vec{\xi}, \vec{U}) \tag{39}$$

or, equivalently, and rather neatly,

$$d^2\vec{\xi}/d\tau^2 = [\nabla_{\vec{U}}, \nabla_{\vec{\xi}}]\vec{U} \tag{40}$$

We can also obtain the covariant geodesic deviation equation (38) directly from the definition of curvature in terms of parallel transport of a 4-vector around a loop as illustrated in figure 9.

This shows that the degree of non-parallelism of two initially parallel particles, with initial separation $\vec{\xi}$, and after being propagated along their world lines a proper time $\Delta\tau$ is

$$\Delta\vec{U} = \mathbf{R}(\ \ , \vec{U}, \vec{\xi}, \Delta\tau\vec{U}). \tag{41}$$

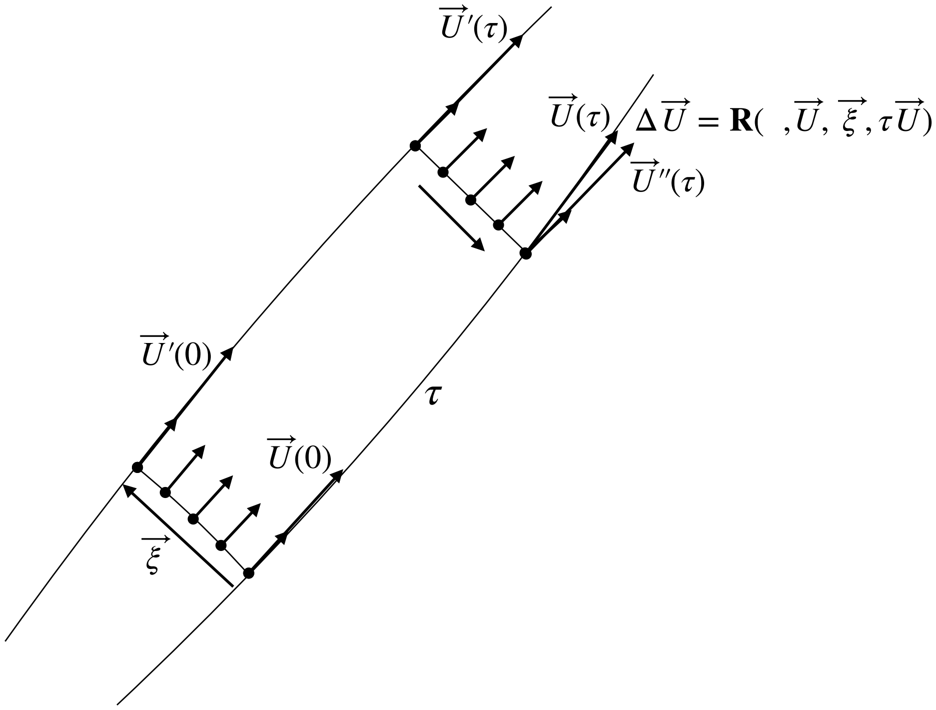


Figure 9: The covariant geodesic deviation equation. Let's start with a test particle at the bottom right with 4-velocity $\vec{U}(0)$. Parallel transport a copy of it along the vector $\vec{\xi}$ to make a cloned (primed) particle with 4-velocity $\vec{U}'(0)$. We now have two parallel test particles. Advance both particles along their world-lines – parallel transporting their 4-velocities, of course – by a proper time τ and then transport the primed particle's 4-momentum $\vec{U}'(\tau)$ back over to the un-primed particle's path. Call this $\vec{U}''(\tau)$ and compare it with the un-primed particle's 4-velocity $\vec{U}(\tau)$. The difference is provided – according to the definition of the Riemann tensor – by $\Delta\vec{U} = \vec{U}(\tau) - \vec{U}''(\tau) = \mathbf{R}(\ ,\vec{U},\vec{\xi},\tau\vec{U})$.

Invoking linearity, we have the rate of change of the 4-velocities of the initially parallel moving particles:

$$d\Delta\vec{U}/d\tau = \Delta\vec{U}/\Delta\tau = \mathbf{R}(\ ,\vec{U},\vec{\xi},\vec{U}) \tag{42}$$

which, with $\Delta\vec{U} = d\vec{x}'/d\tau - d\vec{x}/d\tau = d\vec{\xi}/d\tau$, is equivalent to (38).

Comparing with the Newtonian equation for tidal deflection $d^2\xi_i/dt^2 = -\xi_j\phi_{,ij}$ we can identify certain components of the Riemann tensor with the Newtonian tidal field 3-tensor $\phi_{,ij}$. Moreover, we know that the contraction of this is $\phi_{,ii} = 4\pi G\rho$ (Poisson's equation), so this suggests that the connection between the curvature and the stress-energy tensor – of which ρ is the time-time component – will involve the contraction of the Riemann tensor on its 1st and 3rd indices.

5 The Einstein field equations

Here we will show that there is an essentially unique contraction of the Riemann tensor – known as the 'Einstein tensor' – that has a vanishing 4-divergence. Equating this to the matter stress energy tensor gives Einstein's field equations.

5.1 The Ricci tensor and Ricci scalar

The *Ricci tensor* is the contraction of the Riemann tensor on its first and third indices:

$$\boxed{R_{\alpha\beta} = g^{\gamma\mu}R_{\gamma\alpha\mu\beta} = R^{\mu}{}_{\alpha\mu\beta}} \tag{43}$$

This is symmetric under $\alpha \Leftrightarrow \beta$ and is essentially the only contraction of the Riemann tensor (others either vanishing or being equivalent owing to symmetries).

The Ricci tensor, as we will see, plays an important role in Raychaudhuri's equation that describes the focussing of a beam of geodesics by matter.

The contraction of the Ricci tensor is the *Ricci scalar*

$$\boxed{R = g^{\alpha\beta} R_{\alpha\beta} = R^{\beta}{}_{\beta}} \tag{44}$$

Our goal here is to find some kind of geometric measure of curvature – i.e. something that vanishes in flat space-time – which, in the presence of matter we can equate to some multiple of the stress tensor $T_{\alpha\beta}$ which special relativity tells us ought to be the 'source term' for gravity as it is the generalisation of $\rho = T_{00}$ appearing in Poisson's equation.

