ENS M1 General Relativity - Lecture 5 - Weak Field Gravity

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1 Introduction

Einstein's field equations provide a natural relativistic generalisation of Poisson's equation with the mass density ρ replaced by the stress-energy tensor $\mathbf{T}$ as the 'source'.

The analogue of the Newtonian gravitational potential is the metric, and, if we work in a local inertial frame (LIF), the 2nd derivatives of the metric are the Riemann curvature which, like the 2nd derivatives of the Newtonian potential – i.e. the Newtonian tidal field – are observable through their influence on the trajectories of neighbouring particles or photons. This – as we shall see in more detail below – ties down the constant κ in the theory. The field equations are the simplest such generalisation, though there is the possibility to add the cosmological constant term, with the additional constant Λ.

Solving the field equations, however, is much more difficult than for Newton's gravity. In Newtonian gravity the space-time geometry is given *a priori*, and Poisson's equation and the relations between the potential, gravity and tidal field are all linear. That mean we can write down the potential – and hence obtain the gravity and the tide – for a given mass distribution simply by summing the $\delta\phi = -G\delta m/r$ potentials of all the mass elements.

In Einstein's theory, the equations are non-linear, and the space-time geometry emerges as part of the solution, so it is a great challenge to find a space-time and stress-energy tensor that are compatible with each other. And we further require that that stress-energy tensor be compatible with e.g. the equations of motion[1] of the fields comprising it. And even when we have a solution, the fact that we had complete freedom in choice of the coordinate system in terms of which it is expressed may make it difficult to interpret physically.

Here we will consider the situation where the curvature of space-time is very weak, so we look for solutions where the metric is very close to that of a flat Minkowski 'background', with small perturbations:

$$g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta}. \tag{1}$$

We can then proceed much as in Newtonian gravity where we can search for a solution giving the metric perturbations caused by a given matter source term. Though there are still subtleties to do with the choice of coordinate systems.

In this lecture we will develop this perturbation theory approach, and we will apply it in the Newtonian limit. This is an important application because most things that we can observe – with the exception of the immediate vicinity of black-holes and relativistic stars on one hand and the properties of the universe on the largest scale on the other – are well described by this theory.

The other important application of the weak-field theory developed in the first half of this lecture is to gravitational waves, which will be considered in the next lecture.

2 Geometrized units

It can be convenient in SR to choose units of length and time so that the numerical value of the speed of light is unity. That doesn't mean that $c = 1$ as c has units of length / time or [L/T]. But if we think of formulae as only representing numbers, rather than physical entities, we can then be sloppy and simply omit c from formulae, leaving it up to the reader to figure out that, in a formula like $ds^2 = -dt^2 + dx^2 + dy^2 + dz^2$, for example, dt^2 is really shorthand for $c^2 dt^2$.

Similarly, it can be convenient in GR to choose units of mass such that Newton's constant $G_\mathrm{N} = c^2$. These are called *geometrized* units.

Given that the orbital velocity v for a test particle around a mass is $v^2 = G_\mathrm{N} M/r$, we have $v^2/c^2 = (G_\mathrm{N}/c^2) \times M/r$, which numerically, is $v^2 = M/r$, so we can express masses as equivalent lengths; the length corresponding to a mass M is the radius of the orbit for which the orbital speed would be c. I.e. the Schwarzschild radius for a BH of that mass.

So we can quote the value of mass in (equivalent) metres. For example, 1 solar mass ($M_\odot \simeq 2 \times 10^{30}$kg) is approximately equivalent to 1.5 km.

Einstein's field equations are $\mathbf{G} = 8\pi\kappa\mathbf{T}$ and contain a single dimensionful constant κ. In the convention we are using, $\mathbf{G}$ has units of inverse length squared [L^{-2}] while $\mathbf{T}$ has units of energy density [$\mathrm{ML}^{-1}\mathrm{T}^{-2}$], so κ has units [$\mathrm{M}^{-1}\mathrm{L}^{-1}\mathrm{T}^2$]. Newton's constant, on the other hand, from $v^2 = G_\mathrm{N} M/r$, has units [$\mathrm{M}^{-1}\mathrm{L}^3/\mathrm{T}^2$], different from that of G_N by [$\mathrm{L}^4/\mathrm{T}^4$]. Expressed in terms of G_N, and requiring correspondence between Einstein and Newton for low velocity particles (which is where the 8π comes from) the field equations are $\mathbf{G} = 8\pi(G_\mathrm{N}/c^4)\mathbf{T}$.

So in geometrized units the field equations take the form

$$\mathbf{G} = 8\pi\mathbf{T}. \tag{2}$$

[1]People often describe the equations of energy and momentum conservation $T^{\mu\nu}{}_{;\mu} = 0$ as being the 'equations of motion', but, in general, they are not sufficient to describe the matter and one needs to solve the equations of motion for the fields and/or particles.

This saves a bit of ink and typing. But it is not clear, in the present circumstances, that this is such a great idea. Here we are doing perturbation theory, where we think of $\mathbf{G}$ as being a first order response to a zeroth order matter source term $\mathbf{T}$. So maybe it is helpful to keep the $\kappa = G_{\rm N}/c^4$ visible rather than hidden, in order to remind ourselves that this is facilitated by the weakness of the gravitational interaction. Mostly, we'll keep it (as this is supposed to be an introductory course we want to minimise the burden on the reader), but if it offends you just ignore it.

3 Weak field gravity

3.1 Nearly Minkowskian coordinate systems

We assume there exists a coordinate system $\vec{x}$ such that the proper separation between two events with separation $d\vec{x} \to dx^\alpha$ is, as usual, $ds^2 = g_{\alpha\beta}(\vec{x})dx^\alpha dx^\beta$, with

$$\boxed{g_{\alpha\beta}(\vec{x}) = \eta_{\alpha\beta} + h_{\alpha\beta}(\vec{x})} \tag{3}$$

- where $\eta_{\alpha\beta} = \mathrm{diag}\{-1, 1, 1, 1\}$ is the usual flat space-time 'Minkowski metric'
- and where $h_{\alpha\beta}$ are the components of the 'metric perturbation'
- and where we assume that these are all small: $|h_{\alpha\beta}| \ll 1$

There are a wide range of situations where this is a very good approximation. As we saw, the effect of a gravitating body like the Earth can be described by a metric in which there is 'warping of time' with $g_{00} \simeq -(1 + 2\phi/c^2)$ where ϕ is the Newtonian potential which, for Earth, gives $h_{00} \sim 10^{-9}$. And, as we shall see shortly, the spatial parts of the metric perturbation are of the same order of magnitude.

For galaxies and galaxy clusters the motions are larger, but still $h_{00} \lesssim 10^{-5}$. So the weak-field approximation is very good indeed.

By working only to lowest order in these perturbations we will obtain a great simplification of the theory.

This breaks down, however, for black holes and highly relativistic stars like neutron stars on the one hand, and on the largest scales we can observer in cosmology on the other.

3.2 Transformation of the weak-field metric

A metric, in general, contains information about both the geometry of space-time – described by $\mathbf{g}$ – and the coordinate system we have adopted, which, together with the geometry, fixes the components $g_{\alpha\beta}(\vec{x})$ and hence $h_{\alpha\beta}(\vec{x})$.

The same is true in weak-field gravity. This is both a blessing – because we can judiciously choose coordinates to simplify the equations – and a curse.

Understanding and exploiting this requires, as a first step, figuring out how the components of the metric perturbations $h_{\alpha\beta}(\vec{x})$ change under transformations of the coordinate system

3.2.1 Global 'background' Lorentz transformations

Consider a transformation of coordinates exactly like a Lorentz transformation in flat space:

$$x^{\alpha'}(x^\alpha) = \Lambda^{\alpha'}{}_\alpha x^\alpha \tag{4}$$

where, for a boost along the x^1-axis with velocity v, for example,

$$\Lambda^{\alpha'}{}_\alpha = \begin{bmatrix} \gamma & -\gamma v/c & & \\ -\gamma v/c & \gamma & & \\ & & 1 & \\ & & & 1 \end{bmatrix}. \tag{5}$$

This has an inverse transformation $x^\alpha = \Lambda^\alpha{}_{\alpha'} x^{\alpha'}$ with $\Lambda^\alpha{}_{\alpha'}$ given by the same formula with the sign of v reversed.

More generally, the velocity need not lie along the x-axis, and the transformation matrix might also include rotation, so we have the usual family of Lorentz transformations, parameterised by the three components of $\mathbf{v}$ and the three Euler angles.

Since $ds^2 = g_{\alpha\beta}(\vec{x})dx^\alpha dx^\beta = g_{\alpha\beta}(\vec{x})\frac{\partial x^\alpha}{\partial x^{\alpha'}}\frac{\partial x^\beta}{\partial x^{\beta'}}dx^{\alpha'}x^{\beta'}$, and, since $\partial x^\alpha/\partial x^{\alpha'} = \Lambda^\alpha{}_{\alpha'}$ here, the metric transforms, as usual, according to

$$g_{\alpha'\beta'} = \Lambda^\alpha{}_{\alpha'}\Lambda^\beta{}_{\beta'}g_{\alpha\beta} = \Lambda^\alpha{}_{\alpha'}\Lambda^\beta{}_{\beta'}(\eta_{\alpha\beta} + h_{\alpha\beta}) \tag{6}$$

but $\eta_{\alpha\beta}$ is unchanged by this transformation so, writing $g_{\alpha'\beta'} = \eta_{\alpha'\beta'} + h_{\alpha'\beta'}$, we have

$$\boxed{h_{\alpha'\beta'} = \Lambda^\alpha{}_{\alpha'}\Lambda^\beta{}_{\beta'}h_{\alpha\beta}} \tag{7}$$

so the metric perturbation transforms, under this transformation, just like a tensor transforms in SR.

One might imagine using this if one wanted to know what is the gravitational field of a star as perceived by a rapidly moving observer. One could then calculate the metric perturbation in the frame in which the star is at rest, where one can exploit the symmetry and lack of time variation, and then apply boost matrices as above to transform to the relatively moving frame.

3.2.2 Raising, lowering and contracting indices of the metric perturbation

The components of the inverse metric $g^{\alpha\beta} = (\mathbf{g}^{-1})^{\alpha\beta}$ are given, to zeroth order in the perturbation, by $\eta^{\alpha\beta}$. So the mixed rank metric perturbation $h^\alpha{}_\beta = g^{\alpha\mu}h_{\mu\beta} = \eta^{\alpha\mu}h_{\mu\beta} + \ldots$. So we can use the Minkowski metric to raise or lower indices of, and to perform contractions on, the metric perturbation $h_{\alpha\beta}$. So we have for the contraction $h = h^\alpha{}_\alpha = \eta^{\alpha\beta}h_{\alpha\beta}$ and for the contravariant components $h^{\mu\nu} = \eta^{\mu\alpha}\eta^{\nu\beta}h_{\alpha\beta}$.

Note that the perturbation of the components $g^{\mu\nu}$ of $\mathbf{g}^{-1}$ are *not* $h^{\mu\nu}$. Since $g^{\mu\nu}g_{\nu\beta} = \delta^\mu_\beta$, and writing $g^{\mu\nu} = \eta^{\mu\nu} + p^{\mu\nu}$, it must be that $\delta^\mu_\beta = (\eta^{\mu\nu} + p^{\mu\nu})(\eta_{\nu\beta} + h_{\nu\beta}) = \delta^\mu_\beta + \eta^{\mu\nu}h_{\nu\beta} + p^{\mu\nu}\eta_{\nu\beta} + p^{\mu\nu}h_{\nu\beta}$. Dropping the last term as it is 2nd order, we see that the inverse metric perturbations $p^{\mu\nu}$ satisfy $p^{\mu\sigma}\eta_{\sigma\beta} = -\eta^{\mu\alpha}h_{\alpha\beta}$ or, multiplying by $\eta^{\nu\beta}$, $p^{\mu\sigma}\eta_{\sigma\beta}\eta^{\nu\beta} = p^{\mu\sigma}\delta^\nu_\sigma = p^{\mu\nu} = -\eta^{\nu\beta}\eta^{\mu\alpha}h_{\alpha\beta}$ or $p^{\mu\nu} = -h^{\mu\nu}$ and thus, at linear order, the inverse metric (i.e. the thing one uses to compute scalar products of 1-forms, for instance) is

$$g^{\alpha\beta} = \eta^{\alpha\beta} - h^{\alpha\beta}. \tag{8}$$

3.2.3 Gauge transformations

Another – arguably more useful – type of transformation is that in which the coordinates $x^{\alpha'}(\mathcal{P})$ of a point or event $\mathcal{P}$ in the primed frame are the same as the coordinates $x^\alpha(\mathcal{P})$ of the same point in the un-primed frame plus a small – in a sense to be made precise presently – displacement vector field $\xi^\alpha(x^\beta)$

We can write this as

$$\boxed{x^{\alpha'}(x^\beta) = x^\alpha + \xi^\alpha(x^\beta)} \tag{9}$$

where the meaning of this formally illegitimate equation (as the indices do not balance) is that for any choice of the index α', $x^{\alpha'}$ is given by the right hand side with $\alpha = \alpha'$.

To avoid having to say all of that and if you want to keep the indices balanced you can instead write this transformation more carefully – or perhaps pedantically – as

$$\begin{aligned} x^{\alpha'}(x^\beta) &= r^{\alpha'}(x^\beta) \quad \text{where} \\ r^\alpha(x^\beta) &\equiv x^\alpha(x^\beta) + \xi^\alpha(x^\beta). \end{aligned} \tag{10}$$

Either way, the transformation matrix is $\Lambda^{\alpha'}{}_\beta \equiv \partial x^{\alpha'}/\partial x^\beta = \partial r^{\alpha'}/\partial x^\beta$, or

$$\boxed{\Lambda^{\alpha'}{}_\beta = \delta^{\alpha'}{}_\beta + \xi^{\alpha'}{}_{,\beta}.} \tag{11}$$

One can visualise the situation in 2D if we imagine the un-primed coordinates of events like $\mathcal{P}$ displayed as Cartesian coordinates. The primed coordinates can then be read off from an overlaid transparency on which the lines of constant primed coordinates are slightly distorted with respect to the Cartesian grid.

The corresponding inverse transformation is (being careful and/or pedantic)

$$\begin{aligned} x^\alpha(x^{\beta'}) &= p^\alpha(x^{\beta'}) \quad \text{where} \\ p^{\alpha'}(x^{\beta'}) &\equiv x^{\alpha'} - \xi^{\alpha'}(x^{\beta'} - \xi^{\beta'}) \\ &= x^{\alpha'} - \xi^{\alpha'}(x^{\beta'}) + \xi^{\beta'}\xi^{\alpha'}{}_{,\beta'} - \xi^{\beta'}\xi^{\gamma'}\xi^{\alpha'}{}_{,\beta'\gamma'} + \dots \end{aligned} \tag{12}$$

where we have performed a Taylor expansion.

We can now make precise by what we mean by the gauge transformation being small. It is not in fact necessary that $\vec{\xi}$ itself be small. Provided that the components of the 'distortion tensor' are small:

$$|\xi^\alpha{}_{,\beta}| \ll 1 \tag{13}$$

which means that the primed coordinate grid is only slightly *distorted*, and provided that $|\xi^\gamma\xi^\alpha{}_{,\beta\gamma}| \ll 1$ and so on, then we can ignore all the terms in the expansion involving derivatives, and we have for the inverse transformation

$$\begin{aligned} x^\alpha(x^{\beta'}) &= p^\alpha(x^{\beta'}) \quad \text{where} \\ p^{\alpha'}(x^{\beta'}) &\equiv x^{\alpha'} - \xi^{\alpha'}(x^{\beta'}) \end{aligned} \tag{14}$$

or, taking slight liberties as above,

$$\boxed{x^\alpha(x^{\beta'}) = x^{\alpha'} - \xi^\alpha(x^{\beta'}).} \tag{15}$$

The inverse transformation matrix is $\Lambda^\alpha{}_{\beta'} \equiv \partial x^\alpha / \partial x^{\beta'}$ where

$$\boxed{\Lambda^\alpha{}_{\beta'} = \delta^\alpha{}_{\beta'} - \xi^\alpha{}_{,\beta'}.} \tag{16}$$

3.2.4 Transformation of the metric under a gauge transformation

The transformation of the metric is $g_{\alpha'\beta'} = \Lambda^\alpha{}_{\alpha'}\Lambda^\beta{}_{\beta'}g_{\alpha\beta}$, or

$$g_{\alpha'\beta'} = (\delta^\alpha{}_{\alpha'} - \xi^\alpha{}_{,\alpha'})(\delta^\beta{}_{\beta'} - \xi^\beta{}_{,\beta'})(\eta_{\alpha\beta} + h_{\alpha\beta}) \tag{17}$$

or, keeping only terms which are first order (in the metric perturbation or the gauge distortion tensor) and writing

$$g_{\alpha'\beta'} = \eta_{\alpha'\beta'} + h_{\alpha'\beta'} \tag{18}$$

we obtain the law for the transformation of $h_{\alpha\beta}$ under a gauge transformation:

$$\boxed{h_{\alpha'\beta'} = h_{\alpha\beta} - \xi_{\alpha,\beta} - \xi_{\beta,\alpha}} \tag{19}$$

where the meaning of this formally illegitimate equation is as above for (9) and where, for example, $\xi_{\alpha,\beta} = \eta_{\alpha\gamma}\xi^\gamma{}_{,\beta}$.

We will use such transformations to find coordinate systems in which the metric takes a conveniently simple form. In doing this, the components of the required distortion tensor are typically of the same order of magnitude as those of the metric perturbation tensor.

3.3 The curvature in weak-field gravity

3.3.1 The linearised Riemann tensor

The general expression for the connection in terms of the metric is

$$\Gamma^\gamma{}_{\beta\mu} = \tfrac{1}{2}g^{\gamma\alpha}(g_{\alpha\beta,\mu} + g_{\alpha\mu,\beta} - g_{\beta\mu,\alpha}) \tag{20}$$

so for weak fields, and to first order in $|h_{\alpha\beta}|$,

$$\Gamma^\gamma{}_{\beta\mu} = \tfrac{1}{2}\eta^{\gamma\alpha}(h_{\alpha\beta,\mu} + h_{\alpha\mu,\beta} - h_{\beta\mu,\alpha}) \tag{21}$$

so the connection is a purely 1st order quantity. This means that we can ignore the products of Christoffel symbols in the expression for the Riemann tensor and we have, at leading order,

$$R^{\alpha}{}_{\beta\mu\nu} = -\Gamma^{\alpha}{}_{\beta\mu,\nu} + \Gamma^{\alpha}{}_{\beta\nu,\mu} \tag{22}$$

or, lowering the index α (with the Minkowski metric, of course)

$$R_{\alpha\beta\mu\nu} = [-\tfrac{1}{2}(\cancel{h_{\alpha\beta,\mu\nu}} + h_{\alpha\mu,\beta\nu} - h_{\beta\mu,\alpha\nu})] - \{\mu \Leftrightarrow \nu\} \tag{23}$$

where the slash indicates that the first term will cancel when we subtract the corresponding term with μ and ν flipped.

This gives

$$\boxed{R_{\alpha\beta\mu\nu} = -\tfrac{1}{2}(h_{\alpha\mu,\beta\nu} - h_{\alpha\nu,\beta\mu} + h_{\beta\nu,\alpha\mu} - h_{\beta\mu,\alpha\nu})} \tag{24}$$

which you may remember as it's what we got before for the Riemann tensor in locally inertial coordinates. Here we are not (necessarily) working in a LIF, but the same formula is valid by virtue of the smallness of the metric perturbations.

This is a bit ugly, and maybe a bit hard to remember. I find it easier to think of this as

$$\boxed{R_{\alpha\beta\mu\nu} = [(-\tfrac{1}{2}h_{\alpha\mu,\beta\nu}) - \{\alpha \Leftrightarrow \beta\}] - \{\mu \Leftrightarrow \nu\}} \tag{25}$$

so you just have to remember the 1/2, the minus sign – which is conventional – and the fact that $R_{\alpha\beta\mu\nu}$ is antisymmetric under interchange of either α and β or μ and ν.

3.3.2 Transformation of the linearised Riemann tensor

Applying the law (19) for transforming the metric $h_{\alpha'\beta'} = h_{\alpha\beta} - \xi_{\alpha,\beta} - \xi_{\beta,\alpha}$ to (25) above, the curvature changes under a gauge transformation to

$$R_{\alpha'\beta'\mu'\nu'} = R_{\alpha\beta\mu\nu} + ([\tfrac{1}{2}(\xi_{\alpha,\mu\beta\nu} + \xi_{\mu,\alpha\beta\nu}) - \{\alpha \Leftrightarrow \beta\}] - \{\mu \Leftrightarrow \nu\}). \tag{26}$$

But $\xi_{\alpha,\mu\beta\nu}$ is symmetric under $\mu \Leftrightarrow \nu$ and $\xi_{\mu,\alpha\beta\nu}$ is symmetric under $\alpha \Leftrightarrow \beta$, so both terms vanish when we anti-symmetrise, and we have

$$\boxed{R_{\alpha'\beta'\mu'\nu'} = R_{\alpha\beta\mu\nu}.} \tag{27}$$

Thus, unlike the components of the metric perturbations from which it is constructed, the components of the curvature tensor (25) are invariant under a gauge transformation.

Note that this is not just saying that the curvature tensor $\mathbf{R}$ – considered as a geometric object – is invariant. That is a given, and it is invariant under *any* coordinate transformation, no matter how large they might be, as is the metric $\mathbf{g}$ also for that matter.

This is very different in that it says that provided we only make small amplitude coordinate transformations (with $|\xi^{\alpha}{}_{,\beta}| \sim |h_{\alpha\beta}|$) the *components* of $\mathbf{R}$ do not change (at first order).

3.4 The linearised Einstein field equations in the Lorenz gauge

We will now obtain the Einstein tensor $G_{\alpha\beta}$ – which, like $R^{\alpha}{}_{\beta\mu\nu}$ is gauge invariant – in terms of the metric perturbations $h_{\alpha\beta}$, and then show how this relation can be greatly simplified by applying a gauge transformation to $h_{\alpha\beta}$. This results in the linearised Einstein field equations in the Lorenz gauge.

