ENS M1 General Relativity - Lecture 6 - Gravitational Waves

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1 Introduction

Einstein's field equations are $\mathbf{G} = 8\pi\kappa\mathbf{T}$. In the weak field limit, with $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, and in terms of the trace-reversed metric perturbations $\overline{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}h\eta_{\mu\nu}$, and in the Lorenz gauge, the Einstein tensor is $G_{\mu\nu} = -\Box\overline{h}_{\mu\nu}$ so we have

$$\Box\overline{h}_{\mu\nu} = -16\pi\kappa T_{\mu\nu}. \tag{1}$$

Here we will explore the wave-like solutions to this equation in empty space ($\mathbf{T} = 0$).

2 Propagation of GWs

2.1 Plane-wave solutions

Our starting point is the Einstein field equations

$$\boxed{\Box\overline{h}^{\alpha\beta} = 0} \tag{2}$$

where $\Box\overline{h}^{\alpha\beta} \equiv \overline{h}^{\alpha\beta,\mu}{}_{,\mu} = \left(-c^{-2}\frac{\partial^2}{\partial t^2} + \nabla^2\right)\overline{h}^{\alpha\beta}$.

This was obtained by adopting the *Lorenz gauge* in which the 4-divergence of the metric perturbation vanishes: $\overline{h}^{\alpha\beta}{}_{,\beta} = 0$.

Equation (2) admits travelling planar wave solutions of the form

$$\boxed{\overline{h}^{\alpha\beta} = A^{\alpha\beta}e^{ik_\mu x^\mu}} \tag{3}$$

where $A^{\alpha\beta}$ is a constant tensor with complex components and $\tilde{k} \to (-\omega/c, \mathbf{k})$ is a constant 1-form.

The understanding here, as usual, is that $\overline{h}^{\alpha\beta}$ is the real part of (3). Or that it is shorthand for the less sloppy

$$\overline{h}^{\alpha\beta} = \tfrac{1}{2}(A^{\alpha\beta}e^{ik_\mu x^\mu} + A^{*\alpha\beta}e^{-ik_\mu x^\mu}). \tag{4}$$

With this trial solution, the field equations become

$$\Box h^{\alpha\beta} = \overline{h}^{\alpha\beta,\mu}{}_{,\mu} = \eta^{\mu\nu}\overline{h}^{\alpha\beta}{}_{,\mu,\nu} = \eta^{\mu\nu}(ik_\mu)(ik_\nu)\overline{h}^{\alpha\beta} = 0. \tag{5}$$

which gives the constraint on the 4-wave-vector

$$\boxed{k_\mu k^\mu = 0} \tag{6}$$

Viewed as a *dispersion relation:* (6) says

$$\boxed{\omega^2 = c^2|\mathbf{k}|^2} \tag{7}$$

so the solutions are dispersion free. They have phase velocity $v_{\text{phase}} = \omega/k$ and group velocity $v_{\text{group}} = d\omega/dk$ both equal to c.

Equation (6) also tells us, as expected, that the wave-vector is null, and is therefore determined by the spatial wave-momentum $\mathbf{k}$.

Note that the field equations themselves impose no constraints on the components of the amplitude $A^{\alpha\beta}$ beyond that implied by the choice of gauge.

The general solution of the field equations (2) is the sum of plane waves with different $\mathbf{k}$:

$$\boxed{\overline{h}^{\alpha\beta} = \sum_{\mathbf{k}} A_{\mathbf{k}}^{\alpha\beta}e^{ik_\mu x^\mu}} \tag{8}$$

where, as usual, we assume periodic boundary conditions within a cubical box of side L, so the modes $\mathbf{k}$ live on a cubical lattice with spacing $\Delta k = 2\pi/L$.

2.2 The wave amplitude

The amplitude $A^{\alpha\beta}$, like the metric itself, is symmetric and is therefore determined by 10 numbers.

The gauge conditions $\overline{h}^{\alpha\beta}{}_{,\beta} = 0$, however, that we have used to obtain the field equations in the above form, imply, for a plane wave, where differentiating with respect to x^α is equivalent to multiplying by ik_α, that

$$A^{\alpha\beta} k_\beta = 0 \tag{9}$$

which places 4 linear constraints on $A^{\alpha\beta}$.

For the case that the 3-momentum is aligned with the $z = x^3$ axis, so $k_\beta = (-\omega, 0, 0, \omega)/c$, these are that $A^{\alpha 0} = A^{\alpha 3} = A^{0\alpha} = A^{3\alpha}$ for $\alpha = 0, 1, 2, 3$ or

$$A^{\alpha\beta} = \left[\begin{array}{c|c} A_{33} & A_{zi} \\ \hline & \\ A_{jz} & A_{ij} \\ & \end{array} \right]. \tag{10}$$

That might suggest that the wave amplitude is specified by six numbers, which might be taken to be the diagonal and upper right components of the 3×3 matrix A_{ij}, the other components being determined by symmetry and the gauge constraints.

But, in fact, the gravitational field in vacuum (i.e. GWs) has only *two* physical degrees of freedom and, in fact, can be written as

$$A_{\alpha\beta} = \left[\begin{array}{cccc} & & & \\ & A_+ & A_\times & \\ & A_\times & -A_+ & \\ & & & \end{array} \right]. \tag{11}$$

In order to show this we need to invoke a further coordinate (gauge) transformation to what is called the *transverse-traceless* gauge.