Clearly the Ricci tensor is a candidate for this, since, like the Laplacian of the potential appearing on the LHS of Poisson's equation, it is made from contractions of second derivatives (of the components of the metric)

But this choice is not unique, as we can add to $R_{\alpha\beta}$ some multiple of the Ricci scalar times $g_{\alpha\beta}$. There is, however, something special about $T_{\alpha\beta}$ that is not shared by $R_{\alpha\beta}$, which is that $T_{\alpha\beta}$ has a vanishing 4-divergence: $T_{\alpha\beta}{}^{;\beta} = 0$. What we would like is to find a geometric tensor that shares this property. The key to this is another identity discovered by Bianchi.

5.2 The differential Bianchi identities

Partially differentiating (32) for the fully covariant Riemann tensor (in a local inertial frame) $R_{\alpha\beta\mu\nu}$ with respect to x^λ gives

$$R_{\alpha\beta\mu\nu,\lambda} = \tfrac{1}{2}(g_{\alpha\nu,\beta\mu\lambda} - g_{\alpha\mu,\beta\nu\lambda} + g_{\beta\mu,\alpha\nu\lambda} - g_{\beta\nu,\alpha\mu\lambda}) \tag{45}$$

from which one can verify that

$$R_{\alpha\beta\mu\nu,\lambda} + R_{\alpha\beta\lambda\mu,\nu} + R_{\alpha\beta\nu\lambda,\mu} = 0 \tag{46}$$

This is in locally inertial coordinates at point $\mathcal{P}$ where the Christoffel symbols vanish, and so is equivalent to

$$\boxed{R_{\alpha\beta\mu\nu;\lambda} + R_{\alpha\beta\lambda\mu;\nu} + R_{\alpha\beta\nu\lambda;\mu} = 0} \tag{47}$$

which is a *tensor* identity, and so valid in all frames. These equations are known as *the* Bianchi identities, and sometimes written as $R_{\alpha\beta[\mu\nu;\lambda]} = 0$.

5.3 The Einstein tensor and the Einstein gravitational field equations

We would like to find a tensor $\mathbf{G}$ describing the geometry of the manifold that is 'covariantly conserved' like the stress energy tensor and so has $G^{\mu\nu}{}_{;\mu} = 0$. We can construct a tensor that has these properties, and express it in terms of $R^{\mu\nu}$ and R, by taking the double contraction of the Bianchi identities (to give a vector equation). Though this involves a bit of juggling of indices.

Contracting the Bianchi identities on the indices α and μ gives

$$g^{\alpha\mu}(R_{\alpha\beta\mu\nu;\lambda} + R_{\alpha\beta\lambda\mu;\nu} + R_{\alpha\beta\nu\lambda;\mu}) = 0. \tag{48}$$

We'd like to express this in terms of the Ricci tensor and scalar. This first term is just $R_{\beta\nu;\lambda}$ as it involves the contraction of $\mathbf{R}$ on its 1st and 3rd indices as in the definition of $R_{\beta\nu}$. The second is the contraction on the 1st and 4th indices. But $\mathbf{R}$ is antisymmetric under interchange of 3rd and 4th components, so the second term is $-R_{\beta\lambda;\nu}$, so (48) says

$$R_{\beta\nu;\lambda} - R_{\beta\lambda;\nu} + R^{\mu}{}_{\beta\nu\lambda;\mu} = 0. \tag{49}$$

Contracting this on β and ν gives the 'twice-contracted' Bianchi identities:

$$g^{\beta\nu}(R_{\beta\nu;\lambda} - R_{\beta\lambda;\nu} + R^{\mu}{}_{\beta\nu\lambda;\mu}) = 0. \tag{50}$$

The first term is $R_{;\lambda}$, and the second is $-R^{\nu}{}_{\lambda;\nu}$ (which is the same as $-R^{\mu}{}_{\lambda;\mu}$). What about the 3rd? Exploiting the anti-symmetry of $R_{\alpha\beta\mu\nu}$ under $\alpha \Leftrightarrow \beta$, the third is

$$g^{\beta\nu}R^{\mu}{}_{\beta\nu\lambda;\mu} = g^{\beta\nu}g^{\mu\gamma}R_{\gamma\beta\nu\lambda;\mu} = -g^{\beta\nu}g^{\mu\gamma}R_{\beta\gamma\nu\lambda;\mu} = -g^{\mu\gamma}R^{\nu}{}_{\gamma\nu\lambda;\mu} = -g^{\mu\gamma}R_{\gamma\lambda;\mu} = -R^{\mu}{}_{\lambda;\mu} \tag{51}$$

which is the same as the second. So (50) says $R_{;\lambda} - 2R^{\mu}{}_{\lambda;\mu} = 0$ or

$$(R^{\mu}{}_{\lambda} - \tfrac{1}{2}\delta^{\mu}{}_{\lambda}R)_{;\mu} = 0. \tag{52}$$

which is the desired result; we have constructed a rank two tensor out of contractions of the Riemann tensor – the thing in parentheses – that is covariantly conserved.

It is more commonly expressed in 'fully contravariant' form. To get this we just multiply by $g^{\lambda\nu}$ to get the *Einstein tensor* defined to be

$$\boxed{G^{\mu\nu} \equiv R^{\mu\nu} - \tfrac{1}{2}Rg^{\mu\nu}} \tag{53}$$

which, by virtue of the (twice contracted) Bianchi identities, satisfies

$$\boxed{G^{\mu\nu}{}_{;\mu} = 0.} \tag{54}$$

This is just like the equations $T^{\mu\nu}{}_{;\mu} = 0$ that express continuity of energy and momentum, and this is what led Einstein to his *field equations*

$$\boxed{\mathbf{G} = 8\pi\kappa\mathbf{T}} \tag{55}$$

where κ is a constant (the numerical pre-factor 8π being chosen so that, after requiring that the theory reproduce Newtonian dynamics in the appropriate limit, κ is Newton's constant G).

The path to this equation is somewhat tortuous and it may seem to have involved some arbitrary choices – such as that to contract the curvature on the 1st and 3rd indices – but different choices all lead to essentially the same result, so the theory is essentially unique. This is what makes this theory so compelling.