3.4.1 The Ricci tensor and scalar

Performing the contractions of the linearised curvature tensor (25) on its 1st and 3rd indices we obtain the Ricci tensor:

$$R_{\alpha\beta} \equiv \eta^{\mu\nu} R_{\mu\alpha\nu\beta} = -\frac{1}{2}(h_{,\alpha\beta} - h^{\mu}{}_{\beta,\alpha\mu} + h_{\alpha\beta}{}^{,\mu}{}_{,\mu} - h_{\alpha\mu}{}^{,\mu}{}_{,\beta}) \tag{28}$$

and contracting this gives the Ricci scalar:

$$R \equiv \eta^{\beta\nu} R_{\beta\nu} = h_{\mu\nu}{}^{,\mu\nu} - h^{,\mu}{}_{,\mu}. \tag{29}$$

3.4.2 The trace-reversed metric perturbation

It proves useful to introduce, at this point, the *trace-reversed metric perturbation*:

$$\boxed{\overline{h}_{\alpha\beta} \equiv h_{\alpha\beta} - \eta_{\alpha\beta}h/2} \tag{30}$$

which, as its name implies, has contraction, or 'trace', $\overline{h} \equiv \eta^{\alpha\beta}\overline{h}_{\alpha\beta} = \overline{h}^{\alpha}{}_{\alpha} = h^{\alpha}{}_{\alpha} - \delta^{\alpha}{}_{\alpha}h/2 = -h$ as $\delta^{\alpha}{}_{\alpha} = 4$.

Note that the Einstein tensor is the trace-reversed version of the Ricci tensor.

3.4.3 The Einstein tensor

In terms of $\overline{h}_{\alpha\beta}$ the *linearised Einstein tensor:* is

$$G_{\alpha\beta} = R_{\alpha\beta} - \tfrac{1}{2}\eta_{\alpha\beta}R = -\frac{1}{2}[\overline{h}_{\alpha\beta,\mu}{}^{,\mu} + \eta_{\alpha\beta}\overline{h}_{\mu\nu}{}^{,\mu\nu} - \overline{h}_{\alpha\mu,\beta}{}^{,\mu} - \overline{h}_{\beta\mu,\alpha}{}^{,\mu}] \tag{31}$$

where we have gained some compactification by invoking the 'trace-reversed' metric perturbation.

Equating this to ($8\pi\kappa$ times) some given stress energy tensor $T_{\alpha\beta}(\vec{x})$ gives

$$\overline{h}_{\alpha\beta,\mu}{}^{,\mu} + \eta_{\alpha\beta}\overline{h}_{\mu\nu}{}^{,\mu\nu} - \overline{h}_{\alpha\mu,\beta}{}^{,\mu} - \overline{h}_{\beta\mu,\alpha}{}^{,\mu} = -16\pi\kappa T_{\alpha\beta} \tag{32}$$

which, in principle, can be solved for $\overline{h}_{\alpha\beta}$.

3.4.4 The Lorenz or de Donder gauge

The field equations can be dramatically simplified if we invoke a small (i.e. $|\xi^{\alpha}{}_{,\beta}| \ll 1$) gauge transformation $x^{\alpha'}(x^{\beta}) = x^{\alpha} + \xi^{\alpha}(x^{\beta})$, which in no way changes the Einstein tensor, to obtain a coordinate system in which the 4-divergence $\overline{h}_{\nu\mu}{}^{,\mu}$ of the trace-reversed metric perturbation vanishes .

Proof:

- A gauge transformation changes the metric to
  - $h^{(\text{new})}_{\mu\nu} = h^{(\text{old})}_{\mu\nu} - \xi_{\mu,\nu} - \xi_{\nu,\mu}$
- and, it is easily shown, changes the trace-reversed metric to
  - $\overline{h}^{(\text{new})}_{\mu\nu} = \overline{h}^{(\text{old})}_{\mu\nu} - \xi_{\mu,\nu} - \xi_{\nu,\mu} + \eta_{\mu\nu}\xi^{\alpha}{}_{,\alpha}$
- taking the derivative with respect to x^{ν} we get[2] the 4-divergence
  - $\overline{h}^{(\text{new}),\nu}_{\mu\nu} = \overline{h}^{(\text{old}),\nu}_{\mu\nu} - \xi_{\mu,\nu}{}^{,\nu}$
- or
  - $\overline{h}^{(\text{new}),\nu}_{\mu\nu} = \overline{h}^{(\text{old}),\nu}_{\mu\nu} - \Box\xi_{\mu}$
- where $\Box$ is the *d'Alembertian operator* defined by $\Box f \equiv f_{,\nu}{}^{,\nu}$
- choosing the four ξ_{μ} to be solutions of $\Box\xi_{\mu} = \overline{h}^{(\text{old}),\nu}_{\mu\nu}$ gives $\overline{h}^{(\text{new}),\nu}_{\mu\nu} = 0$

In this gauge all but the first term in (31) vanishes. This gives the linearised Einstein tensor in the so-called *Lorenz gauge*:

$$\boxed{G_{\alpha\beta} = -\tfrac{1}{2}\Box\overline{h}_{\alpha\beta}.} \tag{33}$$

[2] Note that $\xi_{\nu}{}^{,\nu} = \eta_{\nu\alpha}\xi^{\alpha,\nu} = \xi^{\alpha}{}_{,\alpha}$

3.4.5 The field equations in the Lorenz gauge

Equating $G_{\alpha\beta}$ above to $8\pi\kappa T_{\alpha\beta}$ gives the *linearised Einstein field equations in the Lorenz gauge:*

$$\boxed{\Box\overline{h}_{\alpha\beta} = -16\pi\kappa T_{\alpha\beta}} \tag{34}$$

which can be solved, for a given matter 'source term' $T_{\alpha\beta}$ on the RHS, to obtain $\overline{h}_{\alpha\beta}$, from which we obtain the, more physically interesting, non-trace-reversed perturbation $h_{\alpha\beta} \equiv \overline{h}_{\alpha\beta} - \eta_{\alpha\beta}\overline{h}/2$, and from which we can obtain the Christoffel symbols – for use in the geodesic equation, for instance, and to modify other physical equations using the 'comma $\Rightarrow$ semi-colon rule' – and the curvature tensor $\mathbf{R}$ to use to calculate geodesic deviation, for example.

One significant difference between this and Newtonian gravity is that here one can add to the solution any 'homogeneous' solution that satisfies the wave equation $\Box\overline{h}_{\alpha\beta} = 0$. These may describe gravitational waves.

3.5 Comments on gauge transformations in GR

A gauge transformation in electromagnetism is a change to the electromagnetic 4-potential $\vec{A} \to \vec{A}' = \vec{A} + \vec{\nabla}\xi$ where $\xi(\vec{x})$ is an arbitrary function of space-time. This leaves the electric and magnetic fields unchanged[3] and can be used to simplify Maxwell's equations and their solutions, particularly for problems involving radiation from moving charges.

For example, in the Lorenz gauge, we demand that the 4-divergence of the potential vanish: ${A'_\mu}^{,\mu} = 0$. In exploiting this, we don't need to actually solve the equation ($\Box\xi = {A_\mu}^{,\mu}$) that ξ needs to satisfy to effect this simplification; we simply appeal to the fact that a solution exists, and then solve the simplified form of Maxwell's equations in which terms involving ${A_\mu}^{,\mu}$ are dropped.

The EM gauge interaction has the additional property that, when coupled to the Schrödinger equation by means of the substitution $\partial_\mu \Rightarrow \partial_\mu - iq(A_\mu/\hbar)$ – which neatly explains the phenomenology of electrodynamics – the combined Maxwell-Schrödinger system is invariant if the wave function is simultaneously changed to $\psi \Rightarrow \psi' = e^{i(q/\hbar)\xi}\psi$. This is often said to mean that the EM gauge field A^μ exists in order that the world be invariant under a local phase shift of the wave function.

Gauge transformations in linearised GR are quite similar (though without the 'philosophical baggage' relating to the unmeasurability of phase of wave functions). As in electromagnetism, and as we have seen above, gauge transformations are extremely useful as they allow significant simplification of the field equations.

The Einstein field equations relate the Einstein tensor $\mathbf{G}$ to the matter stress tensor through $\mathbf{G} = 8\pi\kappa\mathbf{T}$ and these can, in principle, be solved to obtain a metric $\mathbf{g}$ and a stress tensor $\mathbf{T}$ that are compatible with each other. These are all geometrical objects, entirely independent of any coordinate system. The *components* of all these tensors, on the other hand, do depend on the coordinates.

In perturbation theory, however, the right hand side of the field equations is a 'zeroth order' quantity whose components $T_{\alpha\beta}$ we wish to specify as a function of the coordinates x^α; at lowest order it is independent of any first order 'gauge' coordinate transformations $x^\alpha \Rightarrow x^{\alpha'} = x^\alpha + \xi^\alpha$ that we might apply. For this theory to make any sense, it is a logical necessity that the left hand side of the field equations should also have components that, again at lowest non-vanishing order, are gauge independent, and it is reassuring, but perhaps not surprising, that the components of the Einstein tensor and the Riemann tensor are indeed gauge invariant.

But the components of the metric are not gauge invariant, nor are the components of the connection, that appear in the geodesic equation. The general (complicated) form of the field equations (32) apply for coordinate systems differing by any 1st order gauge change and therefore allow considerable freedom in the metric, as expressed in the law for the transformation of the metric under a gauge shift (19). Equations (32) are valid for a *family* of solutions $h_{\alpha\beta}$ for a given source term. The Lorenz-gauge field equations (34)

[3] As an aside, it was thought historically that the *only* physical effects came from the $\mathbf{E}$ and $\mathbf{B}$ fields and that the potential $\vec{A}$ could not be directly observed. But in 1949, Ehrenberg and Siday, who were using electron wave-optics to study electron microscopes, noted that $\vec{A}$ *could* be directly observed using electron beam interference, as there is a shift of the fringes proportional to the difference in the line integral of $\vec{A}$ along the two interfering paths. As they pointed out, this is a purely classical wave-mechanical effect. Their paper went largely un-noticed. The effect was re-discovered independently by Aharonov and Bohm in 1959. They put the word "quantum" in their title and their paper had a high impact.

impose $\overline{h}_{\alpha\beta}{}^{;\alpha} = 0$ and thereby constrain the coordinate system, and result, with some reasonable additional constraints, in an essentially[4] unique solution.

Q: Explain why the Ehrenberg-Siday/Ahoronov-Bohm effect is 'blind' to a gauge transformation

4 The weak-field metric for stationary or nearly-stationary sources

4.1 The source term for non-relativistic matter

Many astrophysical systems have internal velocities that are very small compared to c. Examples are:

- the solar system: $v \sim 30$ km/s $= 10^{-4}c$
- galaxies: $v \sim 100 - 300$ km/s $= 3 - 10 \times 10^{-4}c$
- clusters of galaxies & and large scale 'streaming' motions: $v \sim 1000$ km/s $= 3 \times 10^{-3}c$

So any *momentum density* or *energy flux density* $T^{0i} \ll T^{00} = \rho c^2$ by a factor $\sim v/c$.

And the *pressure* or *momentum flux density* T^{ij} is smaller still (by factor v^2/c^2 compared to T^{00}).

The same is true for thermal gas pressure $T_{ij} = (P/c^2)\delta_{ij} \sim \rho(\sigma_v/c)^2$, since atom and molecular velocity dispersions are $\sigma^2 \sim GM/r$ from the equation of hydrostatic equilibrium.

And radiation pressure is similarly $\ll \rho c^2$ if the radiation density is small compared to the matter.

As mentioned, these conditions do not hold inside and around relativistic stars or close to black holes. They also become invalid on cosmological scales where the Hubble velocity becomes comparable to c and/or where, it is commonly believed, we are seeing the influence of dark energy, possibly in the form of a 'quintessence' field, in which case the stress of the field is not negligible. It also does not include the effect of gravitational waves. But aside from that it has wide applicability.

So for a very wide range of circumstances, the right hand side of Einstein's equation is

$$T_{\alpha\beta} \simeq \begin{bmatrix} \rho c^2 & & & \\ & & & \\ & & & \\ & & & \end{bmatrix} = \delta^0_\alpha \delta^0_\beta \rho c^2 \tag{35}$$

wherein all the blank entries are zero, to high precision.

The matter density distribution ρ in galaxies and clusters etc. also has slow temporal variation from either internal motions for which $\rho_{,t} \simeq \rho/t_{\rm dyn}$, or from net motions, for which $\rho_{,t} \sim v\rho_{,i}$. Both of these are $\ll \rho_{,i}c$.

4.2 The weak-field metric for stationary sources

A stationary source $T_{\alpha\beta}$ is one for which $\partial_t T_{\alpha\beta} = 0$, so $T_{\alpha\beta}$ is only dependent on the spatial coordinates. This includes matter that has no motion, but also includes e.g. steady beams of matter or radiation or things like a rotating flywheel.

If we postulate that such sources allow solutions where the metric perturbations are also stationary, then the d'Alembertian operator becomes the Laplacian:

$$\Box \overline{h}_{\alpha\beta} = (-c^{-2}\partial^2/\partial t^2 + \nabla^2)\overline{h}_{\alpha\beta} = \nabla^2 \overline{h}_{\alpha\beta} \tag{36}$$

where ∇^2 is the 3D Laplacian operator.

So the weak field equations are

$$\nabla^2 \overline{h}_{\alpha\beta} = -16\pi\kappa T_{\alpha\beta} \tag{37}$$

with solution

$$\overline{h}_{\alpha\beta} = -4\Phi_{\alpha\beta}(\mathbf{x}) \tag{38}$$

[4] As we will see in the following lecture, in asserting the Lorenz gauge conditions we have not yet exhausted all of the gauge freedom at our disposal. However the additional gauge transformations are essentially travelling waves, and we exclude these here because we seek a solution corresponding to a slowly varying or static source term.

where

$$\Phi_{\alpha\beta}(\mathbf{x}) = \frac{G_{\rm N}}{c^4} \int d^3x' T_{\alpha\beta}(\mathbf{x}')/|\mathbf{x} - \mathbf{x}'| \tag{39}$$

so we are just summing the inverse square potentials of the source elements. This has the advantage of, as well as solving Poisson's equation $\nabla^2 \Phi_{\alpha\beta} = 4\pi G_{\rm N} c^{-4} T_{\alpha\beta}$ for each component, it has the boundary conditions that, for a source of finite extent, all of the $\Phi_{\alpha\beta}$ tend to zero at spatial infinity.

Thus, by

1. imposing the divergence-free gauge condition $\overline{h}_{\alpha\beta}{}^{,\beta}$
2. requiring the solution be stationary, and
3. imposing the usual boundary conditions at spatial infinity in Poisson's equation

we end up with a unique solution $\overline{h}_{\alpha\beta}(\mathbf{x})$ and from this, by un-trace-reversing, we obtain the solution for $h_{\alpha\beta}(\mathbf{x})$.

4.3 The Newtonian limit metric

A special case of a stationary source is that of a static mass distribution with zero pressure. As discussed above, this has $T_{\alpha\beta} \simeq \delta^0_\alpha \delta^0_\beta \rho c^2$, so

$$\begin{aligned} \overline{h}_{\alpha\beta}(\mathbf{x}) &= -4\delta^0_\alpha \delta^0_\beta \Phi(\mathbf{x}) \quad \text{where} \\ \Phi(\mathbf{x}) &= \frac{G_{\rm N}}{c^2} \int d^3x' \rho(\mathbf{x}')/|\mathbf{x} - \mathbf{x}'| \end{aligned} \tag{40}$$

so Φ is the dimensionless potential given by $\Phi = \phi/c^2$ where ϕ is the solution of Poisson's equation $\nabla^2\phi = 4\pi G_{\rm N}\rho$ for this density distribution.

In this, $\overline{h}_{00} = -4\Phi$, with all other components being negligibly small, or, in gory detail,

$$\overline{h}_{\alpha\beta}(\mathbf{x}) = \begin{bmatrix} -4\Phi(\mathbf{x}) & & & \\ & & & \\ & & & \\ & & & \end{bmatrix} \tag{41}$$

and the trace is $\overline{h} = \overline{h}^\alpha{}_\alpha = \eta^{\alpha\beta}\overline{h}_{\alpha\beta} = 4\Phi$, from which we obtain the non-trace-reversed metric perturbation

$$h_{\alpha\beta} = \overline{h}_{\alpha\beta} - \eta_{\alpha\beta}\overline{h}/2 = \begin{bmatrix} -2\Phi & & & \\ & -2\Phi & & \\ & & -2\Phi & \\ & & & -2\Phi \end{bmatrix} \tag{42}$$

or, more succinctly,

$$\boxed{h_{\alpha\beta}(\mathbf{x}) = -2\Phi(\mathbf{x})\delta_{\alpha\beta}} \tag{43}$$

to give, finally, the 'line element' $ds^2 = g_{\alpha\beta}(\mathbf{x})dx^\alpha dx^\beta$ in the Newtonian limit of GR:

$$\boxed{ds^2 = -(1 + 2\Phi(\mathbf{x}))c^2dt^2 + (1 - 2\Phi(\mathbf{x}))(dx^2 + dy^2 + dz^2)} \tag{44}$$

This is an important result. It shows that the time-time component of the metric is $g_{00} = -(1+2\Phi)$ as Einstein inferred from his tower and rocket thought experiments. So time is warped; or, at the very least it shows that, for an observer maintaining constant $\mathbf{r} \rightarrow (x, y, z)$ – an allowed world-line for an observer as it is timelike – the proper time and coordinate time intervals are related by $d\tau = (1 + \Phi)dt$ (at first order). But it also shows apparent 'warping of space' with $g_{ij} = (1 - 2\Phi)\delta_{ij}$ which we did not have before. The question is: is this a real effect? Or is it an artefact arising from the choice of coordinates thrust upon us by the gauge choice.

4.4 The weak-field metric for nearly stationary sources

For a nearly stationary source $T_{\alpha\beta} = T_{\alpha\beta}(\mathbf{x}, t)$, but with $|\partial_t T_{\alpha\beta}| \ll c|\boldsymbol{\nabla} T_{\alpha\beta}|$, the stationary solution $\overline{h}_{\alpha\beta} = -4\Phi_{\alpha\beta}$ with $\Phi(\mathbf{x}, t)_{\alpha\beta}$ calculated using the instantaneous $T_{\alpha\beta}(\mathbf{x}, t)$ as the source:

$$\Phi_{\alpha\beta}(\mathbf{x}, t) = \frac{G_{\rm N}}{c^4} \int d^3x' T_{\alpha\beta}(\mathbf{x}', t)/|\mathbf{x} - \mathbf{x}'| \tag{45}$$

should provide a metric with fractional error for the components that are on the order of v^2/c^2 or L^2/c^2T^2 (where L is the size of the system under consideration and T is its time variation scale). This is because, in obtaining it we approximated the d'Alembertian $\Box\overline{h}_{\alpha\beta}$ by the Laplacian $\nabla^2\overline{h}_{\alpha\beta}$ and ignored $c^{-2}\partial_t^2\overline{h}_{\alpha\beta}$. We could, if we liked, compute a better solution perturbatively by adding a solution to $\nabla^2\overline{h}^{(1)}_{\alpha\beta} = c^{-2}\partial_t^2\overline{h}^{(0)}_{\alpha\beta}$.

This (the zeroth order solution that is) is useful if we wish to calculate effects that are of 1st order in the velocity such as those sourced by the momentum density T_{0i}/c, which being equal to the energy flux density (divided by c^2) implies time-variation of the leading order T_{00}.

5 The physical implications of the weak-field metric

We will now mostly specialise to the case of a static (non time-varying) metric, for which $\Phi(\vec{r}) = \Phi(\mathbf{r})$ and explore some of the physical properties of this space-time.

5.1 The light-cone structure and the coordinate speed of light

In inertial coordinates in Minkowski space the light cones have 45-degree opening angle and have as axis the time coordinate.

A first step to tease out the physical meaning of the Newtonian limit metric is to ask what do the light cones look like in this weakly perturbed space-time (with this particular choice of coordinates or 'gauge')?

- consider a point (event) $\mathcal{P} \rightarrow (ct, x, y, z)$
- and a neighbouring event $\mathcal{P}' \rightarrow (c(t + dt), x + dx, y + dy, z + dz)$
- and require that they have a null separation $ds^2 = 0$, so they can be connected by the path of a massless particle. This implies:
  - $dr^2 \equiv dx^2 + dy^2 + dz^2 = (1 + 2\Phi)/(1 - 2\Phi)c^2dt^2$
  - so the light rays at a point have $dr/dt = c\sqrt{(1 + 2\Phi)/(1 - 2\Phi)} \simeq (1 + 2\Phi)c$
  - which is independent of direction, so the light cones still have circular sections and have axis aligned with the t-direction
  - the light-cones are not tilted in this coordinate system (as they would be, for example, if there were a non-zero off-diagonal components like g_{0x} in the metric)
  - since the Newtonian potential Φ is negative – as we require $\Phi \rightarrow 0$ at spatial infinity – the light cones will have $dr/dt < c$
  - so the *coordinate speed of light* is slightly less than c
  - i.e. the opening angle of the light-cones is slightly less that 45-degrees, as illustrated in figure 1 (highly exaggerated)
- for a bounded gravitating system the light cone structure becomes Minkowskian as $r \rightarrow \infty$ where $\phi \rightarrow 0$

Does the fact that $dr/dt < c$ mean that an observer would perceive light to be moving slower that c? Not at all. This is the *coordinate speed*. The physical speed as measured by any observer is still c.

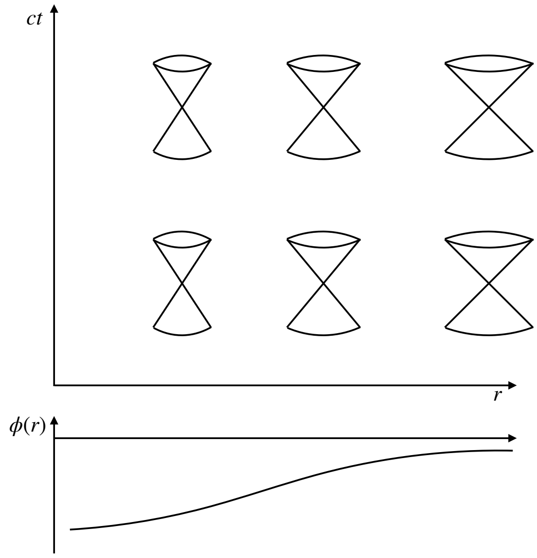


Figure 1: Schematic view of the light cones in the weakly perturbed geometry caused by a static mass distribution with Newtonian potential at bottom. At large r the space-time becomes Minkowskian but more generally the light-cones are squashed and the coordinate speed of light $dr/dt = (1 + 2\Phi)c$ is less that c. Light apparently behaves rather like it does in a medium with refractive index $n = 1/(1+2\Phi) \simeq 1-2\Phi$. Note that, as measured in physical coordinates by any inertial observer the light-cones always have 45 degree opening angle. What this figure is showing us is that, plotted in our coordinate system, the light-cones *appear* to be squashed.