This is possible because the transformation we used above to simplify the field equations is not the most general transformation consistent with $\overline{h}^{\alpha\beta}{}_{,\beta} = 0$.

This is a little complex, but the payoff is substantial as it gives a coordinate system in which the metric shows transparently the physical degrees of freedom of a GW and also makes it relatively easy to infer the physical effects of such waves.

2.3 The Lorenz gauge (review)

Recall that if we make a transformation from an 'old' coordinate system $\vec{x}^{\,\mathrm{o}} \to x^\alpha$, with metric $g^{\mathrm{o}}_{\alpha\beta} = \eta_{\alpha\beta} + h^{\mathrm{o}}_{\alpha\beta}$, to a 'new' system $\vec{x}^{\,\mathrm{n}} \to x^{\alpha'} = r^{\alpha'}$ with $r^\alpha = x^\alpha + \xi^\alpha$, the differentials transform as $dx^{\alpha'} = \Lambda^{\alpha'}{}_\alpha dx^\alpha$ with the transformation matrix $\Lambda^{\alpha'}{}_\alpha = \delta^{\alpha'}{}_\alpha + \xi^{\alpha'}{}_{,\alpha}$, while the inverse transformation is $dx^\alpha = \Lambda^\alpha{}_{\alpha'} dx^{\alpha'}$ with, for $\xi^\alpha{}_{,\mu} \ll 1$, $\Lambda^\alpha{}_{\alpha'} = \delta^\alpha{}_{\alpha'} - \xi^\alpha{}_{,\alpha'}$.

The metric transforms (in order to maintain invariance of the squared proper interval $ds^2 = g^{\mathrm{o}}_{\alpha\beta} dx^\alpha dx^\beta = g^{\mathrm{n}}_{\alpha'\beta'} dx^{\alpha'} dx^{\beta'}$) as

$$\begin{aligned} g^{\mathrm{n}}_{\alpha'\beta'} &= \eta_{\alpha'\beta'} + h^{\mathrm{n}}_{\alpha'\beta'} \\ &= \Lambda^\alpha{}_{\alpha'} \Lambda^\beta{}_{\beta'} g^{\mathrm{o}}_{\alpha\beta} \\ &= (\delta^\alpha{}_{\alpha'} - \xi^\alpha{}_{,\alpha'})(\delta^\beta{}_{\beta'} - \xi^\beta{}_{,\beta'})(\eta_{\alpha\beta} + h^{\mathrm{o}}_{\alpha\beta}) \end{aligned} \tag{12}$$

Or, multiplying these factors and keeping only terms of linear order,

$$h^{\mathrm{n}}_{\alpha\beta} = h^{\mathrm{o}}_{\alpha\beta} - \xi_{\alpha,\beta} - \xi_{\beta,\alpha} \tag{13}$$

from which the transformation of the trace $h \equiv h^\alpha{}_\alpha = \eta^{\alpha\beta} h_{\alpha\beta}$ is

$$h^{\mathrm{n}} = h^{\mathrm{o}} - 2\xi^\alpha{}_{,\alpha} \tag{14}$$

from which the transformation of the trace-reversed perturbation $\overline{h}_{\alpha,\beta} \equiv h_{\alpha,\beta} - \eta_{\alpha\beta} h/2$ is

$$\overline{h}^{\mathrm{n}}_{\alpha\beta} = \overline{h}^{\mathrm{o}}_{\alpha\beta} - \xi_{\alpha,\beta} - \xi_{\beta,\alpha} + \eta_{\alpha\beta} \xi^\mu{}_{,\mu} \tag{15}$$

and from which, finally, the transformation of the 4-divergence $\overline{h}_{\alpha\beta}{}^{,\beta}$ is

$$\overline{h}^{\mathrm{n}}_{\alpha\beta}{}^{,\beta} = \overline{h}^{\mathrm{o}}_{\alpha\beta}{}^{,\beta} - \xi_{\alpha,\beta}{}^{,\beta} = \overline{h}^{\mathrm{o}}_{\alpha\beta}{}^{,\beta} - \Box\xi_{\alpha} \tag{16}$$

To obtain the field equations as in (2) we adopted the 4 Lorenz gauge conditions

$$\boxed{\overline{h}^{\mathrm{n}}_{\alpha\beta}{}^{,\beta} = 0} \tag{17}$$

the existence of such a gauge following from there being solutions ξ_α to $\Box\,\xi_\alpha = \overline{h}^{\mathrm{o}}_{\alpha\beta}{}^{,\beta}$.

2.4 The transverse-traceless gauge

Adopting the Lorenz gauge does not exhaust the possibilities since to any particular set of solutions ξ_α we can add an additional set of solutions ζ_α to the homogeneous equations $\Box\zeta_\alpha = 0$ without spoiling the condition $\overline{h}^{\mathrm{n}}_{\alpha\beta}{}^{,\beta} = 0$.