It is not, however, completely unique. After arriving at (55), Einstein realised that this would not allow static cosmological solutions, so he later proposed to modify the field equations to include a term $\Lambda\mathbf{g}$ where Λ is known as the *cosmological constant*:

$$\boxed{\mathbf{G} + \Lambda\mathbf{g} = 8\pi\kappa\mathbf{T}} \tag{56}$$

which is possible because the metric also obeys $g^{\mu\nu}{}_{;\mu} = 0$.

The field equations express half of John Wheeler's aphorism: *"Matter tells space-time how to curve"*. The other half – that *"Space-time tells matter how to move"* – is encapsulated in the rule that the effect of gravity on any non-gravitational physics is essentially to modify any law of physics expressed as a valid equation in SR by replacing the commas by semi-colons (though, as mentioned, there are some subtleties involving ordering in e.g. electromagnetism).

5.4 Solving the field equations

5.4.1 Parallels with Newtonian gravity

The Einstein field equations are the relativistic analogue of Poisson's equation $\nabla^2\phi = 4\pi G\rho$, with the 'source term' ρ on the RHS replaced by the stress-energy tensor $\mathbf{T}$.

In Newtonian theory, and for a given mass density $\rho(\mathbf{r})$, we can solve Poisson's equation, augmented by the boundary condition that $\phi \to 0$ at infinity, for the potential $\phi(\mathbf{r})$. This is straightforward as we can simply write the potential as the sum of $\phi \propto 1/r$ potentials from all the mass elements and automatically satisfy the desired boundary condition. This is how the matter controls the potential.

Given the potential, we can differentiate it to get the gravity vector $-\boldsymbol{\nabla}\phi$, and differentiating once more gives the gravitational tidal field tensor $\boldsymbol{\nabla}\boldsymbol{\nabla}\phi$. Which, in the Newtonian version of the geodesic deviation equation, is the way the potential 'tells matter how to move'.

Alternatively, we can go in the opposite direction. The tidal field causes observable focussing of orbits of neighbouring test particles, from which we can infer the mass density. We can also use Jeans's equation to measure, from the divergence of the pressure tensor of stars in a galaxy, for instance, the rate at which the gravitational field is delivering momentum and this gives $\boldsymbol{\nabla}\phi$, from which, again, we can infer the density.

In Einstein's theory of gravity the metric tensor $\mathbf{g}$ is the analogue of the Newtonian potential. If we know the metric, taking its derivative provide the Christoffel symbols – the analogue of the gravitational acceleration – and taking derivatives of those gives the curvature tensor $\mathbf{R}$ which, we have seen, is the generalisation of the Newtonian tidal field.

Just as Newtonian gravity the curvature can be measured from geodesic deviation, and a contraction of the generalised tidal field – the rank two tensor $\mathbf{G}$ – is equated to the matter 'source term' – the stress tensor $\mathbf{T}$, being the relativistic analogue of the density – all quite analogous to the Newtonian theory.

But that begs the question: how do we find a solution $\mathbf{g}$ of Einstein's field equations in the first place?

5.4.2 Non-linearity and 'no prior geometry'

There are two things that make *finding* a metric $\mathbf{g}$ that solves the field equations extremely difficult.

One is that the relations between the quantities appearing in Einstein's theory are *non-linear*. Unlike Poisson's equation, in which linearity allows us to simply sum the $1/r$ potentials for all the mass elements, and unlike the relations between the potential, the gravity and the tide, all of which are linear, the relations between the metric, the connection and the curvature all involve non-linearities.

The other is that there is 'no prior geometry'. In Newtonian theory we have a potential on a pre-given space-time (Euclidean space and absolute time). In Einstein's gravity the geometry of space-time and the gravitational field are effectively one and the same thing.

A further complication is that whereas in Newtonian theory it makes sense to solve for the potential at a given instant of time, in Einstein gravity we must find a solution in 4 dimensions.

When Einstein presented his field equations in 1915 he was quite pessimistic about whether solutions would be found.

Historically, progress was made by exploring metrics that had a high degree of symmetry and where the source term $\mathbf{T}$ was very simple. One application was to find solutions for the geometry of black holes, where $\mathbf{T} = 0$. And, in fact, Schwarzschild found his metric in 1916. Another important application was cosmology, where the matter source term was assumed to be that of a spatially homogeneous ideal fluid, the solution often being called the 'FRW' metric[6].

Another area where progress was possible was 'weak-field' gravity in which one searches for solutions in which the metric is close to that of empty space. We will consider that in the next lecture.

5.4.3 Interpreting the solution

A further complication that arises in the relativistic theory that is not present in the Newtonian theory is that while the metric encodes the geometry of the space-time, it is expressed in terms of coordinates and the choice of coordinates is arbitrary. This is called 'diffeomorphism invariance'. Thus there is redundant information in the metric. One can have apparently different solutions that describe the same physical situation, and it may be difficult to say whether two different solutions describe the same space-time.

5.4.4 Number of physical degrees of freedom

It is interesting to look at the physical degrees of freedom and the number of independent components of the various quantities appearing here.

In Newtonian gravity we have a single scalar potential ϕ whose gradient $\boldsymbol{\nabla}\phi$ has three components, while the tide $\boldsymbol{\nabla}\boldsymbol{\nabla}\phi$ has 6, and what appears in the Poisson equation is its contraction, the single scalar Laplacian $\nabla^2\phi$, which gets equated to $4\pi G\rho$ to 'close the loop'.