5.2 Constant-r observers

As mentioned, the paths of constant $\mathbf{r}$ are time-like and are therefore possible world-lines of physical observers.

Let's imagine a family of such observers, each of whom has constant $\mathbf{r}$. If the potential is static, that means that these observers maintain unchanging distances from one another.

We will see shortly that, just as in Newtonian gravity, freely falling – or inertial – observers will *not* remain at fixed $\mathbf{r}$. So constant-$\mathbf{r}$ observers must be accelerated in some way. This could be by means of rocket motors or perhaps by some rigid, but light, scaffolding.

For concreteness, let's imagine that they live in some gravitating system composed of non-interacting matter and that they are maintained in their fixed $\mathbf{r}$ positions by a rigid lattice of rods as illustrated in figure 2.

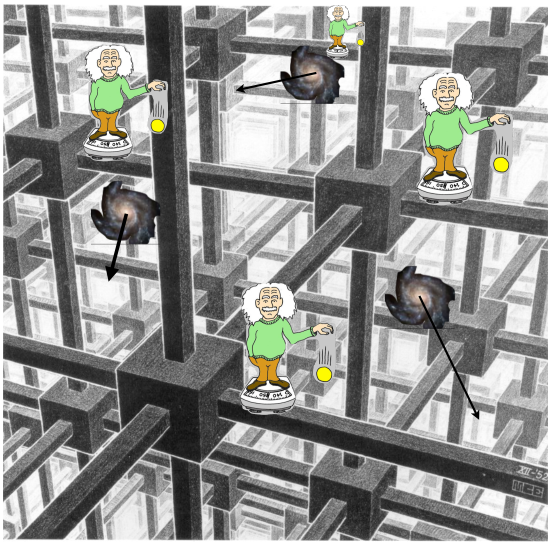


Figure 2: Constant-$\mathbf{r}$ observers. In order to maintain constant $\mathbf{r}$, observers in the weak-field gravity of, say, a cluster of galaxies, or some other cosmic structure, need to be accelerated (their world-lines, as we shall see, are not geodesics). They could be accelerated by rocket motors, or by having some kind of rigid lattice that supports them as shown here (with apologies to M.C. Escher and Albert Einstein). These observers are fully aware of their acceleration. If they stand on a weighing scale it registers their weight and if they release test particles they will see these accelerate with respect to them. As they are accelerated, light signals they exchange will be 'gravitationally' redshifted. If they observe one another's clocks, they will see them drift steadily out of synchronisation.

And let's also assume that they carry clocks measuring proper time τ, and that at some coordinate time t_0 – i.e. at the point on their world-lines that has $t = t_0$ – their clocks all read $\tau = 0$. We can, if we like, imagine that they have synchronised their clocks by exchanging light signals with their neighbours.

5.3 The warping of time

The gravitational potential Φ appearing in $g_{00} = -(1+2\Phi)$, and $d\tau^2 = -ds^2 = -g_{00}dt^2$ causes the coordinate time t to advance at a different rate to the proper time τ measured by constant-$\mathbf{r}$ observers:

$$d\tau/dt = \sqrt{1 + 2\Phi} \simeq 1 + \Phi \leq 1 \qquad (46)$$

If the observers' clocks were set to $\tau = 0$ on some constant-t hyper-surface then the events when their clocks read $\tau > 0$ will not lie on a constant-t hyper-surface, as illustrated in figure 3. This is an observable physical effect.

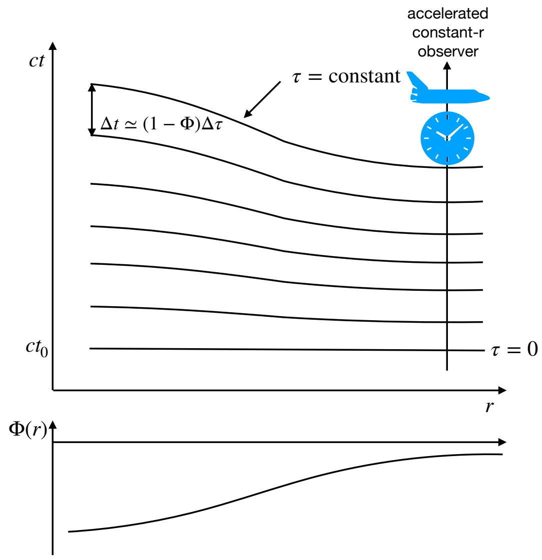


Figure 3: The warping of time: At the bottom is sketched the dimensionless potential $\Phi = \phi/c^2$. Above is a space-time diagram showing surfaces of constant proper time as measured by a set of observers maintaining constant r, and whose clocks were synchronised at coordinate time $t = t_0$. They are accelerated observers, as indicated by the space-shuttle whose rocket is keeping this observer from falling in the potential. These observers' clocks drift steadily out of synch; something they can measure by simply looking at a neighbour's clock. The rate of de-synchronisation they see is in perfect accord with what they would expect if they were in flat space-time with the same acceleration as they perceive.

The deeper the potential, the more advanced will be the coordinate time for a given τ.

This is a 'stretching' of the time coordinate – the longer we wait, the larger becomes the difference between proper and coordinate time. This is the *"gravitational time dilation"* predicted by Einstein for accelerated observers.

While intervals that have $d\tau = 0$ when the clocks read zero are purely space-like in the observers' frames (they 'lie in the rest-frame of the observers'), they do not remain so. At later times pairs of events that have $d\tau = 0$ are not simultaneous in the observer's frame.

5.4 The gravitational redshift

If the potential $\Phi(\mathbf{r})$ is independent of time then world-lines of photons 'falling' into the potential from a stationary observer at large distance will have identical form and will simply be shifted in *coordinate* time as illustrated in figure 4.

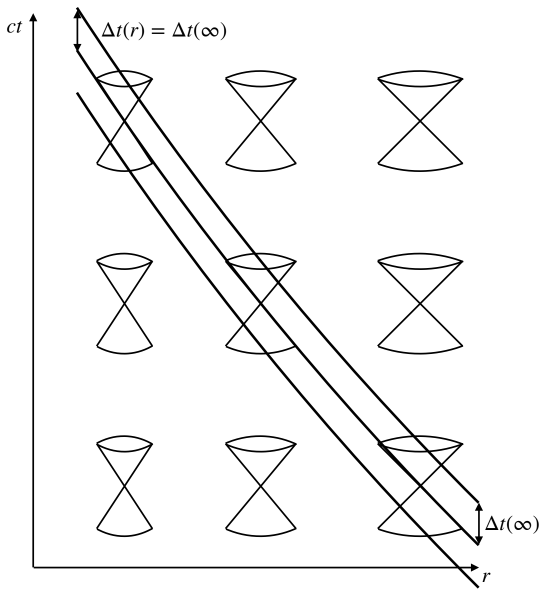


Figure 4: This shows schematically paths of photons or, if you like, pulses of light. They lie locally in the light cones. In a static potential, the paths for successive pulses are identical, being merely shifted in coordinate time. That means that the coordinate time interval between two pulses at the receiver must be the same as at the emitter. But proper time advances at a different rate to coordinate time; the clocks deeper in the potential run slow. This is like the frequency of a quantum mechanical particle; the lower the energy the lower the frequency. It follows that the observers deeper in the potential will observe the pulses from more distant observers to arrive at an increased rate with respect to their proper time. And since the 'pulses' might as well be successive wave-crests of light, that means that they will see the light at a higher frequency; i.e. 'blue-shifted'.

This means that the coordinate time intervals between reception of pulses (or the period of the light) will be the same, regardless of position of the observer.

But because of the warping of time this means that the *proper time* intervals between arrival of light pulses and therefore also the period of the radiation will be different from that in the emitter's frame:

$$\frac{d\tau_{\rm rec}}{d\tau_{\rm em}} = \sqrt{\frac{g_{00}(\mathbf{r}_{\rm rec})}{g_{00}(\mathbf{r}_{\rm em})}} = \sqrt{\frac{1+2\Phi(\mathbf{r}_{\rm rec})}{1+2\Phi(\mathbf{r}_{\rm em})}} \simeq 1 + \Phi(\mathbf{r}_{\rm rec}) - \Phi(\mathbf{r}_{\rm em}). \tag{47}$$

This is a directly observable effect, and therefore shows that the warping of time is a real physical phenomenon, and not simply a coordinate effect.

An observer at constant $\mathbf{r}$ in a potential well will see light from a distant stationary observer blue-shifted, so the measured wave-length will be less than the proper wave-length.

The inverse of this effect was first observed in the light emitted by the white dwarf Sirius-B. It is called the *gravitational redshift* effect. The redshift z is defined as

$$\boxed{1 + z \equiv \frac{d\tau_{\rm rec}}{d\tau_{\rm em}} = \frac{\lambda_{\rm rec}}{\lambda_{\rm em}}.} \tag{48}$$

It is not difficult to intuit what would happen if there is a *time-varying* potential. If we see light that has passed through a static potential somewhere along the light-path we will see the light at the same frequency as emitted. That's because the light is red-shifted on emerging from the potential well by the same amount as it is blue-shifted falling in.

If, on the other hand, the potential well were decreasing with time, the red-shift coming out would not balance the blue-shift falling in, and the result would be a net blue-shift. We would say that the photons had gained energy from a 'gravitational sling-shot' effect. In cosmology this is known as the 'Rees-Sciama' or 'integrated Sachs-Wolfe' (ISW) effect.

We can also attribute this affect to a changing 'optical path length': The same effect would be seen if we observed a source through some material with a time-varying refractive index n. If n is decreasing then so is the number of waves of light within the object. So waves have to emerge from the object at a greater rate than they enter.

5.5 Light deflection from the gravitational redshift

We can use the gravitational redshift formula to (mis)calculate light deflection. As illustrated in figure 5, if we have a beam of light (indicated by the wave-fronts) the wavelengths, as measured by local constant-$\mathbf{r}$ observers will shrink: $\lambda = \lambda_0(1 + \Phi(\mathbf{r}))$ where λ_0 is the proper wavelength as emitted from a source at infinity.

The shrinking is a function of position, being greater for the side of the beam that passes deeper in the potential (closer to the Earth in the figure).

This gives us 'Snell's law' for the deflection of the direction of the light beam:

$$\boxed{d\hat{\mathbf{n}}/d\lambda = -\boldsymbol{\nabla}_\perp \Phi} \tag{49}$$

where λ measures distance along the path.

This gives a deflection in accord with that one would get by applying the equivalence principle (as sketched in the lower left of the figure) to argue that the deflection seen locally by these observers (who must be accelerated to maintain constant-$\mathbf{r}$) would be the same as an identically accelerated observer in empty space would see.

In Newtonian gravity, the deflection of the path of a particle moving with speed v is

$$\boxed{d\hat{\mathbf{n}}/d\lambda = -v^{-2}\boldsymbol{\nabla}_\perp \phi}. \tag{50}$$

So comparison with Snell's law (49) above, together with $\Phi = \phi/c^2$, would lead one to predict that light deflection by the Sun, for example, would be the same as the Newtonian prediction for a particle moving at the speed of light. Whereas the deflection measured by Eddington (and predicted by Einstein) is actually twice that.

This result is also at odds with what we found for the 'coordinate speed of light' in §5.1 above. There we found that, as far as the coordinate speed of propagation is concerned, light in a gravitational potential Φ behaves like light in a refractive medium with refractive index

$$\boxed{n(\mathbf{r}) \simeq 1 - 2\Phi(\mathbf{r})} \tag{51}$$

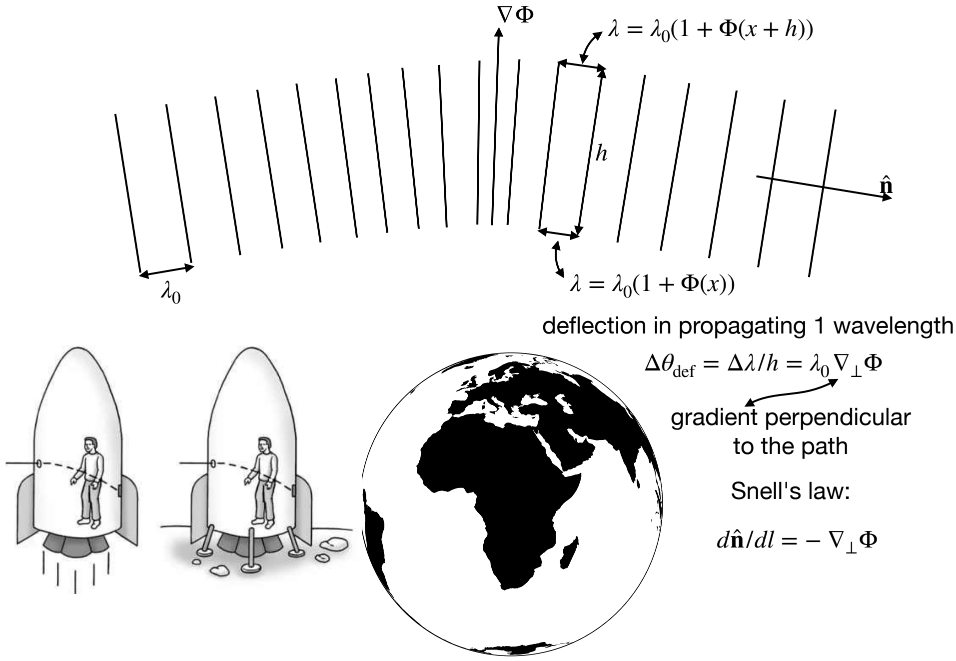


Figure 5: Light deflection from the gravitational redshift. Constant-$\mathbf{r}$ observers near the Earth will see radiation from a distant source blue-shifted. The wavelength will be shrunk in a differential manner, and this allows us to derive 'Snell's law' for the rate of change of direction of a beam with path length. This is in accord with what one would infer using the equivalence principle.

which, in the normal version of Snell's' law $d\hat{\mathbf{n}}/d\lambda = \boldsymbol{\nabla}_\perp n$ also leads to the extra factor 2 observed and predicted from GR.

So what is wrong with the deflection inferred from the gravitational redshift (or from the principle of equivalence)?

5.6 The spatial geometry of $t =$ constant (hyper)surfaces

The reason the above argument gives the wrong answer is that in figure 5 we are implicitly imagining the spatial hypersurfaces of constant coordinate time to be flat. So while the gravitational redshift formula – or the equivalence principle – correctly gives the local deflection one of our accelerated observers would measure, the net bending angle is different because of the warping of space.

In a nutshell, the spatial geometry in a sphere of matter, for instance is positively curved. The geometry of the equatorial plane, for example, is like that of the 2-dimensional surface of a bowl in 3D. This means that there is an additional path length for light paths that go through the centre of the sphere as compared to those that do not probe as deeply.

Think about nearly planar EM waves from a very distant source approaching the sphere. Passing through the potential well of the sphere, the wavelength is reduced as much as illustrated in figure 5. This means that the wavefronts emerging will be distorted, with the parts that passed through the sphere retarded. That means that the light-rays – being normal to the wavefronts – will be converging. But if the space is positively curved inside the sphere there will be an additional retardation. The spatial curvature results in an extra light deflection that is equal to the amount inferred from the gravitational redshift. This is the origin of the famous factor 2 difference between Newtonian and Einsteinian light-bending predictions.

To anlyse this further, consider the line element on the hypersurface $t =$ constant:

$$ds^2 = (1 - 2\Phi(\mathbf{r}))(dx^2 + dy^2 + dz^2) \tag{52}$$

where Φ is the solution of Poisson's equation (divided by c^2).

This, by the way, is an example of what is known as a *'conformal transformation'*, in which have one metric – here the Euclidean line element $ds^2 = dx^2 + dy^2 + dz^2$ in Cartesian coordinates – multiplied by a function of position. The reason for this terminology is that, in this kind of mapping, angles between crossing lines are unchanged.

Let's consider, for simplicity, the spherically symmetric potential created by a spherically symmetric mass distribution: $\Phi(\mathbf{r}) = \Phi(r)$ where $r = |\mathbf{r}|$.

Making a change of variables:

$$\begin{aligned} x &= r \sin\theta \sin\phi \\ y &= r \sin\theta \cos\phi \\ z &= r \cos\theta \end{aligned} \tag{53}$$

in terms of which $dx^2 + dy^2 + dz^2 = dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)$, so the line element becomes

$$ds^2 = (1 - 2\Phi(r))(dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)). \tag{54}$$

Trajectories of photons in this geometry can be taken to lie in the equatorial plane $\theta = \pi/2$, the 2-dimensional surface whose line element is

$$ds^2 = (1 - 2\Phi(r))(dr^2 + r^2 d\phi^2). \tag{55}$$

As this is 2-dimensional, the curvature is described by a single function of radius r, which we might take to be the Ricci scalar.

A simpler, yet highly useful, way to visualise the geometry in spherically symmetric spaces like this is to ask: what is the shape of a circularly symmetric surface in *three* spatial dimensions with vertical displacement from the equatorial plane z that has the same line element? Such a surface is called an *'embedding diagram'*.

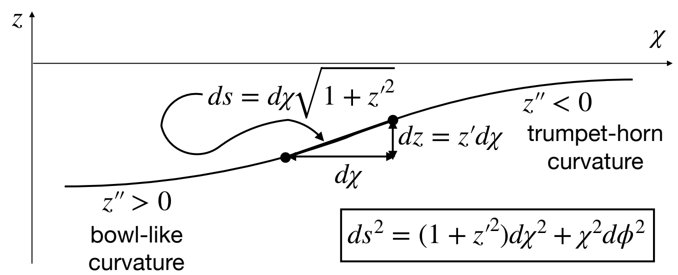


Figure 6: An *'embedding diagram'* is a circularly symmetric surface in 3 spatial dimensions (generated by rotating the line $z = z(\chi)$ shown here about the z-axis) which has the same intrinsic geometry as the equatorial plane ($\theta = \pi/2$) in a spherically symmetric curved 3-space like (54). The line element is given by the boxed formula at the bottom. This matches that for the equatorial plane (55) if $z' = \sqrt{2r\Phi'}$.

To make an embedding diagram, we define a new radial coordinate χ such that $g_{\phi\phi} = \chi^2$, which here is $\chi = (1 - \Phi)r$ (to 1st order in $\Phi \ll 1$). So χ is an 'angular diameter distance'; it is defined such that on object of proper size dl subtends, at the origin, an angle $d\phi = dl/\chi$.

As shown in figure 6, the length of a radial line segment lying in this surface is $ds = d\chi\sqrt{1 + z'^2}$, where $z' \equiv dz/d\chi$ (but can be considered to be dz/dr since z is a first order quantity and, to zeroth order, r and χ are the same).

The total line element in this surface for a displacement in both χ and azimuthal angle is obtained by adding, in quadrature (as they are perpendicular displacements) this ds and a tangential $ds = \chi d\phi$:

$$ds^2 = (1 + z'^2)d\chi^2 + \chi^2 d\phi^2 \tag{56}$$

But $\chi = (1 - \Phi)r$ implies $d\chi = (1 - \Phi - r\Phi')dr$, where Φ is a first order quantity, so we can think of its derivative Φ' as being with respect to either r or χ, or $d\chi^2 = (1 - 2\Phi - 2r\Phi')dr^2$ so this is

$$ds^2 = (1 + z'^2 - 2\Phi - 2r\Phi')dr^2 + (1 - 2\Phi)r^2 d\phi^2. \tag{57}$$

which is the same as (55) if

$$\boxed{z' = dz/dr = \sqrt{2r\Phi'}} \tag{58}$$

where we have taken the positive root as $\Phi' > 0$ and we have taken the sign of z' to be positive since we want z to be negative as in figure 6. This is the equation that the height z of the surface must satisfy in order to have the same intrinsic geometry as the equatorial plane in the 3-space (54).

Of particular interest is the product of first and second derivatives of z, which tells us whether the curvature is positive (like a bowl) or negative (like a trumpet horn). This is

$$\boxed{z'z'' = r\Phi'' + \Phi'.} \tag{59}$$

Alternatively, if we calculate the Christoffel symbols for (56) and from these calculate the curvature tensor we get, for $R^r{}_{\phi r \phi}$ for example:

$$R^r{}_{\phi r \phi} = -3rz'z'' = -3r(r\Phi'' + \Phi'). \tag{60}$$

5.7 Uniform density sphere

Consider a non-expanding sphere of uniform density ρ and radius R.

For $r > R$, the potential that has the sensible boundary condition $\Phi \to 0$ as $r \to \infty$ is $\Phi(r) = -GM/rc^2 = -(4\pi/3)G\rho R^3/rc^2$.

For $r < R$, the potential gradient is $d\Phi/dr = (4\pi/3)G\rho r/c^2$ which we can integrate to get $\Phi = (2\pi/3)G\rho r^2/c^2$+constant. Matching smoothly to the solution for $r > R$ gives the constant to be $-2\pi G\rho R^2/c^2$. So

$$\Phi(r) = \tfrac{2\pi}{3} G\rho R^2/c^2 \begin{cases} r^2/R^2 - 3 & \text{for} \quad r < R \\ -2R/r & \text{for} \quad r > R \end{cases} \tag{61}$$

which is illustrated in figure 7.