Taking ζ_α to be travelling wave solutions with the same wave-vector k_μ as in $\overline{h}_{\alpha\beta} = A_{\alpha\beta}e^{ik_\mu x^\mu}$ allows one to enforce (proof below) the additional constraints on the wave amplitude

- $A^\alpha{}_\alpha = 0$ and
- $A_{\alpha\beta}U^\beta = 0$

where U^β is a constant normalised time-like vector – which we will take to be the 4-velocity of the observer.

Note while $A_{\alpha\beta}U^\beta = 0$ may look like 4 constraints as α can be 0,1,2 or 3, it is really only 3 extra constraints since the linear combination $k^\alpha A_{\alpha\beta}U^\beta = \sum_\alpha k^\alpha(A_{\alpha\beta}U^\beta)$ vanishes by virtue of the original gauge transformation $\vec{x} \to \vec{x} + \xi$ which we used to enforce $\overline{h}^{\alpha\beta}{}_{,\beta} = 0$ which implies $A_{\alpha\beta}k^\beta = 0$.

This gives a total of 8 constraints, leaving 2 degrees of freedom for the wave amplitude.

Taking $U^\beta = \delta^\beta{}_0$ (one can always boost into a frame where this is true) implies $A_{\alpha 0} = 0$ for all α.

And taking the wave to be travelling along the z-axis in that frame, so $\tilde{k} \to (-\omega, 0, 0, \omega)/c$, the conditions $A_{\alpha 0} = A_{0\alpha} = 0$, together with $A_{\alpha\beta}k^\beta = 0$, imply $A_{\alpha z} = 0$ and $A_{z\alpha} = 0$, so the only non-zero components of $A_{\alpha\beta}$ are the four with $\alpha, \beta = (x, y)$:

$$A_{\alpha\beta} = \begin{bmatrix} & & & \\ & A_{xx} & A_{xy} & \\ & A_{yx} & A_{yy} & \\ & & & \end{bmatrix} \tag{18}$$

But the trace condition $A^\alpha{}_\alpha = 0$ (which, incidentally, forces $\overline{h}_{\alpha\beta} = h_{\alpha\beta}$) implies $A_{yy} = -A_{xx}$ so we are led to the metric in the *transverse-traceless* gauge:

$$g_{\alpha\beta} = \eta_{\alpha\beta} + h^{\mathrm{TT}}_{\alpha\beta} = \eta_{\alpha\beta} + A^{\mathrm{TT}}_{\alpha\beta}e^{ik_\mu x^\mu} \tag{19}$$

with amplitude

$$A^{\mathrm{TT}}_{\alpha\beta} = \begin{bmatrix} & & & \\ & A_+ & A_\times & \\ & A_\times & -A_+ & \\ & & & \end{bmatrix} \tag{20}$$

where $A_+ \equiv A_{xx} = -A_{yy}$ and $A_\times \equiv A_{xy} = A_{yx}$.

The curvature tensor is readily calculated in terms of A_+ and $A_\times$ using

$$R_{\alpha\beta\mu\nu} = [(-\tfrac{1}{2}h_{\alpha\mu,\beta\nu}) - \{\alpha \Leftrightarrow \beta\}] - \{\mu \Leftrightarrow \nu\} \tag{21}$$

From we find, for instance, that $R_{0xx0} = (\omega^2/c^2)A_+/2$ and $R_{0xy0} = (\omega^2 c^2)A_\times/2$.

But since $R_{\alpha\beta\mu\nu}$ is gauge invariant, this means that there is no further freedom to 'gauge away' the wave amplitude coefficients thus proving that these travelling wave vacuum solutions do indeed have 2 physical degrees of freedom (or 'polarisation states')

2.5 Proof that we can choose a gauge such that $A^\alpha{}_\alpha = 0$ and $A_{\alpha\beta}U^\beta = 0$

Writing our previous Lorenz gauge (or 'new') solution as $\overline{h}^{\rm n}_{\alpha\beta} = \overline{h}^{\rm O}_{\alpha\beta}$, this now being the 'Old' solution, and applying the transformation $\vec{x} \to \vec{x} + \vec{\zeta}$ to obtain the N=New solution in the TT-gauge, the trace-reversed metric perturbation changes according to

$$\overline{h}^{\rm N}_{\alpha\beta} = \overline{h}^{\rm O}_{\alpha\beta} - \zeta_{\alpha,\beta} - \zeta_{\beta,\alpha} + \eta_{\alpha\beta}\zeta^\mu{}_{,\mu}. \tag{22}$$

Let us consider a single plane wave, and choose $\zeta_\alpha = B_\alpha e^{ik_\mu x^\mu}$ with B_α a complex amplitude.

These are solutions of the homogeneous wave equation $\Box\zeta_\alpha = 0$, so the total gauge transformation $\vec{x} \to \vec{x} + \vec{\xi} + \vec{\zeta}$ should still maintain $\overline{h}^{\mu\nu}{}_{,\nu}$.

Then, for the plane-wave, where differentiation with respect to x^α becomes multiplication by ik_α, we find that the amplitude changes as

$$A^{\rm N}_{\alpha\beta} = A^{\rm O}_{\alpha\beta} - ik_\beta B_\alpha - ik_\alpha B_\beta + i\eta_{\alpha\beta}B^\mu k_\mu. \tag{23}$$

Note that multiplying by k^β the second term vanishes, since $k_\beta k^\beta = 0$, and the last two terms cancel, so this verifies that the Lorenz condition $A_{\alpha\beta}k^\beta = 0$ applies to both old and new amplitudes.