Einstein's gravity starts with geometry measured in terms of the metric $\mathbf{g}$ as the analogue of ϕ. This has 10 independent components at each point in space-time, but there are 4 continuous degrees of freedom in the arbitrary choice of coordinates, so we can say that the metric has 6 physical degrees of freedom. From this, we obtain the connection – the analogue of g_i – which has 40 independent components and from which we form the curvature – which is the analogue of the tidal field – which gives physical answers to questions about how vectors change if parallel transported around loops (and how freely falling particles accelerate towards one another) which is characterised by 20 numbers. The curvature gets contracted to make $\mathbf{G}$ – symmetric and so with 10 independent components – which appears on the left-hand-side of Einstein's equation (the analogue of $\nabla^2\phi$ in Newtonian gravity). This gets 'sourced' by the symmetric stress-tensor $\mathbf{T}$. While both $\mathbf{G}$ and $\mathbf{T}$ can be written as symmetric matrices containing 10 functions of $\vec{x}$, not any choice of these functions is physically allowed as both sides of the field equations have vanishing 4-divergences. This provides 4 constraints. If we specify $\mathbf{G}$ (and therefore also $\mathbf{T}$) on some 'initial' space-like hypersurface then the conservation laws dictate how the energy- and momentum-densities evolve moving forward in time. If we write the field equations as $\mathbf{G}[\mathbf{g}] = 8\pi\mathbf{T}$, where the square brackets represent the sequence of computing the connection, forming the curvature $\mathbf{R}$, and contracting it to make $\mathbf{G}$, then all the quantities appearing here can be said, in a somewhat loose sense, to have 6 physical degrees of freedom.

[6] Friedmann presented the essential result in two papers in 1922 and 1924. This was before the discovery that we live in an expanding universe and these went largely unnoticed at this time. Robertson and Walker made significant contributions in the 30's. But Lemaitre had independently found the solution in 1927. For this reason, and to reflect the historical timeline, it is quite common to refer to this as the FLRW model.

5.5 What is 'relative' about general relativity?

In special relativity, lengths and distances between events are observer-frame dependent, as is simultaneity. And things like the components of the electromagnetic field and the charge current density are also frame dependent. What looks like a magnetic field from an electrically neutral current carrying wire in one frame appears, in a different frame, to have an electric field and non-zero charge density.

But the equations of electromagnetism are expressible in a covariant manner using the language of tensor calculus. And if we think of things like the Faraday tensor and the charge current density vector as geometric entities then the 'relativism' or observer-frame dependence fades away into the background. This is just as in geometry where we are quite happy of thinking of vectors like $\mathbf{r}$ as being real unambiguous entities, while their components, which is what one has to deal with if one wants to calculate things with a computer, may be frame dependent.

Much of the 'heavy lifting' of GR that Einstein established was extending this to 'general covariance' in which the laws of physics are expressed in a covariant manner in an arbitrary coordinate system. While the components of the tensors and vectors representing physical quantities are dependent on the coordinate system, the actual physical entities are not.

Many text-books and web-sites that say that GR is founded on Einstein's equivalence principle which states that gravity is indistinguishable from, or equivalent to, acceleration. But that is dangerous and misleading. It would seem to suggest that there is something ambiguous or 'relative' about the gravitational field; that an accelerated observer would say there is a gravitational field, while a non-accelerated observer would deny its existence.

In GR, the generalised tidal field $\mathbf{R}$ is a tensor, so whether or not it vanishes is an absolute, and empirically verifiable, fact. In a rocket being accelerated in empty space the curvature tensor vanishes, whereas near the surface of the Earth it does not.

I think a lot of confusion here stems from the fact that most of the phenomena we associate with gravity are really the effect of acceleration; the acceleration we feel standing on the Earth being essentially an indirect consequence of the gravitational field. And it is compounded by the fact that we can nicely describe such phenomena using the space-time metric $g_{\mu\nu} = \mathrm{diag}(-(1 + 2\mathbf{a} \cdot \mathbf{x}/c^2), 1, 1, 1)$ where $\mathbf{a}$ is the acceleration of the observer. From this one can obtain the connection and then invoke generalised covariance to obtain the equations of motion for particles observed in a frame of reference tied to the observer. But this isn't gravity, as the Riemann tensor makes no appearance.

A more precise way to state the equivalence principle is to say that if you are being accelerated by the Earth (in order to avoid plunging, freely falling, to its centre) then the local physical effects you will see are the same as if you were being accelerated in empty space. I.e. there are no *extra* effects coming from the gravitational field. In this view, the Pound and Rebka measurement in 1959 was not a measurement of a 'gravitational' redshift *per se*, but a confirmation that, for photons, there is no extra contribution to the redshift – or 'coupling to gravity' – over and above that arising from the acceleration.

5.6 The physical nature of space and space-time in GR

Shortly after Einstein proposed the field equations, Friedmann found a solution for a homogeneous expanding universe. Lemaitre found this independently, and Robertson and Walker studied this further. The FLRW metric is written in terms of coordinates for events which are proper time τ since the 'big-bang' and where the coordinates on spatial surfaces of constant time taken to be a radial coordinate χ and the usual polar coordinates θ and ϕ. The line element, expressed in these coordinates, is

$$ds^2 = -c^2 d\tau^2 + a^2(\tau)(d\chi^2 + S_k^2(\chi)(d\theta^2 + \sin^2\theta d\chi^2)) \tag{57}$$

where the function $S_k(\chi) = \sinh\chi,\ \chi,\ \sin\chi$ for $k = -1,\ 0,\ +1$. The spatial parts of the metric are just like the homegeneous 2-spaces described above, where the space can be flat or positively or negatively curved depending on the 'curvature constant' k, but in 3-dimensions. As will be described in more detail later in the course, the Einstein equations also provide the dynamical equations for the 'scale-factor' $a(\tau)$.

Soon after these solutions were found, Hubble and Slipher and others discovered that the Universe is expanding, and the FLRW model became, and remains, the model in which cosmological observations are interpreted. The essential idea here is that galaxies follow world lines of constant spatial coordinates χ, θ & ϕ. These are called 'fundamental observers' (FOS) in the model, and it follows a) that the proper distance between any pair of observers (on a surface of given τ) is increasing in proportion to the scale factor $a(\tau)$, and b) they have 4-velocities $\vec{U}_{\rm FO} \to (c, \mathbf{0})$.