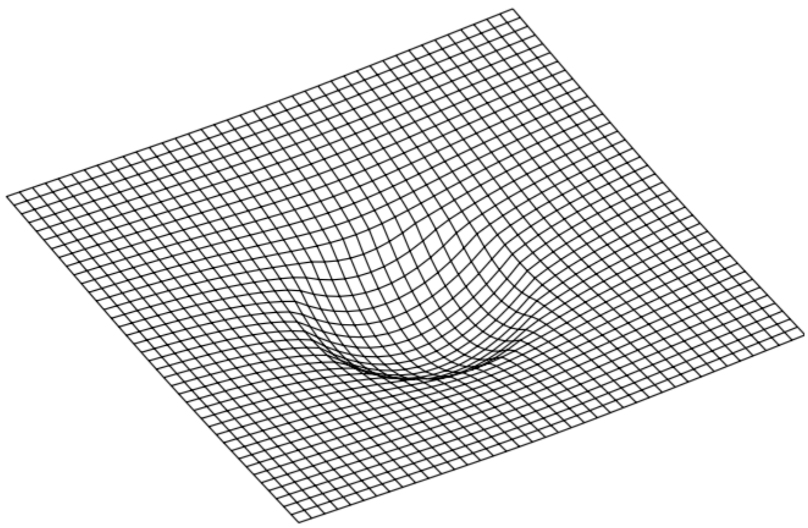


Figure 7: Gravitational potential on the equatorial plane for a uniform density sphere plotted as the depth of a surface. The potential is parabolic within the sphere – and the potential 'surface' has positive (bowl-like) curvature – and falls off as $1/r$ outside the sphere, so the potential surface has negative curvature like a trumpet-horn. The embedding diagram for this – that is to say the circularly symmetric surface in 3D with height $z(r)$ that has the same metric as the equatorial plane metric $ds^2 = (1 - 2\Phi)(dr^2 + r^2 d\phi^2)$ looks qualitatively somewhat similar.

The potential gradient $\Phi' \equiv d\Phi/dr$ is

$$\Phi'(r) = \tfrac{4\pi}{3} G\rho R/c^2 \begin{cases} r/R & \text{for} \quad r < R \\ R^2/r^2 & \text{for} \quad r > R \end{cases} \tag{62}$$

which, as it should be, positive everywhere, and is continuous at the edge of the sphere, while the curvature parameter

$$r\Phi'' + \Phi'(r) = \tfrac{4\pi}{3} G\rho R/c^2 \begin{cases} 2r/R & \text{for} \quad r < R \\ -R^2/r^2 & \text{for} \quad r > R \end{cases} \tag{63}$$

is discontinuous at $r = R$ and evidently we have positively curved, bowl-like, geometry inside the sphere, with $z'z'' \propto r$, and negatively curved, trumpet horn-like (or locally saddle-like), geometry outside with the curvature parameter falling off as $-z'z'' \propto 1/r^2$.

The component of the Riemann tensor $R^r{}_{\phi r \phi} = -3rz'z''$ scales in proportion to r^2 within the sphere and as $1/r$ outside. This component is what appears in the geodesic deviation equation for the second rate of change of radial separation for a pair of spatial geodesics with instantaneously tangential path $\mathbf{U} \to (U^r, U^\theta, U^\phi) = (0, 0, d\phi/d\lambda)$.

$$\frac{d^2 \Delta r}{d\lambda^2} = -R^r{}_{\phi r \phi} \Delta r \frac{d\phi}{d\lambda} \frac{d\phi}{d\lambda} \tag{64}$$

for such trajectories $d\phi/d\lambda = 1/r$, so we have positive focussing $\ddot{\Delta r} \propto -\Delta r$ at a rate which is independent of location within the sphere, and de-focussing outside with $\ddot{\Delta r} \propto \Delta r/r^3$ as one would expect for a tidal field.

5.8 Embedding diagrams for other spherical and cylindrical models

For a spherical structure with a power law density profile $\rho \propto r^{-\gamma}$, the mass enclosed within r is $M = 4\pi \int dr r^2 \rho \propto r^{3-\gamma}$, and the potential gradient is $\Phi' \propto M/r^2 \propto r^{1-\gamma}$. With $\Phi' = \alpha r^{1-\gamma}$ (with α a constant), $\Phi'' = (1-\gamma)\alpha r^{-\gamma}$ so $r\Phi'' + \Phi' = (2-\gamma)\alpha r^{1-\gamma}$. So we have positive (bowl-like) curvature if $\gamma < 2$ and negative (trumpet-horn-like) curvature if $\gamma > 2$.

The dark matter structures that form in numerical simulations of cosmological structure formation are found to be well described by so called 'NFW profiles' (after Navarro, Frenk and White). These have $\gamma = 1$ in the central parts and $\gamma = 3$ at large radii. The transition between these extremes is rather extended, and so they behave, to a crude approximation as 'locally power-law like' and the spatial curvature transitions from positive to negative as one moves outwards. Over a fairly wide range of radii straddling the 'virial radius' (the radius where the interior density is about 200 times the mean cosmological density) the slope is $\gamma \simeq 2$. This radius delineates the transition from the equilibrated interior – which has come to 'virial equilibrium' – and the outskirts, where matter is falling in for the first time.

Consistent with this, a much used earlier model for e.g. clusters of galaxies is the so-called 'isothermal sphere' model, which has $\rho \propto r^{-2}$ and galaxies – which have dark matter halos with roughly flat 'rotation curves' – also have $\rho \propto r^{-2}$.

These $\rho \propto 1/r^2$ models are therefore of considerable practical interest. They are particularly interesting here because they have no curvature. Does that mean that curvature plays no role in light deflection for such structures?

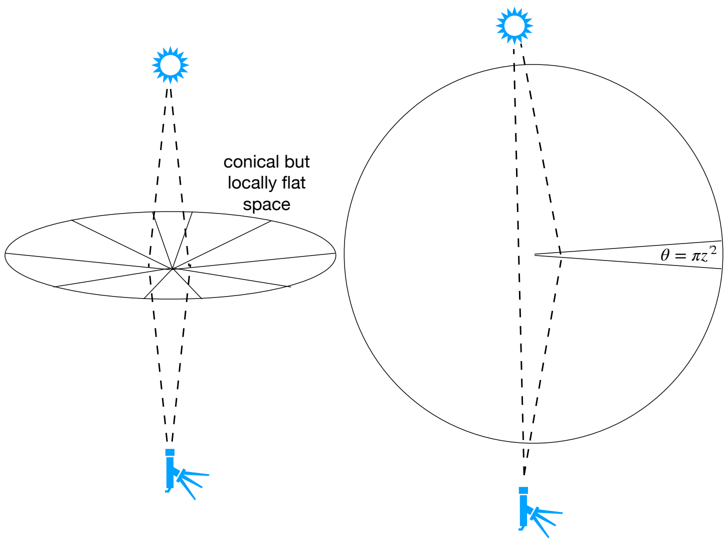


Figure 8: Left side shows an embedding diagram for an 'isothermal-sphere' (or 'flat-rotation curve' halo). The spatial geometry of the equatorial plane is locally flat, but conical. Even without the gravitational redshift effect (which tends to focus light as wavelengths shrink in the potential well) an observer looking at distant sources through such an object would see a deflection and could see multiple images of a background source. The right hand figure shows the geometry flattened out into a plane with the missing 'wedge' of angle on the order of $\theta \sim v^2/c^2$. More precisely, the effect of the geometry is to double the deflection as compared to that predicted from time-dilation alone.

The answer is yes and no. For $\gamma = 2$, $z' = \sqrt{2r\Phi'} =$ constant. That tells us that $z \propto r$, so the embedding diagram is *conical*. So the space is locally flat, but viewed globally there is an *'angular deficit'*. With $z' =$ constant, the metric (56) can be recast – by means of the transformation $r = \sqrt{1+z'^2}\chi$ – to a locally flat metric

$$ds^2 = dr^2 + r^2 d\phi'^2 \tag{65}$$

but where $\phi' = \phi/\sqrt{1+z'^2}$. So there is no local curvature, but, rather than ranging from 0 to 2π, ϕ' ranges from 0 to $2\pi/\sqrt{1+z'^2} \simeq 2\pi - \pi z'^2$ so the geometry is like that of a sheet of paper where we have excised a wedge of angle $\theta = \pi z'^2$ and then glued the edges together to make a cone, as illustrated in figure 8.

The potential gradient for $\rho \propto 1/r^2$ goes like $1/r$ which, integrated, gives a potential $\Phi(r) \sim \log r$. The same logarithmic potential arises for a 'cosmic-string', which behaves like a line of constant linear mass density. Gauss's law tells us the gradient of the potential Φ' integrated over the surface of a cylinder of radius r is proportional to the mass enclosed, so again $\Phi \propto 1/r$ and the potential is logarithmic in r and the spatial geometry of a slice perpendicular to the string is again conical.

5.9 The spatial geometry of $t \neq$ constant hypersurfaces

A word of caution is in order: the spatial curvature depends, in general, on the choice of hyper-surface.

In the appendix (§A) we calculate the curvature in a sphere by comparing the integrated proper radius, and also the area, of the region enclosed within a given circumference. These measures of curvature also depend on the choice of hyper-surface.

Here we have considered the hyper-surfaces of constant coordinate time t. These are orthogonal to the world-lines of the particular family of observers we have chosen, their essential characteristic being that they maintain constant separation from one another.

Had we chosen a *different* family of observers, for example a family of observers that are expanding away from one another, and measured the radius on a hyper-surface that is orthogonal to *their* world-lines we would get a smaller result. That is because, in their frame of reference, the radial distances are length contracted. So, while they would agree on the proper size of the circumference, they would disagree with our observers as to the radius.

This, as we saw previously, is precisely what happens in cosmology where, in the homogeneous FLRW model there is a positive density of matter, so non-expanding observers in some region of space would measure space to be positively curved, but, as measured by so-called 'co-moving' observers, the geometry of hypersurfaces orthogonal to their world-lines may be flat or negatively curved.

However, it is impossible to 'flatten-out' the warping of space and time simultaneously, unless there is no matter present.

Q: flesh out the above. For observers within a constant density sphere of matter calculate the expansion velocity field needed to flatten the spatial geometry. Show how the expansion rate is related to the density.

6 Particle motion in weak field gravity

Physical particles move on geodesics of this curved space-time. We would expect this to conform to the Newtonian behaviour for slowly moving particles. We will show that this is indeed the case, and we will find that the motion of such particles is entirely determined by the time-time part of the metric. We will also develop the geodesic equation for massless particles like photons, and show that these are, in addition, sensitive to the curvature of space, and this explains the famous factor 2 enhancement over the Newtonian prediction. We then look at this from a wave-mechanical perspective.

6.1 Equation of motion for non-relativistic particles

Massive particles parallel transport their 4-velocity $\vec{U} = d\vec{x}/d\tau$ so $0 = \nabla_{\vec{U}}\vec{U} = U^\beta U^\alpha{}_{;\beta} = U^\beta U^\alpha{}_{,\beta} + \Gamma^\alpha{}_{\mu\nu}U^\mu U^\nu$ so $U^\beta U^\alpha{}_{,\beta} = dU^\alpha/d\tau = -\Gamma^\alpha{}_{\mu\nu}U^\mu U^\nu$ or,

$$\boxed{\frac{d^2x^\alpha}{d\tau^2} = -\Gamma^\alpha{}_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}.} \tag{66}$$

As discussed previously, a nice way to show this is to show that these are equivalent to the Euler-Lagrange equations obtained by extremising $S = \int d\lambda L(x^\alpha, \dot{x}^\alpha) = \int d\tau \sqrt{-g_{\alpha\beta}(\vec{x})\dot{x}^\alpha\dot{x}^\beta}$ with $\dot{x}^\alpha = dx^\alpha/d\tau$.

The Christoffel symbols are of first order in the metric perturbations, so we can take the zeroth order approximation to the 4-velocity on the RHS which, for a non-relativistic particle, is $\vec{U} \to U^\mu \simeq (c, \mathbf{0}) = c\delta^\mu_0$ and so

$$dU^\alpha/d\tau = -c^2\Gamma^\alpha{}_{00}. \tag{67}$$

The Christoffel symbols are, to first order in the metric perturbations,

$$\Gamma^\alpha{}_{\mu\nu} = \tfrac{1}{2}\eta^{\alpha\gamma}(h_{\gamma\mu,\nu} + h_{\gamma\nu,\mu} - h_{\mu\nu,\gamma}) \tag{68}$$

from which, using the Newtonian limit metric $h_{\mu\nu} = -2\Phi\delta_{\mu\nu}$, we obtain[5]

$$\begin{aligned} \Gamma^0{}_{00} &= -\tfrac{1}{2}h_{00,0} = \Phi_{,0} \\ \Gamma^i{}_{00} &= -\tfrac{1}{2}h_{00,i} = \Phi_{,i} \end{aligned} \tag{69}$$

so only the time-time component of the metric perturbation plays any role here.

The latter, with $\Phi = \phi/c^2$, gives the equation of motion

$$\boxed{d^2x^i/d\tau^2 = -\phi_{,i}} \tag{70}$$

[5] We write (69) as two separate equations to avoid writing the, formally illegitimate, equation $\Gamma^\alpha{}_{00} = \Phi_{,\alpha}$.

or, if you prefer,

$$\boxed{d^2\mathbf{r}/d\tau^2 = -\boldsymbol{\nabla}\phi} \tag{71}$$

just as in the Newtonian theory.

This shows that freely falling particles do not remain at fixed $\mathbf{r}$; observers maintaining constant $\mathbf{r}$ (as considered above and illustrated in figure 2) must be accelerated.

This is what allows us to connect the parameter κ to Newton's constant of gravitation: $\kappa = G_\mathrm{N}/c^4$.

At linear order, we may replace proper time by coordinate time here. This means that, if we are interested in the deflection of a particle moving with speed v, we can simply project out the components perpendicular to the instanteneous path, and we have, for the 2nd derivative of displacement transverse to the path with respect to path length λ,

$$\boxed{\frac{d^2\mathbf{r}_\perp}{d\lambda^2} = -\frac{1}{v^2}\boldsymbol{\nabla}_\perp\phi} \tag{72}$$

since $d/dt = v d/d\lambda$ and therefore[6] $d^2/dt^2 = v^2 d^2/d\lambda^2$.

6.2 Energy and Hamiltonian of non-relativistic particles

The time component of the non-relativistic geodesic equation (67), together with (69), says $dU^0/d\tau = -c^2\Gamma^0{}_{00} = -c^2\Phi_{,0} = -\phi_{,0} = -c^{-1}\partial\phi/\partial t$. In special relativity, the energy is $E = mcU^0 = \gamma mc^2$. So the geodesic equation would seem to say $dE/d\tau = -m\partial\phi/\partial t$. That is dimensionally correct and looks like the equation for the rate of change of the Hamiltonian, but it doesn't have the right sign. What's wrong with this picture?

6.2.1 Newtonian dynamics

Let's first briefly review the classical mechanics of a non-relativistic particle in a Newtonian gravitational potential.

– The starting point is the *Lagrangian:* the kinetic energy minus potential energy, or

$$\boxed{L(\mathbf{r}, \dot{\mathbf{r}}, t) = m|\dot{\mathbf{r}}|^2/2 - m\phi(\mathbf{r}, t).} \tag{73}$$

– The *action* (a functional of the particle path) is

$$S = \int dt L. \tag{74}$$

– The *momentum* is

$$\mathbf{p} \equiv \partial L/\partial\dot{\mathbf{r}} = m\dot{\mathbf{r}}. \tag{75}$$

– The *Euler-Lagrange equation* obtained by requiring the *action* S to be extremised is

$$d\mathbf{p}/dt = \partial L/\partial\mathbf{r} = -m\boldsymbol{\nabla}\phi. \tag{76}$$

– The *Hamiltonian* is

$$H(\mathbf{r}, \mathbf{p}, t) \equiv \dot{\mathbf{r}}\cdot\mathbf{p} - L(\mathbf{r}, \dot{\mathbf{r}}, t) \tag{77}$$

or

$$\boxed{H(\mathbf{r}, \mathbf{p}, t) = |\mathbf{p}|^2/2m + m\phi(\mathbf{r}, t)} \tag{78}$$

i.e. kinetic energy plus potential energy.
– *Hamiton's equations* are

$$\dot{\mathbf{p}} = -\partial H/\partial\mathbf{r} \qquad \text{and} \qquad \dot{\mathbf{r}} = \partial H/\partial\mathbf{p} \tag{79}$$

and do not, in themselves, give us anything new, but in $dH(\mathbf{r}, \mathbf{p}, t)/dt = \dot{\mathbf{r}}\partial H/\partial\mathbf{r} + \dot{\mathbf{p}}\partial H/\partial\mathbf{p} + \partial H/\partial t$, they tell us that the rate of change of the Hamiltonian obeys $dH/dt = \partial H/\partial t$ or

$$\boxed{\frac{d}{dt}(|\mathbf{p}|^2/2m + m\phi) = m\frac{\partial\phi}{\partial t}} \tag{80}$$

[6] You might think we should have the time derivative of v showing up in d^2/dt^2, which is correct. But here we are applying it to a 1st order quantity, so we can take v to be constant at lowest order.

which we notice has the opposite sign to what the geodesic equation is giving for $d(mcU^0)/dt$.
– Finally, with $d\phi/dt = \partial\phi/\partial t + \dot{\mathbf{r}} \cdot \boldsymbol{\nabla}\phi$, this gives

$$\boxed{d(|\mathbf{p}|^2/2m)/dt = -m\dot{\mathbf{r}} \cdot \boldsymbol{\nabla}\phi} \tag{81}$$

which says the rate of change of the kinetic energy is the rate at which the gravitational force $-m\boldsymbol{\nabla}\phi$ is doing work on the particle.

6.2.2 Correspondence between Newtonian dynamics and weak-field theory

We now establish the correspondence between the foregoing and weak-field gravity theory.
– The relativistic energy-momentum relation is

$$\boxed{g_{\alpha\beta}p^\alpha p^\beta = -m^2c^2} \tag{82}$$

or, here, with $g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta} = \eta_{\alpha\beta} - 2\Phi\delta_{\alpha\beta}$,

$$(1 + 2\Phi)(p^0)^2 = m^2c^2 + (1 - 2\Phi)|\mathbf{p}|^2. \tag{83}$$

– But $|\mathbf{p}|^2/m$ is generally of the same order as $m\phi = mc^2\Phi$ and is therefore also a 1st order quantity, so we can drop the Φ on the right hand side to obtain $(1+2\Phi)(p^0)^2 = m^2c^2+|\mathbf{p}|^2$ so $p^0 = mc\sqrt{(1 + |\mathbf{p}|^2/m^2c^2)/(1 + 2\Phi)}$ or, Taylor expanding and keeping only terms up to 1st order,

$$\boxed{cp^0 = mc^2 + |\mathbf{p}|^2/2m - m\phi} \tag{84}$$

– This is evidently *not* the same as the Hamiltonian. While it has units of energy, the potential enters with the wrong sign. In fact, it is the rest-mass energy mc^2 plus the Newtonian *Lagrangian*[7].
– But lowering the index on p^0 using $p_0 = g_{0\alpha}p^\alpha = g_{00}p^0 = -(1 + 2\Phi)p^0$ gives

$$\boxed{-cp_0 = mc^2 + |\mathbf{p}|^2/2m + m\phi} \tag{85}$$

which *is* the Newtonian Hamiltonian (plus the rest-mass energy)
– So what we would normally think of as the total energy (i.e. the Hamiltonian) should not be associated with the time component of the momentum 4-vector $\vec{p}$, rather it is ($-c$ times) the time component of the momentum 1-form $\tilde{p}$.
– Which fits with the latter, which of course, being a 1-form, has to be the derivative of *something*, being the derivative of the action $S(\vec{x})$ (defined *à la* Hamilton and Jacobi as being the action for a family of particles that started at the same point in space-time with a range of momenta and for which $\mathbf{p} = \boldsymbol{\nabla}S$ and $H = -\partial S/\partial t$):

$$\tilde{p} = \tilde{d}S \tag{86}$$

where

$$\tilde{d}S \to \partial_\alpha S = (c^{-1}\partial_t S, \boldsymbol{\nabla}S) = (-H/c, \mathbf{p}). \tag{87}$$

That all comes from the normalisation of the 4-momentum. To get the rate of change of the energy we can use the covariant form of the geodesic equation:

$$dU_\alpha/d\tau = \tfrac{1}{2}g_{\nu\beta,\alpha}U^\beta U^\nu \tag{88}$$

which tells us that, should the metric be independent of the α^{th} space-time coordinate, the corresponding covariant component of the 4-momentum $p_\alpha = mU_\alpha$ is a constant along the particle trajectory.

So if ϕ, and therefore also the metric, is independent of time, $dp_0/d\tau = 0$ and so $mc^2 + |\mathbf{p}|^2/2m + m\phi$ (or rest mass energy plus Newtonian kinetic plus potential energies) is constant along the particle trajectory.

[7] That there should be a very close connection between p^0 and L is reasonable. In Newtonian mechanics, particle paths are those which extremise $\int L dt$, while geodesics are paths that extremise $\int d\tau = \int (dt/d\tau)^{-1}dt = \int c(dx^0/d\tau)^{-1}dt = \int (mc/p^0)dt$. But this says that, up to an additive constant and some constant multiplicative factor, it is the *inverse* of p^0 that must be equivalent to L. But here $p^0 \simeq mc(1 + (|\mathbf{p}|^2/2m - m\phi)/mc^2)$ so $(mc/p^0) \simeq (1 - (|\mathbf{p}|^2/2m - m\phi)/mc^2) = 1 - L/mc^2$.

On the other hand, if the potential *is* changing with time, the geodesic equation, with $\vec{U} \to (c, \mathbf{0})$ in the factors on the right hand side, as appropriate for a non-relativistic particle, implies

$$\frac{d}{d\tau}(-cp_0) = \frac{d}{d\tau}(|\mathbf{p}|^2/2m + m\phi) = -\tfrac{1}{2}mc^3 g_{00,0} = m\frac{\partial \phi}{\partial t}. \tag{89}$$

As we are working to first order precision we can replace $d/d\tau$ with d/dt, so this is saying, reassuringly, $dH/dt = \partial H/\partial t$ with Hamiltonian $H(\mathbf{p}, \mathbf{x}, t) = |\mathbf{p}|^2/2m + m\phi(\mathbf{x}, t)$, and using the convective derivative $d\phi/dt = \partial\phi/\partial t + \dot{\mathbf{r}} \cdot \boldsymbol{\nabla}\phi$ as above, this in turn implies[8]

$$d(|\mathbf{p}|^2/2m)/dt = -m\dot{\mathbf{r}} \cdot \boldsymbol{\nabla}\phi \tag{90}$$

in accord with the Newtonian result (81) that the change of kinetic energy is equal to the work done by the gravitational force.

Note that while the potential energy $m\phi$ appears in both p_0 and p^0 (though with opposite sign), neither of these is what one of our constant-$\mathbf{r}$ observers would measure. Since these observers have $U^\alpha = (dt/d\tau, 0, 0, 0)$ and $U^0 = dt/d\tau = (1 - \Phi)$, the energy they measure is, to first order,

$$\boxed{E_{\rm obs} = -\tilde{p}(\vec{U}) = -U^0 p_0 = mc^2 + |\mathbf{p}|^2/2m} \tag{91}$$

which, as expected, is just the kinetic energy (plus rest-mass energy).