Demanding that the trace of the new amplitude vanishes: $A^{\rm N} \equiv A^{{\rm N}\alpha}{}_\alpha = \eta^{\alpha\beta}A^{\rm N}_{\alpha\beta} = 0$ gives

$$A^{\rm O} \equiv A^{{\rm O}\alpha}{}_\alpha = \eta^{\alpha\beta}(ik_\beta B_\alpha + ik_\alpha B_\beta - i\eta_{\alpha\beta}B^\mu k_\mu) = 2ik_\alpha B^\alpha - i\delta^\alpha{}_\alpha B^\mu k_\mu = -2ik_\alpha B^\alpha \tag{24}$$

since $\delta^\alpha{}_\alpha = 4$.

This allows one to replace the first and last terms in the expression for $A^{\rm N}_{\alpha\beta}$ above by the trace reverse of $A^{\rm O}_{\alpha\beta}$: $\overline{A}^{\rm O}_{\alpha\beta} \equiv A^{\rm O}_{\alpha\beta} - \frac{1}{2}\eta_{\alpha\beta}A^{{\rm O}\mu}{}_\mu$, giving

$$A^{\rm N}_{\alpha\beta} = \overline{A}^{\rm O}_{\alpha\beta} - ik_\beta B_\alpha - ik_\alpha B_\beta. \tag{25}$$

Note here that, since $A^{\rm O}_{\alpha\beta}$ was defined as the amplitude of the trace-reversed metric perturbation $\overline{h}_{\alpha\beta}$, $\overline{A}^{\rm O}_{\alpha\beta}$ is actually, and confusingly, the amplitude of the non-trace-reversed metric perturbation $h_{\alpha\beta}$.

Contracting with the vector U^β and requiring that $A^{\rm N}_{\alpha\beta}U^\beta = 0$ gives

$$i(k_\beta B_\alpha + k_\alpha B_\beta)U^\beta = \overline{A}^{\rm O}_{\alpha\beta}U^\beta. \tag{26}$$

Now if $\vec{U}$ is time-like and independent of position then we can always perform a global Lorentz boost into the frame where $U^\beta = \delta^\beta_0$ – the rest-frame of $\vec{U}$ – and then solve for $\vec{B}$ in this frame:

- setting $\alpha = 0$ we find $\quad B_0 = \overline{A}^{\rm O}_{00}/2ik_0$
- while for $\alpha = i$ we have $\quad B_i = (\overline{A}^{\rm O}_{i0} + (k_i/2k_0)\overline{A}^{\rm O}_{00})/ik_0$

So this establishes that by making the wave-like coordinate transformation $x_\alpha \to x_\alpha + B_\alpha e^{ik_\mu x^\mu}$, we can indeed obtain a coordinate system in which $A^\alpha{}_\alpha = 0$ and $A_{\alpha\beta}U^\beta = 0$. QED.

The latter condition constrains the amplitude $A_{\alpha\beta}$ for the metric to be the 2×2 symmetric form (18), while the former enforces $A_{xx} = -A_{yy}$ giving the final 'transverse-traceless' amplitude (20).

That was for a single plane wave. In the general situation we have a superposition of plane waves. In that case we use the transformation $\vec{x} \to \vec{x} + \vec{\zeta}$ with $\vec{\zeta}$ being a superposition of plane waves (i.e. a Fourier synthesis) where each component enforces the TT gauge conditions for the corresponding component in $A^{\rm N}_{\alpha\beta}$.

Note that this only works when the metric perturbations are actually travelling waves (as is the case in a vacuum). So one cannot use this procedure to 'gauge away' the metric perturbations generated by some general non-vanishing source $T_{\alpha\beta}$