An important feature of these models is that light we see from distant galaxies is redshifted. This can be shown from the geodesic equation. As usual, this is simplest using the covariant component of the 4-momentum. For a particle moving radially (i.e. with constant θ and ϕ) so $\vec{U} \to (U^\tau, U^\chi, 0, 0)$ we have

$$dU_\chi/d\tau = \tfrac{1}{2} U^\mu U^\nu g_{\mu\nu,\chi} \tag{58}$$

but as only $g_{\tau\tau} = -c^2$ and $g_{\chi\chi} = a^2(\tau)$ are involved in this contraction, and neither depend on χ, it follows that U_χ is a constant of the motion. The normalisation of the 4-momentum $\vec{p} = m\vec{U}$ is $\vec{p}\cdot\vec{p} = g^{\mu\nu} p_\mu p_\nu = -m^2c^2$. In the limit of an extremely relativistic particle moving radially this says $g^{\tau\tau}p_\tau^2 + g^{\chi\chi}p_\chi^2 = 0$ or, since $g^{\tau\tau} = -1$, $g^{\chi\chi} = 1/a^2$ and $p_\chi =$ constant, this says $p_\tau \propto 1/a$. But the energy the fundamental observer sees is $E = -\vec{U}\cdot\vec{p} = -cp_\tau$, so $E \propto 1/a$; the energy of a relativistic particle – or a photon – as would be measured by fundamental observers it is passing scales inversely with the size of the universe.

The redshift effect came to be described, in textbooks and in the broader literature, as being caused by the 'expansion of space'. Galaxies, it is said, are fixed in space, but space is expanding and, as photons propagate their wavelengths get stretched by the expansion of space.

The red-shifting of radiation can also be understood in terms of Maxwell's equations. It's actually simpler to consider the behaviour of a massless scalar field, for which the equations of motion, for disturbances of scale much less that the radius of curvature of the Universe, is

$$\ddot{\phi} + 3(\dot{a}/a)\dot{\phi} - (c^2/a^2)\nabla^2\phi = 0 \tag{59}$$

where dots denote derivative with respect to proper time and $\nabla^2\phi$ is the Laplacian in 'co-moving' coordinates. This equation admits solutions which are standing waves in comoving coordinates, but with a decaying amplitude. The second term looks like a 'friction' and it results in the field disturbances losing energy. It is called the 'Hubble damping' term. The same is true for EM waves. The $3H\dot{\phi}$ term – where $H \equiv \dot{a}/a$ is the expansion rate – is described as giving the coupling of the EM field to the gravitational field of the expanding universe.

The phenomenon that these equations describe is a real one. But the language used in their interpretation is very strange indeed. The expansion of space is said to be somehow coupling to the fields and sapping energy from them. But that seems to be attributing to space an attribute that it does not possess; that of expansion.

In the mathematical development of GR, as described here, there are simply events labelled by – or ascribed – coordinates on a manifold. There are distances between events, from which we can distill the metric, and from that everything else we can know about the manifold like its curvature, but that is all there is. This is Mach's conception of the world; all there are are distances. There isn't any underlying 'fabric' of space that possesses some extra property of expansion that might somehow be influencing the EM waves. The *amount* of space in the volume occupied by a collection of galaxies is increasing. And, in closed models, the space is positively curved and the total amount of space at any time is finite – and this too is increasing (or decreasing). But space itself cannot – and should not – be said to be expanding. The notion of 'expansion of space' and this coupling the EM field equations and photon energies to gravity is misleading.

The problem here is that the metric above encodes, as always, both information about the geometry of space-time – which is undoubtably curved if matter is present – and the choice of coordinates. The Hubble damping term in (59) is better thought of as being a local effect of using non-physical 'co-moving' spatial coordinates (rather than the physical coordinates that a fundamental observer would measure using rulers).

An important step in understanding the physical meaning of the FLRW metric was that of Milne. He pointed out that, in the limit that the mass density is very low (so that the kinetic energy of expansion of a small part of the universe is much greater than its gravitational binding energy), the scale factor grows linearly with time: $a(\tau) \propto \tau$, and the space-time described by the line element (57) is then just Minkowski space-time written in a funny coordinate system.

In fact, it is a coordinate system tied to a cloud of 'shrapnel' from an explosion where τ measures the time since the explosion, as measured by the particles, and the space coordinates are a simple function of the velocities of the particles, which are individually constant. This is illustrated in figure 10.

An intriguing feature of Milne's model is that Maxwell's equations in 'co-moving coordinates', with the 'Hubble damping' term, should be valid, but the space-time is empty, so the normal form of Maxwell's equations must be valid also.

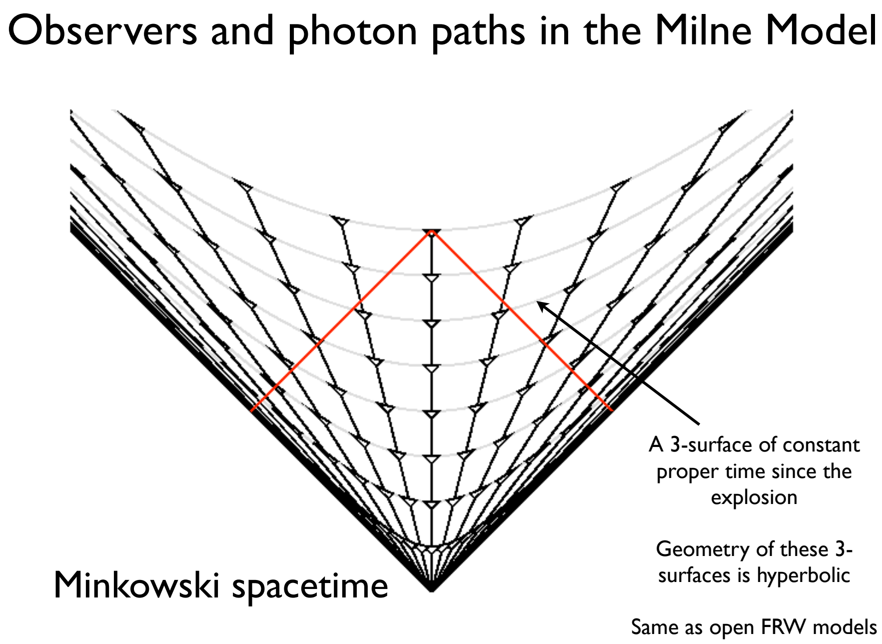


Figure 10: Milne's cosmological model is constructed by considering an explosion at some point in Minkowski space-time from which massless particles emerge with all velocities less than the speed of light. The metric is then written in coordinates where time is proper time since the big bang – which, because of time dilation is a hyperbola – and the spatial radial coordinate χ is a function of velocity (so particles maintain constant χ, θ, & ϕ). The result is formally identical to the FLRW metric (57).