6.3 Relativistic particle dynamics

In the previous section we considered particle dynamics in the non-relativistic limit. Here we relax that restriction and consider motion of particles of arbitrary momentum. As above, we will work very much in '3+1' formalism. We will first develop the formalism for an arbitrary metric and then specialise to weak-field geometry.

6.3.1 Classical mechanics of relativistic particles

The starting point here is the differential action dS for a relativistic massive particle:

$$\boxed{dS = -mc^2 d\tau} \tag{92}$$

where m is the rest mass.

This form of the action guarantees that world-lines that extremise the action are paths of extremal proper time; i.e. geodesics. To justify the factor $-mc^2$, note that in flat space-time, $d\tau = dt/\gamma$, so then

$$dS = -\gamma^{-1} mc^2 dt \simeq dt(-mc^2 + \tfrac{1}{2}mv^2 + \ldots), \tag{93}$$

which says that the Lagrangian $L = dS/dt$ for a free particle is, in the non-relativistic limit $v \ll c$, and aside from the constant $-mc^2$, just equal to the kinetic energy, as one would expect. Another nice thing about the action S defined through (92) is that it is evidently coordinate – and therefore gauge – independent[9]; it is a Lorentz scalar.

In general coordinates, $d\tau = -ds/c$ with $ds^2 = g_{\alpha\beta}dx^\alpha dx^\beta$ so the differential action is

$$dS = -mc\sqrt{-g_{\alpha\beta}\dot{x}^\alpha \dot{x}^\beta}\, dt \tag{94}$$

where $\dot{x}^\alpha \equiv dx^\alpha/dt = (c, \dot{\mathbf{x}})$, or $dS = \int dt L$ where the Lagrangian is

$$\boxed{L(\mathbf{x}, \dot{\mathbf{x}}, t) = -mc\sqrt{-g_{\alpha\beta}(\vec{x})\dot{x}^\alpha \dot{x}^\beta}} \tag{95}$$

[8] In the foregoing, we worked only to first order precision, keeping terms like $m\phi$ and $|\mathbf{p}|^2/2m$ but dropping terms involving their products. We see in (90) that the rate of change of kinetic energy is of higher than 1st order. But there is no inconsistency here. The smallness of $d(|\mathbf{p}|^2/2m)/dt$ is simply because, in the Newtonian limit, things move slowly. For a particle moving in a potential well of size $\sim L$ that is changing on the 'dynamical' (i.e. orbital) time-scale, the change in the kinetic energy of the particle in one dynamical time $\Delta t \sim L/|\dot{\mathbf{r}}|$ is $\Delta|\mathbf{p}|^2/2m \sim m\Delta t|\mathbf{r} \cdot \boldsymbol{\nabla}\phi| \sim mL|\boldsymbol{\nabla}\phi| \sim m\phi$ and is a 1st order quantity.

[9] It is interesting to contrast this with the action for a charged particle in electromagnetism, for which $dS = (-mc^2 - qU^\alpha A_\alpha)d\tau$ which is *not* gauge invariant.

and the space and time dependence derives from the metric.

The canonical momentum is

$$p_i \equiv \frac{\partial L}{\partial \dot{x}^i} = mc \frac{g_{\alpha i}\dot{x}^\alpha}{\sqrt{-g_{\alpha\beta}\dot{x}^\alpha \dot{x}^\beta}} \tag{96}$$

and the Euler-Lagrange equation is

$$\frac{dp_i}{dt} = \frac{\partial L}{\partial x^i} = \tfrac{1}{2} mc \frac{g_{\alpha\beta,i}\dot{x}^\alpha \dot{x}^\beta}{\sqrt{-g_{\alpha\beta}\dot{x}^\alpha \dot{x}^\beta}} \tag{97}$$

or, dividing by $\dot{\tau} = \sqrt{-g_{\alpha\beta}\dot{x}^\alpha \dot{x}^\beta}/c$ and using $\dot{x}^\alpha/\dot{\tau} = dx^\alpha/d\tau = U^\alpha$,

$$\frac{dp_i}{d\tau} = \tfrac{1}{2} m g_{\alpha\beta,i} U^\alpha U^\beta \tag{98}$$

where the right hand side is m times the right hand side of the covariant form of the geodesic equation $dU_i/d\tau = \frac{1}{2} g_{\alpha\beta,i} U^\alpha U^\beta$, leading us to identify the canonical momentum p_i and mU_i:

$$\boxed{p_i \equiv \partial L/\partial \dot{x}_i = mU_i.} \tag{99}$$

The Hamiltonian – or 'canonical energy' – is defined by $H(\mathbf{x}, \mathbf{p}, t) \equiv \dot{\mathbf{x}} \cdot \mathbf{p} - L$ and is

$$H(\mathbf{x}, \mathbf{p}, t) = mc \left[\frac{g_{\alpha i}\dot{x}^\alpha \dot{x}^i}{\sqrt{-g_{\alpha\beta}\dot{x}^\alpha \dot{x}^\beta}} + \sqrt{-g_{\alpha\beta}\dot{x}^\alpha \dot{x}^\beta} \right] \tag{100}$$

this obeys

$$\frac{dH}{dt} = -\frac{\partial L}{\partial t} = -\tfrac{1}{2} mc \frac{g_{\alpha\beta,t}\dot{x}^\alpha \dot{x}^\beta}{\sqrt{-g_{\alpha\beta}\dot{x}^\alpha \dot{x}^\beta}} \tag{101}$$

so, on dividing by $\dot{\tau} = \sqrt{-g_{\alpha\beta}\dot{x}^\alpha \dot{x}^\beta}/c$, and with $g_{\alpha\beta,t} = c g_{\alpha\beta,0}$, we have

$$\frac{dH}{d\tau} = -\tfrac{1}{2} mc g_{\alpha\beta,0} U^\alpha U^\beta \tag{102}$$

where the right hand side is $-mc$ times the right hand side of the covariant form of the geodesic equation $dU_0/d\tau = \frac{1}{2} g_{\alpha\beta,0} U^\alpha U^\beta$, leading us to identify $-H/c$ and mU_0.

$$\boxed{H \equiv \dot{\mathbf{x}} \cdot \mathbf{p} - L = -mcU_0.} \tag{103}$$

So the canonical $-H/c$ and $\mathbf{p}$ are the time and space parts of the 1-form $\tilde{p} = m\tilde{U}$. And 1-forms also emerge as gradients of Lorentz-scalar fields. So what Lorentz-scalar function of $\vec{x}$ might $(-H/c, \mathbf{p})$ be the gradient of? The only Lorentz-scalar we have here is the action S. But, like $-H/c$ and $\mathbf{p}$, it is only defined along the world-line; it is not actually a field. We can, however, *make* a field out of the action if we consider a bundle of particle trajectories emanating from a common starting point. The action then becomes a function of space and time $S(\vec{x})$, and, as shown by Hamilton and Jacobi, it satisfies $H = -\partial S/\partial t$ and $\mathbf{p} = \boldsymbol{\nabla} S$, so

$$\boxed{\tilde{p} = \tilde{d}S \rightarrow \partial_\mu S = (-H/c, \mathbf{p}).} \tag{104}$$

The Hamilton is a function of position, time and 3-momentum, but is given above in terms of 3-velocity. We can obtain an explicit expression for the Hamiltonian as a function of $\mathbf{x}$, t and $\mathbf{p}$ from the normalisation condition $\tilde{U} \cdot \tilde{U} = -c^2$ which implies $\tilde{p} \cdot \tilde{p} = g^{\alpha\beta} p_\alpha p_\beta = -m^2 c^2$ or the quadratic equation for p_0

$$g^{00} p_0^2 + 2 g^{0i} p_i p_0 + g^{ij} p_i p_j = -m^2 c^2 \tag{105}$$

which we can solve to give

$$H(x_i, p_i, t)/c = -p_0 = \pm \sqrt{\frac{m^2 c^2 + g^{ij} p_i p_j}{-g^{00}} + \left[\left(\frac{g^{0i}}{-g^{00}} \right) p_i \right]^2} - \frac{g^{0i}}{-g^{00}} p_i. \tag{106}$$

This can be differentiated to obtain Hamilton's equations for $\dot{\mathbf{p}}$ and $\dot{\mathbf{x}}$, but it is easier to use the geodesic equation, which gives $d\mathbf{p}/d\tau = (dt/d\tau)\dot{\mathbf{p}} = c^{-1}U^0\dot{\mathbf{p}}$ and $d\mathbf{x}/dt = \mathbf{U}/U^0$ and obtain U^0 as a function of momentum using $U^0 = g^{0\alpha}U_\alpha = -g^{00}H/mc + g^{0i}p_i/m$

$$\begin{aligned} \dot{p}_k &= -\frac{\partial H}{\partial x_k} = -\tfrac{1}{2}\gamma mc^2\left(g^{00}{}_{,k} + 2\frac{p_i}{mc}g^{0i}{}_{,k} + \frac{p_ip_j}{m^2c^2}g^{ij}{}_{,k}\right) \\ \dot{x}_k &= \frac{\partial H}{\partial p_k} = \frac{1}{\gamma}(g^{ik}p_i/m - g^{0k}H/mc) \end{aligned} \tag{107}$$

where

$$\gamma(x_i, p_i, t) \equiv (-g^{00}H/mc^2 + g^{0i}p_i/mc). \tag{108}$$

These can be integrated, given some initial position and momentum, to give the trajectory $\mathbf{x}(t)$ and $\mathbf{p}(t)$.

6.3.2 Classical mechanics of relativistic particles in weak-fields

Writing $g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta}$ and defining γ by $\gamma^{-2} = -\eta_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta/c^2$ so $\gamma(\dot{\mathbf{x}}) = 1/\sqrt{1 - |\dot{\mathbf{x}}|^2/c^2}$ (and which, to zeroth order, is $\gamma = 1/\dot{\tau}$ the differential action is

$$dS = -\gamma^{-1}c^2 dt\, m\left(1 - \tfrac{1}{2}h_{\alpha\beta}U^\alpha U^\beta \big/ c^2\right) \tag{109}$$

where we have expanded the square root, keeping only terms up to 1st order in the metric perturbation.

So the action is like that in flat space-time, but with a position (and generally also velocity) dependent 'effective mass'.

$$m_{\rm eff}(\vec{x}, \dot{\vec{x}}) = \left(1 - \tfrac{1}{2}h_{\alpha\beta}(\vec{x})U^\alpha U^\beta \big/ c^2\right)m \tag{110}$$

and, for non-relativistic particles, for which $\dot{x}^\alpha \simeq (c, \mathbf{0})$ and $\gamma \simeq 1$, this is just a function of position

$$m_{\rm eff}(\vec{x}) = \left(1 - \tfrac{1}{2}h_{00}(\vec{x})\right)m \tag{111}$$

and with the Newtonian limit metric, where $h_{00} = -2\Phi$, this is

$$m_{\rm eff} = (1 + \Phi(\vec{x}))m. \tag{112}$$

We will see later that the same is true for a massive scalar field in the appropriate limit.

The Lagrangian is $L = dS/dt$ or

$$L(\mathbf{x}, \dot{\mathbf{x}}, t) = -m_{\rm eff}c^2/\gamma = -\gamma^{-1}c^2 m(1 - \tfrac{1}{2}mh_{\alpha\beta}U^\alpha U^\beta/c^2) \tag{113}$$

and the $i^{\rm th}$ component of the canonical spatial momentum is

$$p_i = \frac{\partial L}{\partial \dot{x}^i} = \gamma m\dot{x}^i - \tfrac{1}{2}\gamma^3 mh_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta\dot{x}^i/c^2 + \gamma mh_{\alpha i}\dot{x}^\alpha \tag{114}$$

which, at zeroth order, is the usual relativistic 3-momentum $\mathbf{p} = \gamma m\dot{\mathbf{x}}$. More generally, the 1st two terms are $\gamma m_{\rm eff}\dot{\mathbf{x}}$. This is somewhat reminiscent of electrodynamics, where the canonical momentum is $\mathbf{p} = \gamma m\dot{\mathbf{x}} + q\mathbf{A}$.

The Hamiltonian – or 'canonical energy' – is

$$H \equiv \dot{\mathbf{x}}\cdot\mathbf{p} - L = \gamma mc^2 - \tfrac{1}{2}\gamma^3 mh_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta + \gamma mh_{\alpha i}\dot{x}^\alpha\dot{x}^i \tag{115}$$

so, as with the canonical momentum, we have the normal zeroth order $H = \gamma mc^2$ but augmented at 1st order by extra terms that are quite analogous to how, in electrodynamics, we have $H = \gamma mc^2 + qA^0$.

Expressed in terms of momenta and position, the Hamiltonian is, at 1st order, and using $g^{\alpha\beta} = \eta^{\alpha\beta} - h^{\alpha\beta}$, so $g^{00} = -(1 + h^{00})$, $g^{ij} = \delta^{ij} - h^{ij}$ and $g^{0i} = -h^{0i}$,

$$\begin{aligned} H(x_i, p_i, t) &= -cp_0 = c\left(-h^{0i}p_i + \sqrt{(1 - h^{00})(|\mathbf{p}|^2 + m^2c^2 - h^{ij}p_ip_j)}\right) \\ &= c\sqrt{|\mathbf{p}|^2 + m^2c^2} - \frac{c}{2}\frac{(|\mathbf{p}|^2 + m^2c^2)h^{00} + h^{ij}p_ip_j}{\sqrt{|\mathbf{p}|^2 + m^2c^2}} - cp_ih^{0i} \\ &= \gamma_{\mathbf{p}}mc^2 - \tfrac{1}{2}(\gamma_{\mathbf{p}}mc^2h^{00} + (p_ip_j/\gamma_{\mathbf{p}}m)h^{ij}) - cp_ih^{0i} \end{aligned} \tag{116}$$

where we have taken the positive square root in order to get a positive energy and, in the last line, we are defining $\gamma_{\mathbf{p}} = \sqrt{1 + |\mathbf{p}|^2/m^2c^2}$.

Hamilton's equations in a general weak-field metric are:

$$\begin{aligned}\frac{dp_k}{dt} &= -\frac{\partial H}{\partial x_k} = \tfrac{1}{2}(\gamma_{\mathbf{p}} mc^2 h^{00}{}_{,k} + (p_i p_j/\gamma_{\mathbf{p}} m) h^{ij}{}_{,k}) + c p_i h^{0i}{}_{,k} \\ \frac{dx_k}{dt} &= \frac{\partial H}{\partial p_k} = \frac{p_k}{\gamma_{\mathbf{p}} m}\left(1 - \tfrac{1}{2}h^{00} + \tfrac{1}{2}\frac{h^{ij}p_i p_j}{\gamma_{\mathbf{p}}^2 m^2 c^2}\right) - \frac{p_i}{\gamma_{\mathbf{p}} m} h^{ik} - c h^{0k}\end{aligned} \tag{117}$$

6.3.3 Classical mechanics of relativistic particles in the Newtonian limit metric

Of considerable interest is the case of the Newtonian limit metric, for which $h^{\alpha\beta} = h_{\alpha\beta} = -2\Phi\delta_{\alpha\beta}$, and for which the $H - \mathbf{p}$ relation is

$$(1 - 2\Phi)H^2 = (1 + 2\Phi)|\mathbf{p}|^2 c^2 + m^2 c^4. \tag{118}$$

from which we obtain, at linear order in Φ,

$$H(\mathbf{p}, \mathbf{x}, t) = \sqrt{m^2c^4 + |\mathbf{p}|^2c^2} + \frac{m^2c^4 + 2|\mathbf{p}|^2c^2}{\sqrt{m^2c^4 + |\mathbf{p}|^2c^2}}\Phi(\vec{x}) \tag{119}$$

or

$$H(\mathbf{p}, \mathbf{x}, t) = \gamma_{\mathbf{p}} mc^2(1 + (2 - 1/\gamma_{\mathbf{p}}^2)\Phi(\vec{x})) \tag{120}$$

from which we obtain Hamilton's equations:

$$\begin{aligned}\frac{d\mathbf{p}}{dt} &= -\frac{\partial H}{\partial \mathbf{x}} = -mc^2(2\gamma_{\mathbf{p}} - 1/\gamma_{\mathbf{p}})\boldsymbol{\nabla}\Phi \\ \frac{d\mathbf{x}}{dt} &= \frac{\partial H}{\partial \mathbf{p}} = \frac{\mathbf{p}}{\gamma_{\mathbf{p}} m}(1 + (2 + 1/\gamma_{\mathbf{p}}^2)\Phi)\end{aligned} \tag{121}$$

which we can integrate to obtain the trajectory $\mathbf{x}(t)$ and $\mathbf{p}(t)$ and hence, from (120) $H(t)$ and therefore the 4-momentum 1-form $\tilde{p}(t) \to (-H(t)/c, \mathbf{p}(t))$ for a particle moving in a given weakly perturbed metric $g_{\alpha\beta} = \eta_{\alpha\beta} - 2\Phi\delta_{\alpha\beta}$.

The energy and momentum are the components of $\tilde{p}$ in a particular coordinate system, as, in order to obtain the Newtonian limit we imposed the Lorenz gauge condition. With a different choice of gauge these would change. To obtain the physical energy $E_{\rm obs}$, for example, we need to 'dot' $\tilde{p}$ with the 4-velocity of the observer doing the measurement. For observers being accelerated so as to maintain constant $\mathbf{r}$, this is $\vec{U}_{\rm obs} \to (U^0_{\rm obs}, \mathbf{0})$ where, at 1st order in the metric, $U^0_{\rm obs} = c/\sqrt{-g_{00}} = c(1-\Phi)$, so we obtain $E_{\rm obs} = (1-\Phi)H$, or

$$E_{\rm obs} = \gamma_{\mathbf{p}} mc^2(1 + (1 - 1/\gamma_{\mathbf{p}}^2))\Phi(\vec{x}). \tag{122}$$

The first of Hamilton's equation (the one giving $d\mathbf{p}/dt$) is quite revealing. It is purely 1st order. So if we consider a rapidly moving particle (one for which $|\mathbf{p}|^2/2m \gg \Phi c^2$) we expect only a small deflection from the 'unperturbed' path, and, to linear order in Φ we can calculate the change in $\mathbf{p}$ by integrating $d\mathbf{p}/dt$ while holding $\mathbf{p}$ constant. This is called the 'Born-approximation'. Taking the particle to be moving along the z-axis, the rate of change of $\mathbf{p}$ with z is $d\mathbf{p}/dz = (d\mathbf{p}/dt)/(dz/dt)$ while the second of Hamilton's equations tells is that, at zeroth order, $dz/dt = |\mathbf{p}|/\gamma_{\mathbf{p}} m$, so $d\mathbf{p}/dz = -(2\gamma_{\mathbf{p}}^2 - 1)|\mathbf{p}|^{-1}m^2c^2\boldsymbol{\nabla}\Phi = -(2\gamma_{\mathbf{p}}^2 - 1)m^2|\mathbf{p}|^{-1}\boldsymbol{\nabla}\phi$ (since $\Phi \equiv \phi/c^2$). Thus, in the Born approximation, $\Delta\mathbf{p} = \int dz(d\mathbf{p}/dz) = -(2\gamma_{\mathbf{p}}^2 - 1)m^2|\mathbf{p}|^{-1}\int dz\boldsymbol{\nabla}\phi$. Dividing this by $|\mathbf{p}|$, we obtain the fractional change in the momentum

$$\frac{\Delta\mathbf{p}}{|\mathbf{p}|} = -\frac{2\gamma_{\mathbf{p}}^2 - 1}{|\mathbf{p}|^2/m^2}\int dz\, \boldsymbol{\nabla}\phi \tag{123}$$

So for a non-relativistic particle, for which $\gamma_{\mathbf{p}} \simeq 1$, and $|\mathbf{p}|/m \simeq v$, this is $\Delta\mathbf{p}/|\mathbf{p}| = -(1/v^2)\int dz\, \boldsymbol{\nabla}\phi$, while for an ultra-relativistic particle, for which $|\mathbf{p}|/m \simeq \gamma_{\mathbf{p}} c$, we have $\Delta\mathbf{p}/|\mathbf{p}| = -(2/c^2)\int dz\, \boldsymbol{\nabla}\phi$, so the deflection is twice what the non-relativistic limit formula would predict for a particle moving at $v = c$.

6.4 The geodesic equation for massless particles

The proper time τ is ill-defined for a massless particle, as is the 4-velocity $\vec{U} = d\vec{x}/d\tau$ and therefore also the geodesic equation $dU^\alpha/d\tau = -\Gamma^\alpha{}_{\mu\nu}U^\mu U^\nu$.

But the 4-momentum $\vec{p} = m\vec{U}$ *is* a well-defined entity. Multiplying the geodesic equation by m^2, on the right hand side we find $m^2 U^\mu U^\nu = p^\mu p^\nu$, while the left hand side is $m^2 dU^\alpha/d\tau = mdp^\alpha/d\tau = dp^\alpha/d\lambda$ where we have introduced, as an alternative to proper time τ as affine parameter for the world-line, λ, defined such that $d\lambda = d\tau/m$.

The contravariant geodesic equation is then equivalent to

$$dp^\alpha/d\lambda = -\Gamma^\alpha{}_{\mu\nu}p^\mu p^\nu \tag{124}$$

or equivalently, since $\vec{p} = m\vec{U} = md\vec{x}/d\tau = d\vec{x}/d\lambda$,

$$\frac{d^2x^\alpha}{d\lambda^2} = -\Gamma^\alpha{}_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}. \tag{125}$$

So while τ is not defined for a massless particle, the alternative affine parameterisation in terms of $\lambda = \tau/m$ works for either massive or massless particles (a massless particle of finite energy E behaving the same as a massive particle with energy $E = \gamma mc^2$ in the limit of $m \to 0$ and $\gamma \to \infty$ with $\gamma m = E/c^2$ finite).

What's more, if we take the photon to be moving along the z coordinate axis with unit 3-momentum: $\mathbf{p} = d\mathbf{x}/d\lambda \to (0,0,1)$, the affine parameter λ measures coordinate distance z along the path. This is a convenient choice as we can then use (125) to calculate, for example, the curvature of the path (i.e. the 2nd rate of change of the other coordinates (x, y) with respect to z). And this path is independent of the momentum of the photon.