3 Response of particles and matter to GWs

3.1 Geodesic motion:

- the geodesic equation is
  - $dU^\alpha/d\tau + \Gamma^\alpha{}_{\mu\nu} U^\mu U^\nu = 0$
- for a particle initially at rest in this coordinate system $\vec{U} \to (c, 0, 0, 0)$ so
  - $dU^\alpha/d\tau = -c^2 \Gamma^\alpha{}_{00} = -\frac{c^2}{2}\eta^{\alpha\beta}(h_{0\beta,0} + h_{\beta 0,0} - h_{00,\beta})$
- but this vanishes in the TT-gauge, so in a space-time perturbed by gravitational waves (and in the coordinate system obtained via the aforesaid gauge transformations) initially stationary particles remain at constant coordinate location
- thus, it might seem, the wave has no effect on freely falling particles
- that is incorrect; this calculation only shows that the spatial *coordinates* of the particles do not change. What is relevant observationally is the *physical* or *proper* separation between e.g. a pair of freely falling particles.
- if we consider a pair of particles, with (fixed) coordinate separation $\vec{dx} \to (0, dx, 0, 0)$, for example, their proper separation is
  - $dl = \sqrt{g_{\alpha\beta} dx^\alpha dx^\beta} = \sqrt{g_{xx} dx^2} \simeq (1 + h_+^{\mathrm{TT}}/2) dx$
- this is most revealing:
  - the proper separation of two freely falling test particles fluctuates with time as the gravitational wave passes
    * this is directly measurable, for example by comparison with a ruler – that maintains a fixed proper length
  - and the perturbation to their separation scales linearly with the separation dx
    * this is the hallmark of a *tidal field*
- we say that the wave produces a 'strain' – given by the amplitude h of the metric perturbation
- the amplitude of the tidal distortion (defined as the relative acceleration $d^2 l/dt^2$ divided by dx) is $\omega^2 h/2$
  - this is very similar to what is found with the Newtonian limit form of the metric, where the components of h are twice the (dimensionless) Newtonian gravitational potential $\Phi = \phi/c^2$ and the tide, being second spatial derivative $\boldsymbol{\nabla\nabla}\phi = \boldsymbol{\nabla\nabla} h/2c^2$. Here we have wave $h = Ae^{i(\mathbf{k}\cdot\mathbf{x} - \omega t)}$) so the components of $\boldsymbol{\nabla\nabla} h/2c^2$ is essentially the same as $\omega^2 h/2$.
  - note however that a ripple of ϕ is different in that it would cause *longitudinal* tidal distortion whereas the distortion from a GW is transverse
- the identification of these metric fluctuations with ripples of the tidal field – that have somehow escaped from the emitting system – meshes rather nicely with the form of the TT metric perturbations. Similarities include:
  - the Newtonian tidal field tensor ϕ_{ij} is, in empty space, traceless – like A_{ij}
  - which means that it's action on cloud of test particles does not change their volume – like A_{ij}
  - so the sum of the three eigenvalues of ϕ_{ij} vanishes, so the tide in a vacuum is characterised by two numbers (and a 3D rotation matrix giving the spatial orientation)

3.2 The reality and observability of gravitational waves

Questions sometimes arise about the reality and observability of gravitational waves. Even Einstein had some second thoughts about the reality of GWs.

A question that Schutz poses (rhetorically) is this: If rulers are mostly empty space, with nuclei and electrons occupying a tiny volume, will not the atoms in the ruler be pushed apart or compressed by a gravitational wave as it passes? Thus rendering the change in separation for test particles unobservable?

The answer is no: The atoms in solid materials are kept at a fixed physical separation from one another by electrostatic forces and their size is determined by quantum mechanics, so their proper separation does not change (at least if the wave is at a frequency less than the inverse of the sound-crossing time in the ruler). Put another way, the inter-atomic forces mean that the atoms do not follow geodesics.

This means that, for example, two massive weights free to slide along on a metre ruler will definitely move relative to the rule (albeit by a very small amount for realistic amplitude waves). And if there is friction, this motion will generate heat in the ruler. So there is no question that the effect is real and, in principle, observable.

A question of a similar nature arises with regard to interferometric GW detection – we return to that below

I think that the potential for misunderstanding what happens is enhanced by the description of gravitational waves as 'ripples in the geometry of spacetime' or, worse still, ripples in the 'fabric' of space-time. This is very much like the misconception surrounding 'expanding space' in cosmology. It is safer to say that these are ripples of tidal field.

4 Properties of GWs

4.1 Polarisation of gravitational waves

- If only the A_{xx} component is present the separation between a pair of particles with separation in the plane of the wave varies as $\cos 2\varphi$ where ϕ is the angle between the separation vector and the x-axis
- so with the phase of the wave chosen to that $h^{\alpha\beta} = 0$ at $t = 0$ a set of particles that are initially on a circle will deform into an ellipse elongated along the x-axis after a quarter of a cycle and after a further half cycle will be stretched along the y-axis
- similarly, the A_{xy} component causes oscillatory stretching and compression along an axis rotated by 45 degree. I.e. varying as $\sin 2\varphi$
- this is illustrated in figure 1

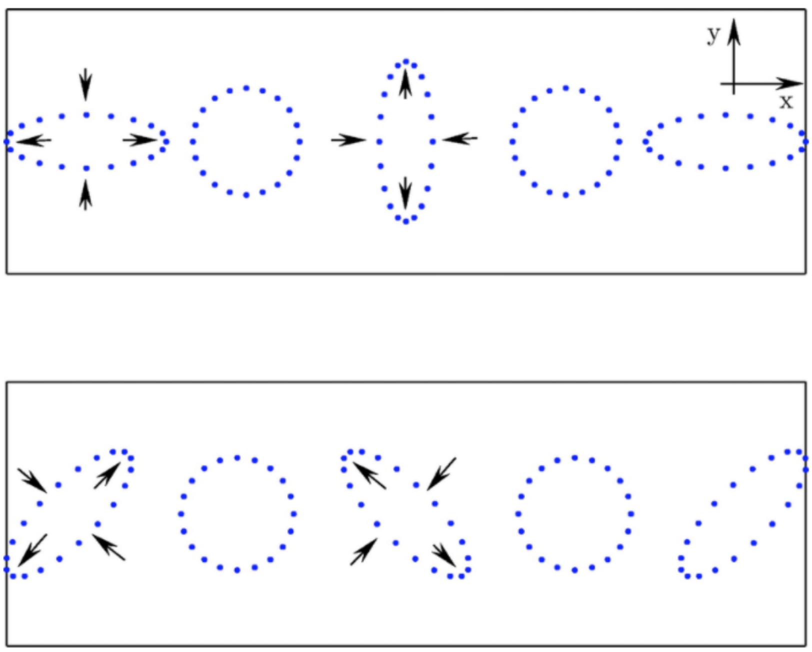

Figure 1: Polarisation of gravitational waves. Upper panel shows a ring of freely test particles in the $x - y$ plane at successive times spaced by on quarter of a period as they are being displaced by '+' polarised wave with wave-vector pointing into the plane of the figure. The lower panel is for the '×' polarisation. The coordinates are 'physical'; i.e. what would be measured with respect to rigid rulers.