So we should be able to find a solution of the ordinary Maxwell equations in non-expanding coordinates that is like an 'expanding fireball', or like background radiation fields in the expanding universe. Or if we write down a solution of the EM field equations in comoving spatial coordinates (i.e. solutions (59)) as a sum of standing plane waves which evolve with time losing energy because of the Hubble damping, this should also be a solution of Maxwell's equations in non-expanding coordinates. In figure 11 we show an example in 1 spatial dimension (i.e. 1+1 dimensional space-time).

The purpose of this example is to make clear that the $3H\dot{\phi}$ Hubble-damping term in (59) is not in any sense a 'coupling to gravity' as here there is no gravity. Nor does it represent a red-shifting of the radiation field 'caused by the expansion of space'. It simply arises as a local coordinate effect coming from a non-physical choice of coordinates. *Space* is not expanding in figure 11 – it is the radiation that is expanding.

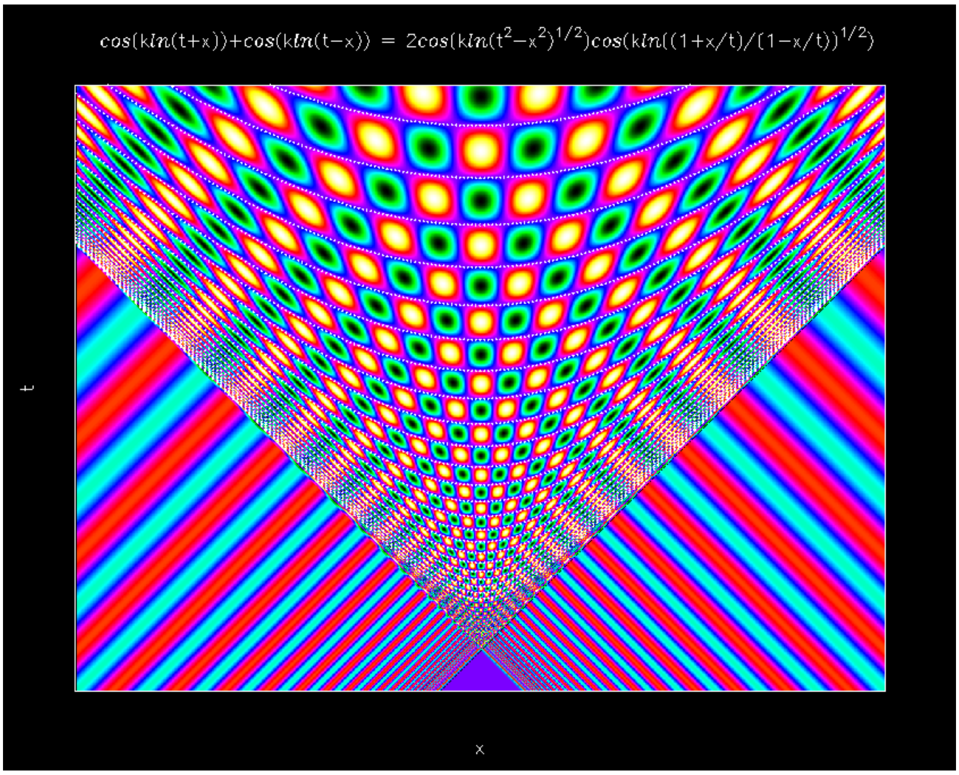


Figure 11: Expanding radiation in Milne's model. This shows that we can make a solution of the usual flat space-time Maxwell equations – it is a particular case of a d'Alembertian solution composed of two oppositely propagating 'chirps', that is also a standing wave solution of Maxwell's equations written in an expanding coordinate frame. If we had Milne observers (the massless particles of shrapnel) present, they would say that the energy flux density in their frame vanishes, but they would find the energy density of the radiation decreasing, and consistent with this they would note that the spatial divergence of the energy flux density is non-vanishing and positive. But it is important to appreciate that what is expanding here is the radiation itself.

6 Problems

6.1 Parallel transport on the unit sphere

The intrinsic metric of the unit sphere in Euclidean 3-space is

$$ds^2 = d\theta^2 + \sin^2\theta d\phi^2 \quad \text{or} \quad g_{\alpha\beta} = \begin{bmatrix} 1 & 0 \\ 0 & \sin^2\theta \end{bmatrix} = \text{diag}(g_{\theta\theta}, g_{\phi\phi}) = \text{diag}(1, \sin^2\theta) \tag{60}$$

- What is the inverse metric (index raising operator) $g^{\alpha\beta}$?
- A: $g^{\alpha\beta} = \text{diag}(1, 1/\sin^2\theta)$
- Calculate the non-vanishing Christoffel symbols
- A: using $\Gamma^\mu{}_{\alpha\beta} = \frac{1}{2}g^{\mu\lambda}(g_{\alpha\lambda,\beta} + g_{\beta\lambda,\alpha} - g_{\alpha\beta,\lambda})$
- so $\Gamma^\phi{}_{\phi\theta} = \Gamma^\phi{}_{\theta\phi} = \cot\theta$ and $\Gamma^\theta{}_{\phi\phi} = -\sin\theta\cos\theta$, all others vanish

A vector $\vec{V} = V^\alpha \vec{e}_\alpha$ is parallel transported along a path – not necessarily a geodesic – with tangent vector $\vec{U} = (dx^\alpha/d\lambda)\vec{e}_\alpha$ on the sphere

- Give the equation for $V^\alpha{}_{,\beta}$ in terms of components of $\vec{V}$ and the Christoffel symbols.
- A: the equation of parallel transport is $\nabla_{\vec{U}}\vec{V} = 0$ or $U^\beta V^\alpha{}_{;\beta} = U^\beta(V^\alpha{}_{,\beta} + V^\mu\Gamma^\alpha{}_{\mu\beta}) = 0$
- so $V^\alpha{}_{,\beta} = -V^\mu\Gamma^\alpha{}_{\mu\beta}$ or
  - $V^\theta{}_{,\theta} = 0$
  - $V^\theta{}_{,\phi} = V^\phi\sin\theta\cos\theta$
  - $V^\phi{}_{,\theta} = -V^\phi\cot\theta$
  - $V^\phi{}_{,\phi} = -V^\theta\cot\theta$

A nice way to reformulate these is to define $\mathbf{Y} \equiv Y^\alpha = (V^\theta, V^\phi\sin\theta)$ (where we are using the bold-face vector notation since $\mathbf{Y}$, while having 2 components $\{Y^\alpha\}$, does not transform in the same way as a vector like $\vec{V}$).