Perhaps the main conceptual difference in dealing with massless vs. massive particles is that whereas for the latter one thinks primarily in terms of their proper time as being the natural parameterisation of their world-lines, for massless particles this has no utility and it is actually more useful to think in terms of distance along the path. This may seem strange, as this is coordinate distance dz, rather than a 'proper' quantity. But remember, it's only for a particle that instantaneously has $\mathbf{p} \to (0,0,1)$ that λ measures z-distance travelled along the path. More generally, the 'proper' quantity is (coordinate) distance travelled per unit 3-momentum. As both 3-displacement and 3-momentum transform in the same way, this is frame independent and is a proper quantity. Also, if in doubt about the meaning of the λ as the affine parameterisation one can always fall back on the definition of $d\lambda$ as the limit, as $m \to 0$ and $\gamma \to \infty$ of $d\tau/m$.

6.5 Light deflection from the geodesic equation for massless particles

As for a massive particle, we can calculate the 1st order equations of motion using the zeroth order momentum on the RHS since the Christoffel symbols are first order.

As usual, it is simpler to use the covariant geodesic equation:

$$p^\beta p_{\alpha;\beta} = p^\beta(p_{\alpha,\beta} - \Gamma^\mu{}_{\alpha\beta}p_\mu) = 0 \tag{126}$$

as two of the terms in the connection then cancel by symmetry when contracted with the symmetric combination $p^\beta p^\mu$, and we are left, using $p^\beta p_{\alpha,\beta} = dp_\alpha/d\lambda$, with

$$dp_\alpha/d\lambda = \tfrac{1}{2}g_{\nu\beta,\alpha}p^\beta p^\nu. \tag{127}$$

This is quite general. Specialising to the weak-field metric, we have $g_{\nu\beta,\alpha} = h_{\nu\beta,\alpha}$, so

$$\boxed{dp_\alpha/d\lambda = \tfrac{1}{2}h_{\nu\beta,\alpha}p^\beta p^\nu.} \tag{128}$$

Specialising further to the Newtonian limit – for which $h_{\nu\beta} = -2\Phi\delta_{\nu\beta}$ is diagonal – and considering, without much loss of generality, $p^\alpha = (1,0,0,1)$ – i.e. a photon $(m = 0)$ moving instantaneously along the $z = x^3$ axis – in the momenta on the right hand side, we have

$$\boxed{dp_\alpha/d\lambda = \tfrac{1}{2}(h_{00,\alpha} + h_{33,\alpha}) = -2\Phi_{,\alpha}} \tag{129}$$

in which we see that both space-space and time-time components of $h_{\alpha\beta}$ play a role for massless particles.

Using $\Phi \equiv \phi/c^2$, and taking α to be the x or y axis, we have the rate of change of the transverse displacement:

$$\boxed{d^2\mathbf{r}_\perp/d\lambda^2 = -2\boldsymbol{\nabla}_\perp\phi/c^2} \tag{130}$$

which, comparing to (72) we see to be twice the Newtonian prediction for a particle moving with $v = c$.

Equivalently, we can write this as

$$\boxed{d\hat{\mathbf{n}}/d\lambda = -2\boldsymbol{\nabla}_\perp\phi/c^2} \tag{131}$$

in accord with Snell's law $d\hat{\mathbf{n}}/d\lambda = \boldsymbol{\nabla}_\perp n$ for the case that the effective refractive index for a gravitating system is $n = 1 - 2\Phi = 1 - 2\phi/c^2$, which in turn is consistent with the fact that the coordinate speed of light is $(1 + 2\Phi) \times c$. It is also consistent with what we found above using Hamilton's equations.

6.6 Gauge invariance of light deflection

You might well object, at this point, that the geodesic deviation equation is not gauge-invariant. Equation (130) tells us that the 2nd rate of change of the transverse *coordinate* is twice the Newtonian prediction for a particle moving at speed $v = c$. What's more, it is apparently in conflict with the prediction based on the equivalence principle, which seems to be on fairly firm grounds. What's to say that (130) isn't a 'gauge-artefact' arising from the coordinate system we have adopted. After all, as we have already noted, the connection components appearing in the geodesic equation are gauge dependent.

In the (contravariant) geodesic equation $dp^\alpha/d\lambda = -\Gamma^\alpha{}_{\mu\nu}p^\mu p^\nu$, for instance, we find, from the weak field expression for the connection (21) and the law for transformation of the metric $h_{\alpha\beta} \Rightarrow h_{\alpha'\beta'} = h_{\alpha\beta} - \xi_{\alpha,\beta} - \xi_{\beta,\alpha}$, that the connection transforms like $\Gamma^\alpha{}_{\mu\nu} \Rightarrow \Gamma^{\alpha'}{}_{\mu'\nu'} = \Gamma^\alpha{}_{\mu\nu} - \xi^\alpha{}_{,\mu\nu}$.

But that's exactly as it should be, since the transformed 4-momentum is $p^{\alpha'} = dx^{\alpha'}/d\lambda = d(x^\alpha + \xi^\alpha)/d\lambda = p^\alpha + p^\mu\xi^\alpha{}_{,\mu}$ and therefore, on differentiating this (and using the fact that $d(p^\mu\xi^\alpha{}_{,\mu})/d\lambda = p^\mu d(\xi^\alpha{}_{,\mu})/d\lambda$ at first order) $dp^{\alpha'}/d\lambda = dp^\alpha/d\lambda + p^\mu p^\nu\xi^\alpha{}_{,\mu\nu}$ which is the same as the transformed geodesic equation $dp^{\alpha'}/d\lambda = -\Gamma^{\alpha'}{}_{\mu'\nu'}p^{\mu'}p^{\nu'}$, again taking into account that on the right hand side, since the connection is first order, we can ignore any difference between $p^{\mu'}p^{\nu'}$ and $p^\mu p^\nu$.

Thus, with a different choice of gauge, the modified geodesic equation

$$\frac{dp^{\alpha'}}{d\lambda} = \frac{d^2x^{\alpha'}}{d\lambda^2} = -(\Gamma^\alpha{}_{\mu\nu} - \xi^\alpha{}_{,\mu\nu})\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda} \tag{132}$$

describes the *same* photon trajectory, just in the different coordinate system $x^{\alpha'} = x^\alpha + \xi^\alpha$.

So there *is* some ambiguity in e.g. the equation for the rate of change of the transverse displacement of the path $\mathbf{r}_\perp$ (130). This displacement is in a particular coordinate system; that dictated by our choice of the Lorenz gauge. For some other choice of gauge we would get a different answer.

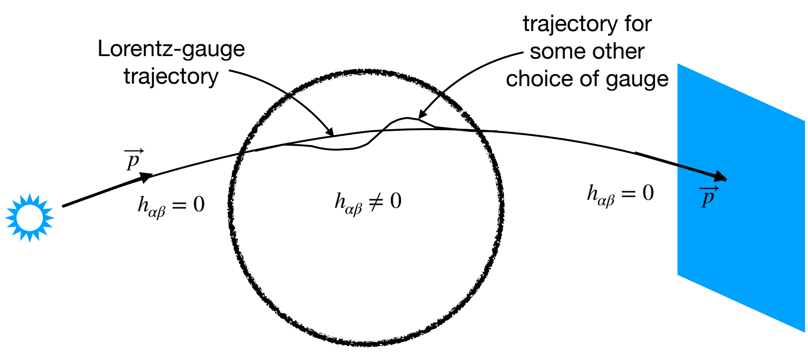
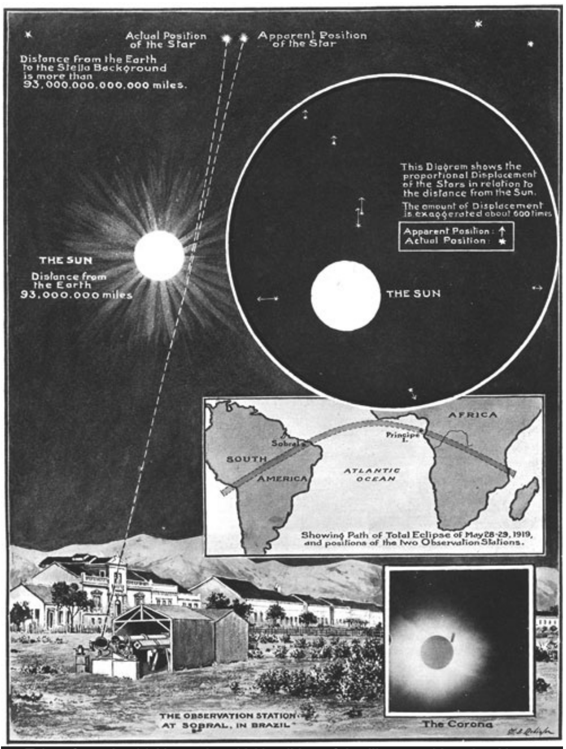
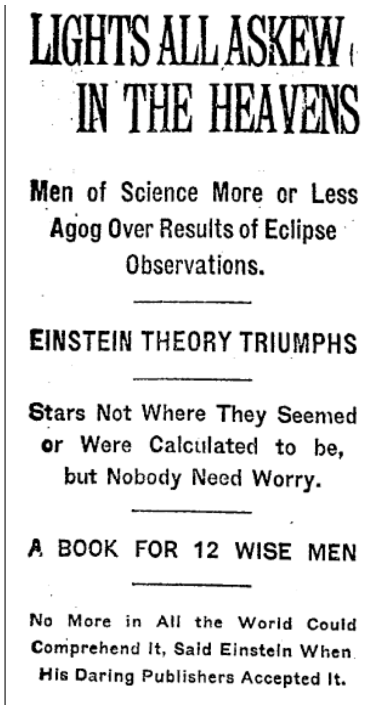


Figure 9: Gauge invariance of light deflection. The geodesic equation provides, for instance, the 2nd rate of change of the transverse displacement $\mathbf{r}_\perp$ as a function of distance along the path (i.e. the curvature of the light-ray). But this is a coordinate displacement and is therefore gauge dependent. With a different choice of 'gauge' (i.e. coordinates), we will have a different $d^2\mathbf{r}_\perp/d\lambda^2$. So the same physical photon trajectory will appear different when drawn in coordinate-space as above. But provided the change in gauge only applies in the region where the metric perturbations are non-zero, the integrated change in the transverse momentum will be entirely independent of whatever choice of gauge we adopt.

But that should not worry us too much, since what we are probably most interested in is the result of integrating this equation to get the change of direction of the photon after passing the lens (e.g. the Sun in the case of Eddington's expedition – see figure 10). If we integrate (132) to get $\Delta p^{\alpha'} = \int d\lambda \frac{dp^{\alpha'}}{d\lambda}$ we get the Lorenz-gauge result plus $\int d\lambda \xi^\alpha{}_{,\mu\nu} \frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda} \simeq p^\mu \int dx^\nu \partial_\nu (\xi^\alpha{}_{,\mu})$ which is just the difference between $p^\mu \xi^\alpha{}_{,\mu}$ at the beginning and end points, and is independent of any changes to the coordinate system within the lens itself.



Taken from the 22 November 1919 edition of the Illustrated London News.


LIGHTS ALL ASKEW IN THE HEAVENS

Men of Science More or Less Agog Over Results of Eclipse Observations.

EINSTEIN THEORY TRIUMPHS

Stars Not Where They Seemed or Were Calculated to be, but Nobody Need Worry.

A BOOK FOR 12 WISE MEN

No More in All the World Could Comprehend It, Said Einstein When His Daring Publishers Accepted It.

Coverage in the (more excitable) New York Times

Figure 10: The Eddington 1919 eclipse expedition. The deflection of light by the Sun was measured by comparing the positions of stars behind the Sun during the eclipse to a reference image taken when the Sun was elsewhere on the sky. The effect can be analysed using the geodesic equation, but that involves 'gauge-issues'. An alternative way to test GR is to measure the 'image shear' of the star-field. That involves considering the separation between neighbouring geodesics to get the 'so-called' geodesic deviation. This can be analysed in a gauge invariant manner.

Another way to side-step the issue of gauge invariance of the deflection is to use the geodesic *deviation* equation to calculate, rather than the deflection itself, the image shear (the gradient of the deflection). As discussed earlier, this gives the 2nd rate of change of the separation vector $\vec{\xi}$ with respect to affine distance as an integral involving the Riemann tensor, which, as we saw above, has components that are gauge independent. Integrating this gives the observable image shear in a manifestly gauge invariant manner.

Regarding the conflict between (130) and the equivalence principle, there is nothing wrong with the latter. A constant-**r** observer *would* measure a local deflection of light perfectly in accord with the equivalence principle using the acceleration he measures, and in *disaccord* with (130). So would a freely falling observer – who would see no local light deflection. So in that sense, the geodesic equation is 'wrong' in that it does not describe what either of those observers sees locally for the deflection measured in physical coordinates tied to their state of motion. One can say that the factor 2 is a 'coordinate-artefact'; it is expressing the local deflection in a different coordinate system to those used by either the constant-**r** or freely falling observer and giving a different answer. But the advantage of the geodesic equation is that it allows us to integrate the effect along the photon trajectory and, as we have seen, gives a gauge-independent result – provided we don't try to modify the coordinates in the vicinity of the observer or the source – that can be directly compared to what was actually observed.

7 Matter waves in weak-field gravity

The dark matter observed in galaxies and galaxy clusters etc. may be the axion – a scalar field whose excitations have a rest-mass (energy) usually taken to be on the order of $mc^2 \sim 10^{-5}$eV – or it may be an ultra-light axion-like field – so called 'fuzzy dark matter' – with a mass $mc^2 \sim 10^{-22}$eV.

So it is of interest to understand how such matter would behave in the potential wells of cosmological structure; these being well described by the weak-field theory we have developed above.

We can also think of non-scalar particles as being described, via Schrödinger's equation, in terms of a quantum mechanical wave function $\psi(\vec{x})$ and it is of interest to understand how this is coupled to weak-field gravity also.

Below, we first obtain the equations of motion – the Klein-Gordon equation – for a massive scalar field in a gravitational field. It is very simple; to a very good approximation the effect of gravity is to give the field a position dependent mass (actually the Compton wave-number here) $m(\mathbf{x}) = m \times (1 + \Phi(\mathbf{x}))$. We then look at nearly monochromatic waves, which are analogous to beams of particles with well defined momentum (and the scalar wave is like the wave-function for a particle with well defined momentum and poorly defined position) and show the relation between the phase of the field and the action $S(\mathbf{x})$ for such beams.

We show how such waves behave in a very similar fashion to EM waves in a cold plasma, and how one can think of scalar matter being trapped in potential wells much as EM waves are trapped by the ionosphere.

We then show how the Klein-Gordon equation becomes the (classical) Schrödinger equation in the non-relativistic limit, which is useful as it provides a more efficient way to numerically simulate the behaviour of wave-like matter. It is also provides a nice way to show that, along with energy and momentum conservation $T^{\mu\nu}{}_{,\mu} = 0$, in this limit there is a 5th conserved quantity, with the same units as action, which is conserved, and this corresponds to conservation of number of particles. It also provides a very nice way to understand the behaviour of scalar fields in the highly non-linear regime – analogous to what happens with particles in the 'multi-streaming' regions that develop in gravitational collapse – in a manner very similar to the Kirchoff-Frensel formalism in optics,

We also describe in an appendix the alternative Madelung equation, which is a re-parameterisation of the complex Schrödinger field ψ into modulus (squared) $\rho = |\psi|^2$ and phase $\theta = \arg(\psi)$. This also been applied in simulations. We discuss some peculiarities of this approach.

7.1 The Klein-Gordon equation in weak-field gravity

In flat space-time, the Lagrangian density of a classical massive scalar field ϕ (not to be confused with the Newtonian potential) is

$$\mathcal{L}(\phi_{,\mu}, \phi) = -\tfrac{1}{2}(\phi^{,\mu}\phi_{,\mu} + m^2\phi^2) \tag{133}$$

where the 'mass' m has units of $[\mathrm{L}^{-1}]$ and is the Compton wavenumber for the bosonic particles of mass $M = \hbar m/c$ that are the quantum mechanical excitations of this field. The Lagrangian density is a Lorentz scalar and has units of energy density or $[\mathrm{ML}^{-1}\mathrm{T}^{-2}]$ so the field has units of $[\mathrm{M}^{1/2}\mathrm{L}^{1/2}\mathrm{T}^{-1}]$. We take (133) as the fundamental definition of the scalar field.

Extremising the action $S = \int dt L = \int d^4x \mathcal{L}(\phi_{,\mu}, \phi)$ gives the Klein-Gordon equation:

$$\phi^{,\mu}{}_{,\mu} = m^2\phi. \tag{134}$$

To obtain the action in curved space time, we simply replace the invariant space-time volume element $d^4x = \sqrt{-|\boldsymbol{\eta}|}d^4x$, where $|\boldsymbol{\eta}| = -1$ is the determinant of the Minkowski metric, by its equivalent in a general coordinate system: $\sqrt{-|\mathbf{g}|}d^4x$, and $\phi^{,\mu}{}_{,\mu}$ by its equivalent $g^{\mu\nu}\phi_{,\mu}\phi_{,\mu}$, to obtain

$$S = \int d^4x \underbrace{\sqrt{-|\mathbf{g}|}(-\tfrac{1}{2}g^{\mu\nu}\phi_{,\mu}\phi_{,\mu} - \tfrac{1}{2}m^2\phi^2)}_{\mathcal{L}(\phi_{,\mu},\phi,\vec{x})} \tag{135}$$

so we see that, since $\mathbf{g} = \mathbf{g}(\vec{x})$, the Lagrangian density is now a function of position as well as of ϕ and its derivatives.

The variation of the action under a variation of the field $\phi(\vec{x}) \Rightarrow \phi(\vec{x}) + \delta\phi(\vec{x})$ is

$$\delta S = \int d^4x \left[\frac{\partial \mathcal{L}}{\partial \phi_{,\mu}}\delta\phi_{,\mu} + \frac{\partial \mathcal{L}}{\partial \phi}\delta\phi\right] = \int d^4x \delta\phi \left[\frac{\partial}{\partial x^\mu}\frac{\partial \mathcal{L}}{\partial \phi_{,\mu}} + \frac{\partial \mathcal{L}}{\partial \phi}\right] \tag{136}$$

where we have integrated by parts and discarded a boundary term. Setting $[\ldots] = 0$ in the last expression gives the equations of motion, which looks, with the more complicated Lagrangian density in (135), intimidating. But it isn't really; the equations of motion are local, and we can always express them in a local inertial frame so the extra derivatives of the metric components vanish, and the determinant factor is $\sqrt{-|\mathbf{g}|} = 1$, and the EoM is simply (134), and this, in general coordinates, is

$$\phi^{;\mu}{}_{;\mu} = m^2\phi \tag{137}$$

where we don't need to modify the first derivative as $\phi^{;\mu} = \phi^{,\mu}$.

Of course we could have simply invoked the 'comma becomes semi-colon' mantra, to obtain (137). The purpose of the foregoing is to show how this comes about, starting from the fundamental Lagrangian density (133).

Writing out the covariant derivative in terms of Christoffel symbols, the general form of the Klein-Gordon equation is

$$g^{\mu\nu}(\phi_{,\mu\nu} - \Gamma^{\alpha}{}_{\mu\nu}\phi_{,\alpha}) = m^2\phi. \tag{138}$$

If we specialise to weak-fields $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \Rightarrow g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu}$, and work at linear order, the connection is $\Gamma^{\alpha}{}_{\mu\nu} = \frac{1}{2}\eta^{\alpha\gamma}(h_{\gamma\mu,\nu} + h_{\gamma\nu,\mu} - h_{\mu\nu,\gamma})$, and we can replace the index raising operator $g^{\mu\nu}$ acting on this in (138) by $\eta^{\mu\nu}$, to obtain

$$g^{\mu\nu}\Gamma^{\alpha}{}_{\mu\nu} = \tfrac{1}{2}\eta^{\mu\nu}\eta^{\alpha\gamma}(h_{\gamma\mu,\nu} + h_{\gamma\nu,\mu} - h_{\mu\nu,\gamma}) = \eta^{\alpha\gamma}\left(h_{\gamma\mu}{}^{,\mu} - \tfrac{1}{2}h_{,\gamma}\right) = \eta^{\alpha\gamma}\left(h_{\gamma\mu} - \tfrac{1}{2}\eta_{\gamma\mu}h\right)^{,\mu}. \tag{139}$$

But we recognise the term in brackets as our friend the trace-reversed metric perturbation, $\overline{h}_{\gamma\mu} \equiv h_{\gamma\mu} - \frac{1}{2}\eta_{\gamma\mu}h$, so $g^{\mu\nu}\Gamma^{\alpha}{}_{\mu\nu} = \eta^{\alpha\gamma}\overline{h}_{\gamma\mu}{}^{,\mu}$ and, if we work in the Lorenz (or de Donder) gauge, this vanishes.

Thus the Lorenz gauge proves useful, not just for simplifying Einstein's equations, but it also banishes the connection term from the curved space-time Klein-Gordon equation (138).

So it appears that in this instance, and in the Lorenz gauge, the comma remains a comma and this might lead one to think that there is no coupling of gravity to a scalar field. But not quite. We still have the metric perturbation is the first term in (138). With $\eta^{\mu\nu}\phi_{,\mu\nu} = \Box\phi$, the Klein-Gordon equation in weakly curved space-time becomes

$$\boxed{\Box\phi - h^{\mu\nu}\phi_{,\mu\nu} = m^2\phi} \tag{140}$$

which is a non-covariant equation, but is valid nonetheless, but only in the coordinate system implied by our choice of gauge.

Using the Newtonian limit metric $h_{\mu\nu} = -2\Phi\delta_{\mu\nu}$, which implies $h^{\mu\nu} = -2\Phi\delta_{\mu\nu}$, we get

$$-(1 - 2\Phi)\ddot{\phi}/c^2 + (1 + 2\Phi)\nabla^2\phi = m^2\phi \tag{141}$$

or

$$\Box\phi = m^2\phi - 2\Phi\phi_{,\mu\mu} \tag{142}$$

where, in this formally illegitimate and non-covariant equation, we sum over the repeated indices even though they are both downstairs.