Figure 2 raises a puzzling feature of these waves. What if we had a rigid box of depth $\lambda/2$ in the z-direction. This picture would suggest that a pair of particles at the front and back of the box would, unless they happen to be on the axis of the wave at $(x, y) = (0, 0)$, have a relative transverse displacement proportional to $\sqrt{x^2 + y^2}$. Does that make any sense? What if we have a perfectly planar wave – i.e. of infinite extent in the transverse directions – and made the ring of particles very large. Would there be a correspondingly large observable effect? What, for that matter, is special about the 'axis' $(x, y) = (0, 0)$ if the wave is infinite in extent?

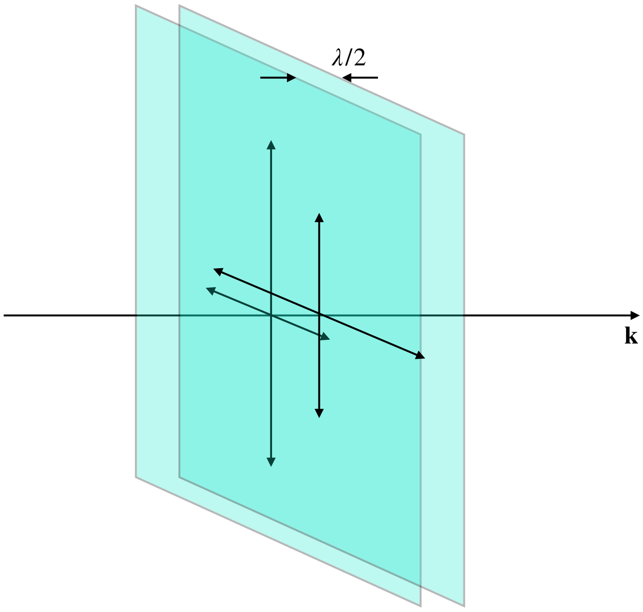

Figure 2: This shows two rigid planes, separated by half a wavelength on which there are test particles free to move, with positions as indicated by the ends of the arrows. It seems that there is a relative transverse displacement that becomes arbitrarily large as one lets the distance from the axis become large. Is that reasonable?

4.2 Energy density of gravitational waves

How much energy is carried by gravitational waves?

The energy density of an *electromagnetic field* is

$$\mathcal{E} = (\epsilon_0|\mathbf{E}|^2 + \mu_0^{-1}|\mathbf{B}|^2)/2 \tag{27}$$

as can be determined by considering the work done in creating the fields. By pulling the plates of a capacitor apart for the $\mathbf{E}$ field or by increasing the current flowing in a solenoid for the case of the $\mathbf{B}$ field.

We can do something similar for a gravitational field. Consider letting a self-gravitating shell of mass contract. That creates some $\mathbf{g} = -\boldsymbol{\nabla}\phi$ gravitational acceleration field where there was none before. But it *releases* energy in the process, and one would infer that the energy density of the field[1] is

$$\mathcal{E} = -\frac{1}{8\pi G}|\boldsymbol{\nabla}\phi|^2. \tag{28}$$

One can also show, as Maxwell did[2] that the momentum flux density – i.e. the stress 3-tensor – for a Newtonian gravitational field is very similar to the Maxwell stress for a $\mathbf{E}$ or $\mathbf{B}$ field:

$$T_{ij} = \frac{1}{8\pi G}(g_i g_j - g^2\delta_{ij}/2) \tag{29}$$

so these seem to give most of the stress energy tensor this way. The problem is that the energy density above is *negative*.

Another reason that defining the energy density associated with the gravitational field is somewhat tricky in GR is that one can one can always 'transform away' the Christoffel symbols – the analogue of the field – at any point.

[1] One can also obtain this from the fact that the gravitational energy of a Newtonian system is $E = \frac{1}{2}\int d^3r\ \phi\rho$ or, since $\nabla^2\phi = 4\pi G\rho$, $E = \frac{1}{8\pi G}\int d^3r\ \phi\nabla^2\phi$ so on integrating by parts, and assuming that the field dies off at infinity, we get $E = -\frac{1}{8\pi G}\int d^3r\ |\boldsymbol{\nabla}\phi|^2$.

[2] It is rather sad that Maxwell said he was put off from pursuing this approach to gravity further by the fact that above the surface of the earth this stress – the momentum flux density – is about 32,000 tons per square inch (a value that Maxwell believed space would not be able to withstand!)

Nonetheless a gravitational wave has a measurable effect on matter. And, for example, massive beads on a rod will oscillate back and forth and, if there is friction, energy will be dissipated and will heat the rod, and that energy has to come from the gravitational wave.