- What are the corresponding equations for $Y^\alpha{}_{,\beta}$ (expressed in terms of Y^α)?
  - $Y^\theta{}_{,\theta} = V^\theta{}_{,\theta} = 0$
  - $Y^\theta{}_{,\phi} = V^\theta{}_{,\phi} = V^\phi\sin\theta\cos\theta = Y^\phi\cos\theta$
  - $Y^\phi{}_{,\theta} = (V^\phi\sin\theta)_{,\theta} = 0$
  - $Y^\phi{}_{,\phi} = (V^\phi\sin\theta)_{,\phi} = -Y^\theta\cos\theta$
- Show thereby that $\partial\mathbf{Y}/\partial\theta = 0$ and write $\partial\mathbf{Y}/\partial\phi = \mathbf{M}\cdot\mathbf{Y}$ where $\mathbf{M}$ is a 2 by 2 matrix. (Hint: it involves the 90-degree rotation matrix $\mathbf{R}_{\pi/2}$)
  - $\mathbf{Y}_{,\theta} = 0$
  - $\mathbf{Y}_{,\phi} = \begin{bmatrix} Y^\theta{}_{,\phi} \\ Y^\phi{}_{,\phi} \end{bmatrix} = \begin{bmatrix} 0 & \cos\theta \\ -\cos\theta & 0 \end{bmatrix}\begin{bmatrix} Y^\theta \\ Y^\phi \end{bmatrix} = \cos\theta\,\mathbf{R}_{\pi/2}\cdot\mathbf{Y}$
- Verify that a solution of $\partial\mathbf{Y}/\partial\phi = \mathbf{M}\cdot\mathbf{Y}$ is $\mathbf{Y}(\vec{x}) = \mathbf{R}_{\psi(\vec{x})}\cdot\mathbf{Y}_0$ where $\mathbf{Y}_0 \equiv \mathbf{Y}(\vec{x}_0)$ and give the angle $\psi(\vec{x}) = \int d\phi\ldots$ as an integral along the path.
- A: with this ansatz:
- $d\mathbf{Y} = d\begin{bmatrix} \cos\psi & \sin\psi \\ -\sin\psi & \cos\psi \end{bmatrix}\cdot\mathbf{Y}_0 = d\psi\begin{bmatrix} -\sin\psi & \cos\psi \\ -\cos\psi & -\sin\psi \end{bmatrix}\cdot\mathbf{Y}_0 = d\psi\mathbf{R}_{\pi/2}\cdot\mathbf{R}_{\psi(\vec{x})}\cdot\mathbf{Y}_0 = d\psi\mathbf{R}_{\pi/2}\cdot\mathbf{Y}$
- but $d\mathbf{Y} = \mathbf{Y}_{,\theta}d\theta + \mathbf{Y}_{,\phi}d\phi = \mathbf{Y}_{,\phi}d\phi = \cos\theta\,\mathbf{R}_{\pi/2}\cdot\mathbf{Y}d\phi$

- so these are consistent if
- $d\psi = d\phi \cos\theta \to \psi(\vec{x}) = \int d\phi\ \cos\theta$

Thus parallel transport of a vector $\vec{V}$ on a sphere corresponds to a simple rotation of the associated components in $\mathbf{Y}$-space.

- Show graphically how, for a closed loop, the rotation angle ψ is simply related to the enclosed solid angle $\int d\Omega = \int \sin\theta d\theta d\phi$. (Your figure should show how a finite loop integral can be expressed as a sum of integrals around small rectangles in ϕ, θ-space, and you should then evaluate $\oint d\phi \ldots$ for such a rectangle.)
  - Summing the loop integrals for a grid of cells the internals all cancel.
  - The integral for a cell of sides $d\phi$, $d\theta$ is $\oint d\phi \cos\theta = d\phi \times (\cos(\theta) - \cos(\theta + d\theta)) = d\phi d\theta \sin\theta = d\Omega$
  - so the sum for a finite loop integral is just equal to the solid angle enclosed.
- Use the latter formula for ψ to calculate the result of parallel transporting an arbitrary vector $\vec{V}$, starting on the equator at longitude 0, moving up to the North pole, back to the equator along longitude $\pi/2$ and then back to the starting point via the equator.
- A: This is one eighth of the total sphere or $d\Omega = 4\pi/8 = \pi/2$ steradians. So the vector gets rotated by 90 degrees.
- Draw a diagram showing the parallel transport around this loop of a vector that initially points along the equator. Does the result agree with what you got from the parallel transport machinery?
  - Yes.

6.2 Riemann curvature tensor of the unit sphere

The Riemann curvature tensor is

$$R^\alpha{}_{\beta\mu\nu} = \Gamma^\alpha{}_{\beta\nu,\mu} - \Gamma^\alpha{}_{\epsilon\nu}\Gamma^\epsilon{}_{\beta\mu} - (\mu \leftrightarrow \nu) \tag{61}$$

where $(\ldots)$ indicates the same thing with indices μ and ν reversed.