This is quite general, and could be used to describe the propagation of scalar waves of arbitrary momentum (i.e. wave-number). If, however, the field gained its 3-momentum by 'falling' into a Newtonian potential well with $\Phi(\vec{x}) \ll 1$, it's 3-momentum will be small and we can use $\phi_{,\mu\mu} \simeq \phi_{,00} = -m^2\phi$ and so we have.

$$\boxed{\Box\phi = (1 + 2\Phi(\vec{x}))m^2\phi} \tag{143}$$

in which we see that to an excellent approximation the effect of the gravitational field is simply to modulate the effective mass:

$$\boxed{m_{\text{eff}}(\vec{x}) = (1 + \Phi(\vec{x}))m.} \tag{144}$$

The final result is very pleasing and intuitively reasonable. The KG equation in flat space time admits solutions corresponding to particles at rest where ϕ oscillates in time with frequency (c times) m. In curved space-time the 'coordinate-frequency' (i.e. the frequency of oscillation as a function of coordinate time t) is simply $cm \times (1 + \Phi)$, so the scalar field oscillations, like those of any good clock, run slow, as compared to coordinate time, in a potential well.

7.2 The dispersion relation for scalar waves

If the Newtonian gravitational potential[10] $\varphi = \Phi c^2$ in which the matter waves reside is varying on a sufficiently large length and time scale the Klein-Gordon equation (143) will admit locally monochromatic solutions where $\phi(\vec{x}) = a\cos(k_u x^\mu + \Psi_0)$ where a is the amplitude and the constant Ψ_0 is the phase. This can also be expressed as

$$\phi(\vec{x}) = \phi_0 e^{ik_u x^\mu} \tag{145}$$

[10] We will use φ for the Newtonian potential in this section as ϕ denotes here the scalar field.

with complex amplitude $\phi_0 = ae^{i\Psi_0}$ that encodes both the phase and the amplitude, the above being shorthand for

$$\phi(\vec{x}) = \frac{1}{2}(\phi_0 e^{ik_u x^\mu} + \text{c.c.}). \tag{146}$$

With this trial solution (with $\tilde{k} \to k_\alpha = (-\omega_{\mathbf{k}}/c, \mathbf{k})$), and considering Φ to be (locally) constant, the Klein-Gordon equation becomes the dispersion relation linking frequency and wave-number:

$$(1 - 2\Phi)\omega_{\mathbf{k}}^2 = c^2(m^2 + (1 + 2\Phi)|\mathbf{k}|^2) \tag{147}$$

which, multiplied by $\hbar^2$ and using the fact that m is the compton wave-number, $m = Mc/\hbar$, gives

$$(1 - 2\Phi)H^2 = M^2c^4 + (1 + 2\Phi)|\mathbf{p}|^2c^2 \tag{148}$$

(with $H = \hbar\omega$ and $\mathbf{p} = \hbar\mathbf{k}$) which is the same as (118) the relativistic energy-momentum relation for a particle with mass M in the Newtonian limit metric.

Alternatively, if we write

$$\phi(\vec{x}) = \phi_0 e^{i\Psi(\vec{x})} \tag{149}$$

which is equivalent to the above if we expand the phase as $\Psi(\vec{x}) = \Psi_0 + \Psi_{,\mu}x^\mu + \ldots$. The dispersion relation – or energy-momentum relation – relates the components of the phase-derivative 1-form $\tilde{d}\Psi \to \Psi_{,\mu}$, and is then

$$(1 - 2\Phi)(-\partial\Psi/\partial t)^2 = c^2(m^2 + (1 + 2\Phi)(\boldsymbol{\nabla}\Psi)^2) \tag{150}$$

which we see is equivalent to the Hamilton-Jacobi equation for a beam of relativistic particles emanating from a common starting point with classical action $S(\vec{x}) = \hbar\Psi(\vec{x})$ and with $\tilde{d}S \to \partial_\mu S = (-H/c, \mathbf{p}) = \hbar k_\mu$.

This is all very reminiscent of the Dirac-Feynman picture where the wave-function ψ for a particle that would have a classical action S is locally proportional to $\exp(iS/\hbar)$. But the field $\phi(\vec{x})$ here is to be thought of as purely classical, and $\hbar$ appears here simply as a parameter as we have chosen to express the mass M of the field in terms of the Compton wave-number $m = Mc/\hbar$ and in no way indicates anything quantum-mechanical. The field *should* of course be treated quantum mechanically and there would then be a wave-function, but it wouldn't be a function of position like $\phi(\vec{x})$, rather is would be a functional of $\phi(\vec{x})$. The Klein-Gordon equation would then emerge as the equation of motion satisfied by the expectation value of $\phi(\vec{x})$.

There is a very close correspondence between a planar scalar wave and a collimated beam of particles. One can readily calculate the stress tensor for such waves (see below). The components of this have temporal and spatial fluctuations, but if one averages over these, the waves have energy flux-density and momentum density and momentum flux density just like a beam of particles. And, as we will show presently, wave-packets and beams of scalar waves are deflected by a gravitational field much as are particles.

This wave-particle correspondence leads one to expect that, for waves that have gained their momentum falling into a weak-field potential with $\Phi(\vec{x}) \ll 1$ the wave-number $\mathbf{k}$ will be on the order of the inverse of the de Broglie wavelength for a particle of mass M in a Newtonian potential $\varphi = c^2\Phi$. I.e. $|\mathbf{k}| \sim |\mathbf{p}|/\hbar$ with $|\mathbf{p}|^2/M \sim M\varphi$ so $|\mathbf{k}| \sim M\sqrt{\varphi}/\hbar$ or $|\mathbf{k}| \sim m\sqrt{\Phi} \ll m$. Under that assumption we can ignore the term involving the product of Φ and $|\mathbf{k}|^2$ and the dispersion relation becomes:

$$\omega_{\mathbf{k}}^2 = c^2((1 + 2\Phi)m^2 + |\mathbf{k}|^2). \tag{151}$$

7.3 The wave- and group-velocities for scalar waves

The speed with which the wave-crests travel is called the 'phase-velocity' and is

$$v_{\rm p} = \omega_{\mathbf{k}}/|\mathbf{k}| = c\sqrt{1 + (1 + 2\Phi)m^2/|\mathbf{k}|^2} \simeq cm/|\mathbf{k}| \gg c \tag{152}$$

where we have assumed $|\mathbf{k}| \ll m$ which is valid in the non-relativistic regime.

The speed with which the groups of waves or wave-packets – or 'beats' in a wave-train constructed from a superposition of 2 waves of slightly different frequencies – travel is called the 'group-velocity' and is

$$v_{\rm g} = d\omega_{\mathbf{k}}/d|\mathbf{k}| = c/\sqrt{1 + (1 + 2\Phi)m^2/|\mathbf{k}|^2} \simeq c|\mathbf{k}|/m \ll c \tag{153}$$

This is also the speed at which information can be propagated. As m is the Compton wave-number: $m = Mc/\hbar$, $v_{\rm g} \simeq \hbar|\mathbf{k}|/M$ which is the speed of a particle of mass M with momentum $\mathbf{p} = \hbar\mathbf{k}$.

7.4 The stress-energy tensor for scalar waves

The stress-energy tensor for a relativistic scalar field is

$$T_{\mu\nu} = \phi_{,\mu}\phi_{,\nu} + \eta_{\mu\nu}\mathcal{L} = \phi_{,\mu}\phi_{,\nu} - \tfrac{1}{2}\eta_{\mu\nu}(\phi_{,\gamma}\phi^{,\gamma} + m^2\phi^2) \tag{154}$$

or

$$T_{\mu\nu} = \begin{bmatrix} \frac{1}{2}(\dot\phi^2/c^2 + |\boldsymbol{\nabla}\phi|^2 + m^2\phi^2) & \frac{1}{c}\dot\phi\boldsymbol{\nabla}\phi \\ \frac{1}{c}\dot\phi\boldsymbol{\nabla}\phi & \boldsymbol{\nabla}\phi\boldsymbol{\nabla}\phi + \frac{1}{2}(\dot\phi^2/c^2 - |\boldsymbol{\nabla}\phi|^2 - m^2\phi^2)\mathbf{I} \end{bmatrix} \tag{155}$$

where $\mathbf{I}$ is the 3×3 unit matrix. This is valid in flat space-time or in a locally inertial frame (as only 1st derivatives appear).

For a plane wave ϕ all of the components of $\mathbf{T}$ contain fluctuating parts. But if we average over these, we have

$$T_{\mu\nu} = \langle\phi^2\rangle \begin{bmatrix} \omega^2/c^2 & -\omega\mathbf{k}/c \\ -\omega\mathbf{k}/c & \mathbf{kk} \end{bmatrix} = k_\mu k_\nu \langle\phi^2\rangle \tag{156}$$

since $\vec{k} \to (-\omega/c, \mathbf{k})$.

In the non-relativistic limit, the dispersion relation $\omega^2 = c^2(m^2 + |\mathbf{k}|^2)$ says $\omega \simeq mc$ so, raising the indices (which changes the sign of the off-diagonal components)

$$T^{\mu\nu} = \langle\phi^2\rangle \begin{bmatrix} m^2 & m\mathbf{k} \\ m\mathbf{k} & \mathbf{kk} \end{bmatrix} \tag{157}$$

so the energy density is $\mathcal{E} = T^{00} = m^2\langle\phi^2\rangle$, corresponding to a mass density $\rho = \mathcal{E}/c^2 = m^2\langle\phi^2\rangle/c^2$, and the components of the momentum density are $\pi_i = T^{0i}/c = mk_i\langle\phi^2\rangle/c$, so $\boldsymbol{\pi} = (c\mathbf{k}/m)\rho = \rho\mathbf{v}_\mathrm{g}$, i.e. precisely as for a beam of particles of density ρ moving with a velocity equal to the group velocity $\mathbf{v}_\mathrm{g} = c\mathbf{k}/m$.

7.5 The analogy with EM waves in a plasma

The dispersion relation for non-relativistic scalar waves in a gravitational potential (151) is identical to that for EM waves in a cold plasma

$$\omega^2 = \omega_\mathrm{p}^2 + c^2|\mathbf{k}|^2 \tag{158}$$

with $(1+\Phi)mc$ playing the role of the plasma frequency ω_p. The physics of this is illustrated in the left-hand panel of figure 11. The density of electrons defines a frequency ω_p which is the frequency that the plasma would 'ring' at if the electrons in some region were displaced (the resulting charge imbalance creating a restoring force). Maxwell's equations – here in integral form – show that travelling wave solutions are only allowed at frequencies above ω_p. Here the analogous frequency $\omega = cm(1+\Phi)$ arises for completely different reasons, but, as for EM waves in an inhomogeneous plasma, it varies with position and this results in interesting refractive effects on the propagation of waves.

If we turn the dispersion relation around and use it to determine the wave-number $\mathbf{k}$ for a wave of a specified frequency ω we get $|\mathbf{k}|^2 = (\omega^2 - \omega_\mathrm{p}^2)/c^2$. This has real solutions – corresponding to travelling waves – only for $\omega > \omega_\mathrm{p}$. For $\omega < \omega_\mathrm{p}$, the wave-number $\mathbf{k}$ is imaginary and any fluctuations of the field at these frequencies are evanescent.

As one enters the ionosphere, the density of electrons rises at first and then decreases, so the plasma frequency $\omega_\mathrm{p} = \sqrt{nq^2/\epsilon_0 m_e}$ has a maximum, which, it turns out, is at $\nu_\mathrm{p} = \omega_\mathrm{p}/2\pi \simeq 30$MHz.

Terrestrial EM waves of frequency less than this get trapped and reflected as illustrated in figure 11. For a given temporal frequency, the spatial frequency decreases with altitude, so the wavelength increases, and it is this stretching of the wavelengths with height that causes the refraction of such waves[11]. This is how 'short-wave' radio transmissions can be detected around the world.

If the dark matter is the axion or an ultra-light axion-like field then it is trapped in the gravitational potential wells of galaxies and clusters etc. in much the same manner. The situation is somewhat different from short-wave radio waves in that whereas in the ionosphere the plasma frequency increases from zero to its maximum value and then drops again, in the galaxy the effective plasma frequency is everywhere very nearly constant, being equal to cm at infinity and with only a small suppression within bound systems (of at most about 1 part in 10^5 – this being in clusters of galaxies). So the situation we have is that, within a cluster, we have waves just above the local plasma frequency but just below its asymptotic value at infinity.

[11]Note that, for such waves, the group velocity *decreases* with altitude. What matters for refraction is the phase-velocity, which increases with altitude, and one can think of the (wave-crests in the) upper part of of a beam from a transmitter as out-running those in the lower parts and thus causing the beam to turn.

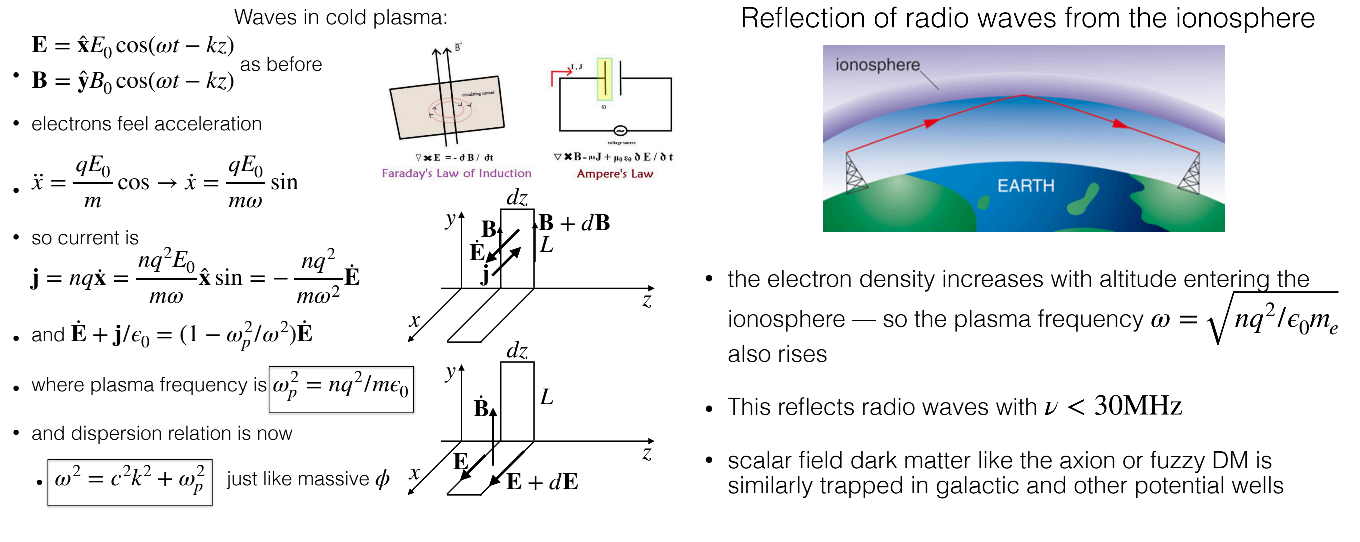


Figure 11: Scalar matter waves are trapped in a gravitational potential well much as EM waves get trapped by the ionosphere.

7.6 Focussing of scalar matter waves

We have seen that, as far as the *phase* of matter waves is concerned one can always find solutions – in potential with sufficiently slow spatial variation – that correspond to a beam of particles with action $S(\vec{x})$ by setting the phase to be $S/\hbar$. What about the *amplitude* of such waves?

If we have a beam of matter waves – analogous to a beam of particles – in empty space that encounters a potential well this will cause focussing of the beam.

But the beam satisfies the conservation laws $T^{\mu\nu}{}_{,\mu} = 0$ and, in particular, $T^{\mu 0}{}_{,\mu} = 0$. This says that the rate of change of the energy density $\mathcal{E} = T^{00}$ (proportional to $\langle\phi^2\rangle$) with respect to coordinate time $x^0 = ct$ is minus the 3-divergence of a vector field $\mathbf{S}$:

$$\dot{\mathcal{E}} = -\boldsymbol{\nabla}\cdot\mathbf{S} \tag{159}$$

where the energy flux density $\mathbf{S}$ has components $S^i = cT^{i0}$.

But recall that the momentum density – denoted here by $\boldsymbol{\pi}$ – has components $\pi_i = T^{i0}/c$ and is given, for a non-relativistic beam, to $\boldsymbol{\pi} = \rho\mathbf{v}$ where $\rho = \mathcal{E}/c^2$ and $\mathbf{v} = c\mathbf{k}/m$, which is the group velocity. So the above equation says that

$$\dot{\rho} = -\boldsymbol{\nabla}\cdot\boldsymbol{\pi} = -\boldsymbol{\nabla}\cdot(\rho\mathbf{v}). \tag{160}$$

But this is the same as the continuity equation for particles in a fluid of density ρ and with velocity field $\mathbf{v}$. So the equivalent mass mass density $\rho = (m^2/c^2)\langle\phi^2\rangle$ changes in the focussing beam in exactly the same manner as would the density of particles with that density and velocity.

This can also be understood from the equation of motion – the Klein-Gordon equation – which links changes of amplitude and phase. If we apply $\Box\phi = m^2\phi$ to a trial solution $\phi(\vec{x}) = A(\vec{x})e^{i\Psi(\vec{x})}$ we obtain

$$A(i\Box\Psi - \Psi_{,\mu}\Psi^{,\mu}) + 2iA_{,\mu}\Psi^{,\mu} + \Box A = m^2 A. \tag{161}$$

We see from this that a phase $\Psi = \Psi_0 + \Psi_{,\mu}x^\mu$ is compatible with wave-like solution with constant amplitude A, provided $\Psi_{,\mu}\Psi^{,\mu} = -m^2$ or, equivalently $\omega^2 = c^2(|\mathbf{k}|^2 + m^2)$ where $\Psi_{,\mu} = (-\omega/c, \mathbf{k})$. On a hyper-surface of constant t, this phase is $\Psi(\mathbf{x}) = \text{constant} + \mathbf{x}\cdot\mathbf{k}$. If we add to this some small, and smoothly varying, 'phase error' $\delta\Psi(\mathbf{x})$ – which will, in general, have some non-vanishing $\Box\delta\Psi = \nabla^2\delta\Psi$ – this will be compatible with a slowly varying amplitude $A = A(t)$ provides the 3rd term above compensates for the first. I.e. provided

$$2\omega\dot{A} + Ac^2\nabla^2\delta\Psi = 0 \tag{162}$$

or

$$d\ln A^2/dt = -\frac{c^2}{\omega}\nabla^2\delta\Psi = -\boldsymbol{\nabla}\cdot\mathbf{v} \tag{163}$$

using $\nabla^2\delta\Psi = \boldsymbol{\nabla}\cdot\boldsymbol{\nabla}\mathbf{k}$ (since $\boldsymbol{\nabla}\mathbf{k} = \boldsymbol{\nabla}\delta\Psi$) and $\mathbf{v} = c\mathbf{k}/m = c^2\mathbf{k}/\omega$ (since $\omega \simeq m$ for non-relativistic waves). Thus again we see that, if a wave passes through some inhomogeneous potential that imposes a phase-error

$\delta\Psi$ – and hence a divergence (or convergence) of the wave-vector $\mathbf{k}$ (and hence velocity) then the squared amplitude of the wave 'downstream' will change in just the same way as would the density of particles being focussed or defocussed.

This is all assuming waves that are locally nearly monochromatic and that the scale of variation of the potential is sufficiently slow. What we are seeing here is directly analogous to the 'geometric optics' limit in electromagnetism, where, if we are dealing with light of sufficiently short wavelength, it behaves like particles with energy density changing just as the density of particles would change.

7.7 The Fresnel scale for matter waves

Just as in optics, there is a length-scale analogous to the 'Fresnel-scale' $r_{\rm F} \sim \sqrt{L\lambda}$ where, in optics, L is the distance from the 'scattering plane' and λ is the wavelength of light. According to Fresnel-Kirchoff theory, the field on the observer plane – the focal plane in an optical system perhaps – is a convolution of the field on the scattering plane with the Fresnel function. The picture here is that there are 'Fresnel wavelets' on the scattering plane that irradiate the observer. These interfere with phases given by the optical path length, and only for particular places on the scattering plane will this be constructive. The result is that the field at a point on the observer plane is sensitive only to a patch on the scattering plane of size of order $r_{\rm F}$ as only the 'Fresnel wavelets' from this region interfere constructively. And if the scale of lenses etc. that are 'scattering' the light is larger than the Fresnel scale then geometric optics provides a good approximation.

Similarly, for scalar waves, we might consider a volume of linear size R centred on the origin $\mathbf{r} = \mathbf{0}$ in which there is a wave corresponding to zero 3-momentum particles, but outside of which the field vanishes. I.e. a 3-dimensional wave-packet in which $\phi \simeq \cos(mct)$. The finite size of the wave-packet means that it is made of waves that actually have a range of momenta $\Delta k \sim 1/R$. The group velocity of the mean momentum $\overline{\mathbf{k}} = 0$ is zero, but the components for which the amplitude is significant will have $v_{\rm g} \sim c\Delta k/m$, so after a time t the packet will have spread by an amount $\Delta R(t) \sim v_{\rm g}t \sim ct/mR$. Solving for the size of the packet that will have spread by its own width in this time – i.e. for which $\Delta R = R$ – gives $R \sim \sqrt{ct/m} \sim \sqrt{ct\lambda_{\rm C}}$ where $\lambda_{\rm C} = 2\pi c/m$ is the Compton wavelength.

It follows that, for an infinite zero 3-momentum wave, the field at $\mathbf{r} = \mathbf{0}$ is only sensitive to the initial field within a volume around the origin of linear size $\sim r_{\rm F} \equiv \sqrt{ct\lambda_{\rm C}}$ (since we would truncate the field outside of this volume at the initial time without affecting the final field at $\mathbf{r} = \mathbf{0}$).

In a self-gravitating object of size $R_{\rm obj}$ with velocity dispersion $v \sim \sqrt{GM/R_{\rm obj}}$ the dynamical (or orbital) time is $t_{\rm dyn} \sim R/v$ and the Fresnel scale after N dynamical times is $r_{\rm F} \sim \sqrt{NR\lambda_{\rm dB}}$ where $\lambda_{\rm dB} = (c/v)\lambda_{\rm C}$ is the. de Broglie scale. The Fresnel scale, in units of the size of the object, is therefore $r_{\rm F}/R_{\rm obj} \sim \sqrt{N\lambda_{\rm dB}/R_{\rm obj}}$.