Schutz shows how the radiation re-radiated by a slab of dissipative systems acts – by interfering destructively – to reduce the amplitude of the wave 'downstream'. From this he finds that the energy density for a gravitational wave is

$$\boxed{\mathcal{E} = \frac{c^2}{32\pi G}\omega^2\langle h_{\mu\nu}h^{\mu\nu}\rangle} \tag{30}$$

where $\langle\ldots\rangle$ denotes an average over a volume containing many waves.

That the energy density would take this form is quite reasonable on various grounds:

- it agrees (aside from the sign) with the Newtonian result above if $k^2\langle h_{\mu\nu}h^{\mu\nu}\rangle = |2\boldsymbol{\nabla}\Phi|^2 = |\boldsymbol{\nabla}(2\phi/c)|^2$ which, since multiplying a wave by ik is the same as taking the spatial derivative, matches nicely with what we get in the Newtonian limit metric, where $h = 2\Phi$.

- so the formula for GWs is formally very similar with $\langle\phi^2\rangle$ replaced by the metric perturbation squared.

- it is formally very similar to what one gets for e.g. a massless scalar field, with Lagrangian density $\mathcal{L} = -\phi_{,\mu}\phi^{,\mu}/2$ the stress energy tensor has time-time component – the energy density – with a time average $\omega^2\langle\phi^2\rangle$ or for the EM field.

- it is also broadly in accord with what one would obtain using the linearised metric to estimate the 2nd order terms in the Einstein tensor (recall that in linearising the curvature we dropped terms involving products of Christoffel symbols).

The presence of the factor c^2/G means that a wave can have a lot of energy for a very small strain amplitude.

To put this into dramatic perspective, the observable region of the universe contains about 10^9 galaxies that each contain about 10^{10} stars that are converting mass to light with an efficiency of about 1 percent or less over some fraction of the age of the universe of $\sim 3\times10^{17}$s. So that's about one solar mass of luminous energy per second.

The binary black-hole merger LIGO150914, by comparison, emitted about 3 solar masses of energy in gravitational waves in a small fraction of a second. So it 'outshone' the entire visible universe for that brief time, yet resulted in a strain here of only $h \sim 10^{-21}$

5 Generation of Gravitational Waves

5.1 Generation of EM waves

5.1.1 Dipole radiation

- Accelerating electrical charges radiate EM radiation. For an electrically neutral system the most effective way for it to radiate is if it has an oscillating *dipole moment* $\mathbf{d} = \sum q\mathbf{r}$.

- Larmor's classical formula says the power radiated is $\boxed{P \sim |\ddot{\mathbf{d}}|^2/\epsilon_0 c^3}$

- this can be understood qualitatively as follows:
  - a dipole produces an electrical field that falls off as $1/r^3$: To order of magnitude $E \sim d/\epsilon_0 r^3$ in the 'near-field' region $r \ll \lambda = c/\omega$
  - at the limit of the near-field region – i.e. in the transition to the 'wave-zone' at $r > \lambda$ where the radiation field falls as $E \propto 1/r$ – the field is $E \sim d/\epsilon_0\lambda^3$
  - the field energy in that volume is $\mathcal{E} \sim \epsilon_0 E^2\lambda^3 \sim d^2/\epsilon_0\lambda^3$
  - if we assume that this energy is released as an EM wave in time $t \sim \lambda/c$ the power output is $P = \mathcal{E}/t \sim cd^2/\epsilon_0\lambda^4$ or, since $\omega \sim c/\lambda$ and $\ddot{d} = -\omega^2 d$ the power is $P \sim \ddot{d}^2/\epsilon_0 c^3$ QED

5.1.2 Quadrupole radiation

- if the system does not have a dipole moment, the dominant emission is *electric quadrupole* emission
  - The field around a quadrupole is $E \sim Q/\epsilon_0 r^4$ where the quadrupole moment is $Q \sim \sum qr^2$
    - it is only the fluctuating part of the quadrupole that counts here
    - but what we have in mind is things like binary systems with similar masses in which case we can use $Q = \sum mr^2$ to order of magnitude
  - the EM field energy in the 'transition region' is then $\mathcal{E} \sim Q^2/\epsilon_0\lambda^5$
  - so the power radiated, by the same line of argument as for the dipole, is $\boxed{P \sim \dddot{Q}^2/\epsilon_0 c^5}$
  - for a system of size r – which we have assumed to be smaller than the wavelength of the radiation emitted – the quadrupole power is smaller than the dipole power by a factor $\sim (r/\lambda)^2$