- Compute $R^\theta{}_{\phi\theta\phi}$, $R_{\theta\phi\theta\phi}$ and $R^\phi{}_{\theta\phi\theta}$
- A: $R^\theta{}_{\phi\theta\phi} = \Gamma^\epsilon{}_{\phi\theta}\Gamma^\theta{}_{\epsilon\phi} - \Gamma^\theta{}_{\phi\phi,\theta} - \Gamma^\epsilon{}_{\phi\phi}\Gamma^\theta{}_{\epsilon\theta} + \Gamma^\theta{}_{\phi\theta,\phi} = \cot\theta(-\sin\theta\cos\theta) - (-\cos\theta\sin\theta)_{,\theta} - 0 + 0$
- or $R^\theta{}_{\phi\theta\phi} = -\sin^2\theta$
- A: $R_{\theta\phi\theta\phi} = g_{\epsilon\theta}R^\epsilon{}_{\phi\theta\phi} = g_{\theta\theta}R^\theta{}_{\phi\theta\phi} = -\sin^2\theta$
- A: $R^\phi{}_{\theta\phi\theta} = g^{\phi\epsilon}R_{\epsilon\theta\phi\theta} = g^{\phi\phi}R_{\phi\theta\phi\theta} = g^{\phi\phi}R_{\theta\phi\theta\phi} = -1$
- Are there any other independent components (i.e. ones that cannot be obtained from $R^\theta{}_{\phi\theta\phi}$ and $R^\phi{}_{\theta\phi\theta}$ using the anti-symmetry under interchange of either first or second pair of indices).
- A: No
- Compute the Ricci tensor $R_{\alpha\beta} = R^\epsilon{}_{\alpha\epsilon\beta}$
- A: $R_{\alpha\beta} = \begin{bmatrix} R^\epsilon{}_{\theta\epsilon\theta} & R^\epsilon{}_{\theta\epsilon\phi} \\ R^\epsilon{}_{\phi\epsilon\theta} & R^\epsilon{}_{\phi\epsilon\phi} \end{bmatrix} = \begin{bmatrix} R^\phi{}_{\theta\phi\theta} & \\ & R^\theta{}_{\phi\theta\phi} \end{bmatrix} = \begin{bmatrix} -1 & \\ & -\sin^2\theta \end{bmatrix} = -g_{\alpha\beta}$
- Compute the Ricci scalar $R \equiv R^\alpha{}_\alpha$ item A: $R = R^\alpha{}_\alpha = g^{\alpha\beta}R_{\alpha\beta} = -2$
- Compute the Einstein tensor $G^{\alpha\beta} = R^{\alpha\beta} - \frac{1}{2}g^{\alpha\beta}R$
- A: $G^{\alpha\beta} = R^{\alpha\beta} - \frac{1}{2}g^{\alpha\beta}R = 0$

6.3 Rindler space-time

Consider the metric in 2D

$$ds^2 = e^{2\rho}(-dv^2/(1-v^2)^2 + d\rho^2). \tag{62}$$

1. sketch some light cones in the ρ, v plane, and also some null rays
2. calculate the connection and the curvature tensor
3. consider alternative coordinates t, x defined by $\rho(t,x) = \log\left(\sqrt{x^2 - t^2}\right)$ and $v(t,x) = t/x$
   - compute the metric in t, x coordinates
   - sketch lines of constant ρ
4. interpret this physically (hint: let $\rho = \log(1/a)$).

6.4 Local flatness

Show that a transformation

$$x^\alpha = \overline{x}^\alpha - \tfrac{1}{2}\Gamma^\alpha{}_{\nu\lambda}\overline{x}^\nu\overline{x}^\lambda \tag{63}$$

renders the metric $\overline{g}_{\alpha\beta}$ locally flat.

Box 11.1 A proof of the flatness theorem

The flatness theorem, as first stated in Section 4.2.3, asserts that at any point P one can always make a coordinate transformation $x^\mu \to \bar{x}^\mu$ and $g^{\mu\nu} \to \bar{g}^{\mu\nu}$ where the metric tensor $\bar{g}^{\mu\nu}$ is a constant, up to a second order correction (i.e. the first order terms vanish):

$$\bar{g}^{\mu\nu}(\bar{x}) = \bar{g}^{\mu\nu}(0) + b^{\mu\nu\lambda\rho}\bar{x}_\lambda\bar{x}_\rho + \cdots, \tag{11.38}$$

where for simplicity we have taken the point P to be at the origin of the coordinate system and the position vector $\bar{x}^\mu$ is assumed to be infinitesimally small. We shall prove this result by explicit construction. Namely, we display a coordinate transformation

$$\frac{\partial x^\mu}{\partial \bar{x}^\nu} = \delta^\mu_\nu - \Gamma^\mu_{\nu\lambda}\bar{x}^\lambda \tag{11.39}$$

that is shown to lead to the result of (11.38).

Here is the proof: According to (11.39) and (11.8), the relation between the new and old coordinates can be written as $x^\mu = \bar{x}^\mu - \frac{1}{2}\Gamma^\mu_{\nu\lambda}\bar{x}^\nu\bar{x}^\lambda + \cdots$. Now, substitute (11.39), as well as the power series expansion $g^{\mu\nu}(x) = g^{\mu\nu}(0) + \partial_\lambda g^{\mu\nu}x^\lambda + \cdots$, into the metric transformation equation

$$\bar{g}_{\mu\nu}(\bar{x}) = \frac{\partial x^\lambda}{\partial \bar{x}^\mu}\frac{\partial x^\rho}{\partial \bar{x}^\nu}g_{\lambda\rho}(x), \tag{11.40}$$

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we have

$$\begin{aligned}\bar{g}_{\mu\nu}(\bar{x}) &= (\delta^\lambda_\mu - \Gamma^\lambda_{\mu\alpha}\bar{x}^\alpha)(\delta^\rho_\nu - \Gamma^\rho_{\nu\beta}\bar{x}^\beta)(g_{\lambda\rho}(0) + \partial_\gamma g_{\lambda\rho}x^\gamma + \cdots)\\ &= g_{\mu\nu}(0) - [\Gamma^\lambda_{\mu\alpha}g_{\lambda\nu}(0) + \Gamma^\lambda_{\alpha\nu}g_{\mu\lambda}(0) - \partial_\alpha g_{\mu\nu}]x^\alpha + \cdots.\end{aligned}$$

The coefficient of x^α (square bracket) vanishes because of (11.34): the metric is covariantly constant. Thus the transformation in (11.39) indeed has the claimed property of leading to a metric having the form of (11.38). ■

Figure 12: Proof of the flatness theorem (from Cheng's book).