The smallest mass deemed feasible for 'fuzzy' DM has, in the Milky-Way, a de Broglie wavelength $\lambda_{\rm dB} \sim 200{\rm pc} \sim R_{\rm obj}/500$ taking $R_{\rm obj} \sim 100{\rm kpc}$. So over one dynamical time, the Fresnel scale is much smaller than $R_{\rm obj}$ and geometric optics should be a very good approximation. The dynamical time, however, is about $t_{\rm dyn} \sim R_{\rm obj}/v \sim 3 \times 10^{15}{\rm sec}$, as compared to the age of the universe $t_{\rm U} \simeq 1/H_0 \simeq 4 \times 10^{17}{\rm sec}$ or about $N \sim 100$ dynamical times. If we track a geodesic path back to turnaround – at which time the galaxy was perhaps a factor 2 times larger than it's present time – this suggests that the Fresnel scale was smaller than the size of the object, but not by an enormous factor.

This allows us to make more precise what is implied by 'sufficiently slow' above. If we have scalar matter waves propagating in a gravitating system – a galaxy or galaxy cluster perhaps – with size R, then geometric optics will be a good approximation if $R \gg r_{\rm F}$.

7.8 Speckly nature of scalar DM in the multi-streaming regime

If we have a potential well – that of a spherical mass concentration say – and we have test particles released from rest then they will fall into the potential. A scalar wave that starts off spatially homogeneous – corresponding to zero momentum particles – will, initially, start to fall in exactly the same way. One can see that the wave will oscillate at a lower frequency (as a function of coordinate time, that is) deeper into the potential. Different parts of the wave will get out of phase with each other. So the initially spatially homogeneous field will develop ripples. Unsurprisingly these give the field a momentum density and, guess what, it is just that that the corresponding beam of particles would develop.

But the particles will, eventually, reach the centre of the potential, and they will meet up with particles that are coming the other way. We say that a 'multi-streaming' region has developed. What happens to the waves in the same analogous situation?

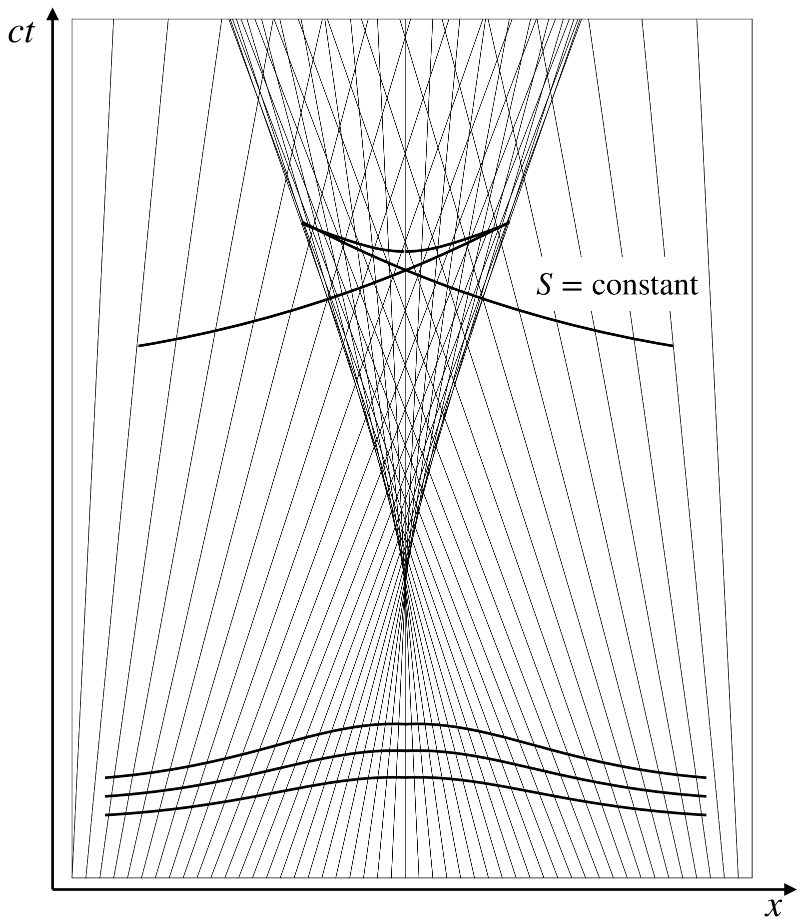


Figure 12: Space-time diagram of trajectories of particles (thin lines) that are focussing shows that, in general, caustics will form. These bound the multi-streaming region. Note that the density ρ of trajectories becomes very large close to the caustics; one can easily show that the density is singular and falls off inversely as the square root of the distance from the caustic for these so-called 'fold-catastrophes'. The heavy curves show hypersurfaces of constant action S for these particles. These are orthogonal to the trajectories of the particles in the special relativistic sense. A classical scalar field $\phi \propto \sqrt{\rho}\cos(S(\vec{x})/\hbar)$ solves, in the geometric optics limit, the Klein-Gordon equation. So wave-fronts (or nodes) of the field are surfaces of constant action. Outside the caustics, there is, locally, a single beam. Inside, in the multi-streaming region, we have a superposition of multiple beams (three here). These beams will interfere wave-mechanically and the density will show interference patterns. If we have many overlapping beams the energy density will be 'speckly'.

The answer is that there will be interference. At any point in the multi-streaming region, if we were to make a Fourier transform of the field within some region (much bigger than the wavelength of the waves, lets say) then we will see spikes at the frequencies of the different overlapping streams – indeed, one can show that the power spectrum obtained by squaring this corresponds to the phase-space density of the corresponding particles. This is illustrated in figure 13 which shows the result of a numerical simulation of structure formation in a universe dominated by so-called 'fuzzy' dark matter (a classical scalar field with Compton wavelength of order a fraction of a parsec).

7.9 Evolution of classical scalar fields via the Schrödinger equation

The Klein-Gordon equation was proposed by Schrödinger to describe ψ, the quantum mechanical wave function of a particle. He obtained it by replacing H and $\mathbf{p}$ in the relativistic energy-momentum relation by the operators $i\hbar\partial_t$ and $-i\hbar\boldsymbol{\nabla}$. In response to the problem that the probability density $\rho = \psi\psi^\star$ is not generally conserved he dropped this in favour of what we usually call the Schrödinger equation, which is obtained by taking the non-relativistic limit. Here the KG equation is considered as that obeyed by a classical scalar field ϕ but, if we are considering such a field as the dark matter, we can similarly use the Schrödinger equation. This is useful in numerical simulation as the Schrödinger field evolves less rapidly than the scalar field. It is also useful conceptually as it shows that, in this limit, the field has, in addition to the 4-conserved quantities $\int d^3r T^{0\mu}$ (the total energy and 3-momentum), a 5th conserved quantity that corresponds to particle number. It is also useful as it makes it somewhat simpler to understand phenomenology such as the speckly nature of the DM in the multi-streaming regime.

7.9.1 From Klein-Gordon equation to the Schrödinger equation

The KG equation in flat space-time is $\Box\phi = m^2\phi$ which, for slowly spatially varying fields becomes $\ddot{\phi} = -c^2m^2\phi$ with solutions $\phi \propto \mathrm{Re}\, e^{i\mu t} = \cos(\mu t)$ where $\mu = mc$ is the Compton (angular) frequency.

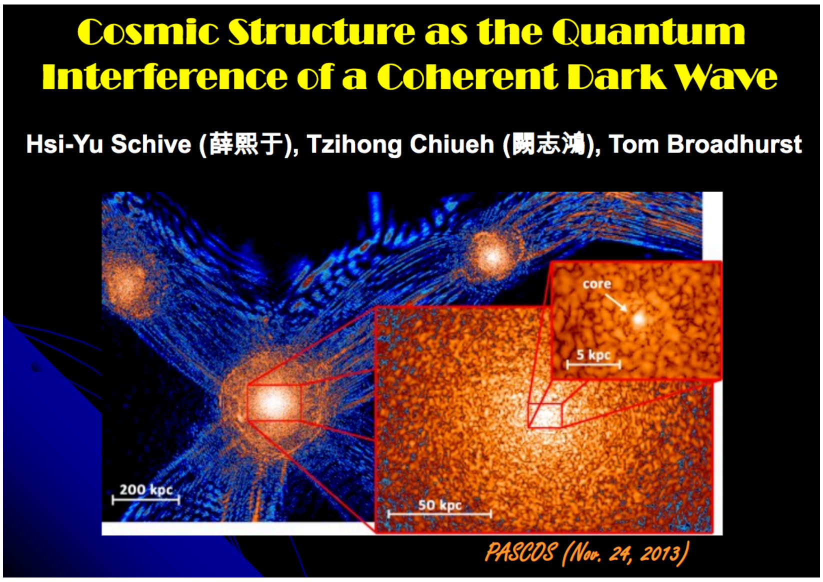


Figure 13: Result of a numerical simulation that evolves a classical scalar field in the gravitational potential that is the solution of Poisson's equation sourced by the effective mass density $\rho = m^2\langle\phi^2\rangle/c^2$ of the scalar field. In fact, these results were obtained by solving the Schrödinger equation, and using $\psi\psi^\star$ for the density. In the low density regions (blue) one can see the interference of 3-waves in what would be, for particles, the 3-stream region. The denser regions (orange) are where, for particles, there would be many overlapping streams of particles so we have interference of multiple 'beams' and this is what gives rise to the characteristic speckly pattern.

To obtain the Schrödinger equation from the KG equation we simply 'factor out' the rapidly oscillating factor $e^{i\mu t}$ and set

$$\phi(\mathbf{r}, t) = \psi(\mathbf{r}, t)e^{-i\mu t} + \psi^\star(\mathbf{r}, t)e^{i\mu t} \tag{164}$$

where $\psi(\mathbf{r}, t)$ is a slowly varying field. By construction ϕ is real. Taking the time derivative of the field gives

$$\dot\phi = -i\mu\psi e^{-i\mu t} + i\mu\psi^\star e^{i\mu t} + \dot\psi e^{-i\mu t} + \dot\psi^\star e^{i\mu t} \tag{165}$$

where the first two terms here are much larger than the last two. Taking a further time derivative yields

$$\ddot\phi = -\mu^2(\psi e^{-i\mu t} + \psi^\star e^{i\mu t}) - 2i\mu(\dot\psi e^{-i\mu t} - \dot\psi^\star e^{i\mu t}) + \mathcal{O}(\ddot\psi e^{i\mu t}). \tag{166}$$

The Laplacian of the field is

$$\nabla^2\phi = \nabla^2\psi e^{-i\mu t} + \nabla^2\psi^\star e^{i\mu t} \tag{167}$$

and combining these in the KG equation $\Box\phi = m^2\phi$ or $\ddot\phi - c^2\nabla^2\phi + \mu^2\phi = 0$, and dropping the term in $\ddot\phi$ involving $\ddot\psi$ gives

$$2i\mu(\dot\psi e^{i\mu t} - \dot\psi^\star e^{-i\mu t}) + c^2\nabla^2\psi e^{i\mu t} - c^2\nabla^2\psi^\star e^{-i\mu t} = 0. \tag{168}$$

Since ψ is supposed to be relatively slowly varying compared to $e^{i\mu t}$ this requires that both the coefficient of $e^{i\mu t}$ and of $e^{-i\mu t}$ must vanish, which means that $2i\mu\dot\psi = -c^2\nabla^2\psi$ or

$$i\dot\psi = -\frac{c}{2m}\nabla^2\psi \tag{169}$$

which is just the Schrödinger equation for a free particle of mass $M = \hbar m/c$.

In a Newtonian gravitational potential φ, the Klein-Gordon equation becomes $\Box\phi = (1 + 2\Phi)m^2\phi$ with $\Phi = \varphi/c^2$ or $\ddot\phi - c^2\nabla^2\phi + (1 + 2\Phi)\mu^2\phi = 0$ and, again ignoring the term involving $\ddot\psi$, this requires $2i\mu\dot\psi = -c^2\nabla^2\psi + 2\mu^2\Phi\psi$ or equivalently

$$\boxed{i\hbar\dot\psi = \frac{1}{2M}|-i\hbar\boldsymbol{\nabla}|^2\psi + V\psi} \tag{170}$$

where $V = m\varphi$ is the gravitational potential energy of a particle and we recognise this as the operator equivalent of $H\psi = (|\mathbf{p}|^2/2M + V)\psi$.

It may seem strange that we have been able to replace the KG equation, which is second order in time, and therefore requires that one specify both ϕ and $\dot\phi$ as initial conditions to obtain a solution, by one that is first order in time, and therefore only requires that one specify the initial field ψ. But this is quite reasonable when we count degrees of freedom since ψ has both a real and imaginary part. Also, one may note that our starting point (164) does not allow one to determine ψ given the initial field ϕ alone, but, if augmented by $\dot\phi = -i\mu\psi e^{-i\mu t} + i\mu\psi^\star e^{i\mu t}$, which we obtain by taking the dominant terms in (165), we have, at $t = 0$, $\psi = (\phi - \dot\phi/i\mu)/2$.

7.9.2 The 5th conservation law: conservation of particle number

A scalar field obeys, in general, 4 conservation laws (or continuity equations); those of energy and the 3 components of spatial momentum. But, for non relativistic particles – where the energy of the particles is too small to create new particles in collisions – there is a 5th conservation law; that of the number of particles.

The corresponding law for a non-relativistic *field* is the law of conservation of total probability (if we think of ψ as a wave-function whose squared modulus gives the probability to find the particle). This is

$$\dot{\rho} + \boldsymbol{\nabla} \cdot \mathbf{j} = 0 \tag{171}$$

where

$$\begin{aligned} \rho &= \psi\psi^\star \\ \mathbf{j} &= \frac{ic}{2m}(\psi^\star \boldsymbol{\nabla}\psi - \psi\boldsymbol{\nabla}\psi^\star) \end{aligned} \tag{172}$$

the latter being the usual Schrödinger 3-current density.

7.9.3 Speckles and phase vortices from the Schrödinger perspective

The Schrödinger formalism gives an interesting perspective on the structure of the density field as illustrated in figure 14 and described in the caption.

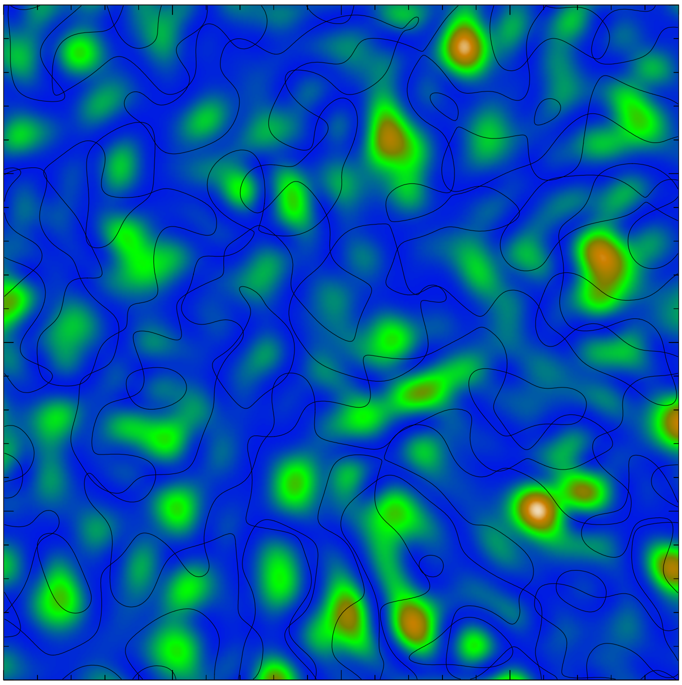


Figure 14: In the strongly multi-streaming regime the Schrödinger field will be the sum of many independent 'beams' coming from various directions. The real and imaginary parts of ψ will then behave – by virtue of the central limit – as Gaussian random fields. The coherence length of these randomly fluctuating field is on the order of the de Broglie wavelength. The real part $\mathrm{Re}\,\psi$ will vanish on one set of 2-dimensional surfaces and $\mathrm{Im}\,\psi$ vanishes on another. In the single stream region these are interleaved so the density $\rho = |\psi|^2$ can never vanish. But in a multi-stream region, $\mathrm{Re}\,\psi$ and $\mathrm{Im}\,\psi$ become effectively statistically independent, and the surfaces will cross, which means that ρ will vanish on a set of lines. This is illustrated at left, where the colour image is the density $\rho = |\psi|^2$ on a 2-dimensional slice, and the lines are contours of zero $\mathrm{Re}\,\psi$ and $\mathrm{Im}\,\psi$.

It is not difficult to show that, if we write the field as $\psi = \sqrt{\rho}e^{i\theta}$, the phase θ will wrap by 2π if one follows a loop around one of the lines where ρ vanishes; these lines are *'phase-vortices'*.

One can also show that the Schrödinger current is proportional to ρ times the gradient of θ. That means that if we define the velocity as $\mathbf{v} = \mathbf{j}/\rho$, this is divergent; tending to infinity inversely with distance from the $\rho = 0$ line.

A Curvature of $t =$ constant surfaces for a uniform density sphere

- We consider again the uniform density sphere
  - this could be a useful approximation for observers in the centre of a dark-matter dominated galaxy or cluster
- the potential is then $\Phi(\mathbf{r}) = \Phi_0 + (2/3)\pi G\rho r^2/c^2$ where $\Phi_0 \equiv \phi(\mathbf{r} = 0)$

- consider the equatorial plane $z = 0$
- and define polar coordinates $r = \sqrt{x^2 + y^2}$ and $\theta = \tan^{-1} y/x$
- in terms of which $dx^2 + dy^2 = dr^2 + r^2 d\theta^2$
- the *proper circumference* of a disk with boundary $\sqrt{x^2 + y^2} = r$ is
  - $l_\theta = \int (ds/d\theta) d\theta = \int d\theta \sqrt{r^2(1 - 2\phi(r))} \simeq 2\pi r(1 - \phi(r)) = 2\pi r(1 - \phi_0 - (2/3)\pi\rho r^2)$
  - where we have used $\sqrt{1 - 2\phi} \to 1 - \phi$ as we are working to first order precision in ϕ
- while the *proper radius* is
  - $l_r = \int dr(ds/dr) = \int dr \sqrt{1 - 2\phi(r)} = \int dr(1 - \phi_0 - (2/3)\pi G\rho r^2) = r(1 - \phi_0 - (2/9)\pi G\rho r^2)$
- so the ratio of the circumference to the radius is, keeping only terms linear in ϕ,
  - $\boxed{l_\theta / l_r = 2\pi(1 - (4/9)\pi G\rho r^2/c^2)}$
  - $l_\theta / l_r < 2\pi$ so evidently these 2D spatial surfaces are *positively curved*
  - the same being true for the 3D hyper-surfaces of constant t – as one can infer from a similar calculation of the volume contained within some coordinate and of the surface area bounding that volume
  - this is something our observers would appreciate in constructing the lattice that supports them – they would have to adjust the length of the rods to accommodate the curved spatial geometry

B The Madelung equation for scalar fields

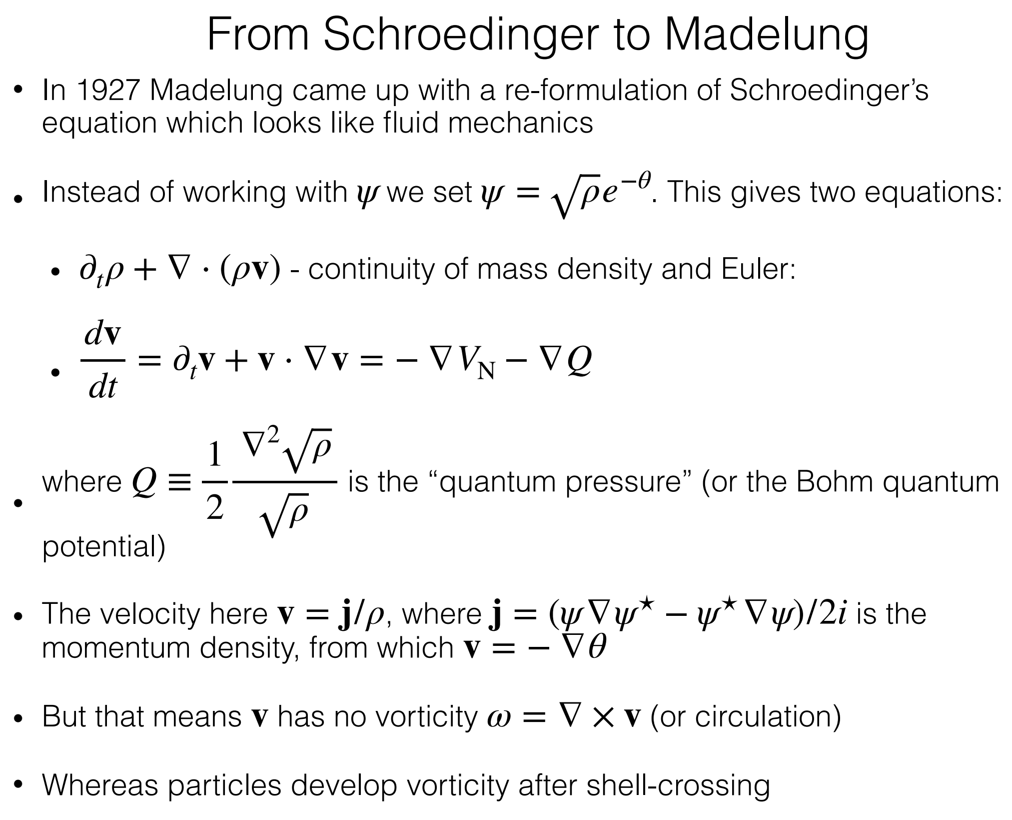


Figure 15: An alternative approach in the non-relativistic regime is to use the Madelung equation.

C Problems

C.1 Problem: self-focusing of a beam of light

Q: Consider a flash-light in the lab emitting a cylindrical beam of light with uniform energy density. Write down the stress tensor and solve for the weak-field metric in the Lorenz gauge. Use this to calculate the geodesic focussing of a set of (transparent) test particles within the beam lying on a circle perpendicular to the beam. Consider two cases: a) where the ring of test particles is initially at rest in the lab-frame and b) where the ring is moving at some velocity v along the beam axis. Compute 2nd derivative of the radius with respect to coordinate time t and relate this to the 2nd derivative with respect to proper time. What does this imply for the rate of self-focusing of the beam? Interpret the result physically (hint: think of the beam as composed of highly relativistic massive particles, rather than massless photons and relate the energy density in the frame of the beam to the lab-frame energy density.) Consider the case of two identical overlapping beams propagating in opposite directions. Do these beams focus one another?