5.2 Quadrupole gravitational radiation

- Electrically charged systems can have a dipole moment by virtue of the fact that the charge-to-mass ratio is different for electrons and ions
- A gravitating system – like a binary star – has no dipole moment, by virtue of the equivalence principle
- so the emission comes primarily from the quadrupole moment
  - The fluctuating part of the gravity at distance $\lambda = c/\omega$ is $g \sim GQ/\lambda^4 \sim GQ\omega^4/c^4$ where the mass quadrupole moment is $Q\sum mr^2$ (or $Q \sim mr^2$ for a binary)
  - with energy density $\epsilon \sim g^2/G$ the energy in volume λ^3 is $\mathcal{E} \sim g^2c^3/G\omega^3$ and if this is emitted in time $1/\omega$ the power is $P \sim G(\omega^3 Q)^2/c^5$ or
    - $\boxed{P \sim G\dddot{Q}^2/c^5}$
- the radiation is emitted at a frequency that is twice the orbital frequency
  - the velocity is $v \sim \sqrt{Gm/r}$ so the frequency is $\omega \sim v/r \sim \sqrt{Gm/r^3}$ (Kepler's law).
  - so $\omega^3 Q \propto r^{-5/2}$ and the power scales with radius as r^{-5}
  - so the power radiated is very weak for large (i.e. non-relativistic) systems

5.3 Estimate of the expected strain

- the strain d in the wave-zone falls off with distance R as $1/R$ (so the energy flux $\propto d^2$ falls of as $1/R^2$) and is equal, to order of magnitude, to the dimensionless Newtonian quadrupolar potential $\phi_{\rm N}/c^2$ at distance $\sim \lambda$ times λ/R, or
  - $h \sim (GMr^2/c^2\lambda^3) \times (\lambda/R) = (GM/rc^2) \times (r^3/\lambda^2 R)$
  - for a black-hole or neutron star binary the first factor reaches unity as the objects merge while the wavelength at that time is on the order of r as the objects are moving at $v \sim c$
  - so for LIGO150914 where $M \sim 30M_\odot$, for which $r \sim 5 \times 10^4$m, at a distance of 400Mpc $\sim 10^{25}$m this very rough estimate predicts a strain $h \sim$ a few $\times 10^{-21}$, not too far from the observed strain which peaked at $h \simeq 10^{-21}$

6 Detection of Gravitational Waves

There are various ways that gravitational waves can be detected in principle. Here we will consider only two: Interferometers – which in the case of LIGO has achieved detections at kilo-Hertz frequencies – and pulsar timing arrays, which are close to being able to detect the nano-Hertz waves expected from merging of massive black hole binaries. First we will briefly mention how the existence of gravitational radiation was inferred observationally from the energy loss from a gravitating system

6.1 The Hulse-Taylor binary pulsar

We saw that the power radiated scales as $1/r^5$. The orbital energy scales as $1/r$. So the 'orbital decay rate' goes like $1/r^4$. It was first measured by Hulse and Taylor for a neutron star binary system called PSR B1913+16, discovered in 1974. One of the stars is a pulsar, which allowed precise measurement of the orbital properties and revealed that, over time, the orbit is shrinking.

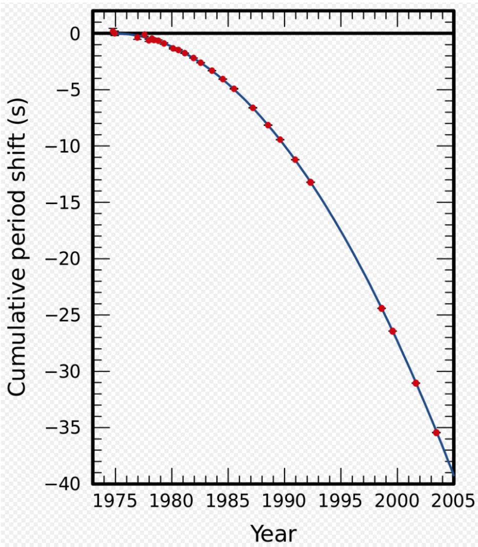

Figure 3: The Hulse-Taylor binary pulsar. The red points are measurements of the cumulative change in the orbital period plotted vs. time. The blue line is the prediction of general relativity. Hulse and Taylor got the Nobel prize for this in 1993.

6.2 Interferometric detection of gravitational waves

- The Michelson interferometer provides a very precise way to measure distances: in its simplest form, light is divided at a beam splitter (a half silvered mirror) and the two perpendicular beams are reflected off mirrors and then recombined on the same beam-splitter plate and the two possible outputs paths monitored. If the path lengths are exactly the same no light will come out of one of the ports as the beams interfere destructively. Any slight difference in the path length will mean that there are slightly more wavelengths along one arm than the other, so the interference will not be perfectly destructive and this will allow a slight amount of light to emerge.
- this type of experiment is well suited to GW detection as waves with an appropriate polarization will stretch one arm and compress the other
- At LIGO the mirrors are suspended and are free to move in the direction along the beam direction.
- The lengths of the beams are of order a few km, so the strain results in an incredibly small change in path length. The amplitude of the effect is enhanced by having the radiation bounce back and forth along the arms many times to multiply the effective arm length.

- This raises an interesting question:
  - gravitational waves are often describes as 'ripples of space-time'
  - in this view, space is expanding and contracting as the wave passes and this is what is causing the change in the separation of the mirrors
  - but expansion of space is also invoked as the cause of the cosmological redshift
    * the wavelength of light, it is often said, is 'stretched' by the expansion of space
  - doesn't this mean that the light will be stretched by the expansion of space by the same amount that the mirrors are moved apart?
  - so the number of waves along each arm will be unaffected by the waves?
  - and there will therefore be no phase shift and therefore no observable effect
  - what is wrong with this argument?

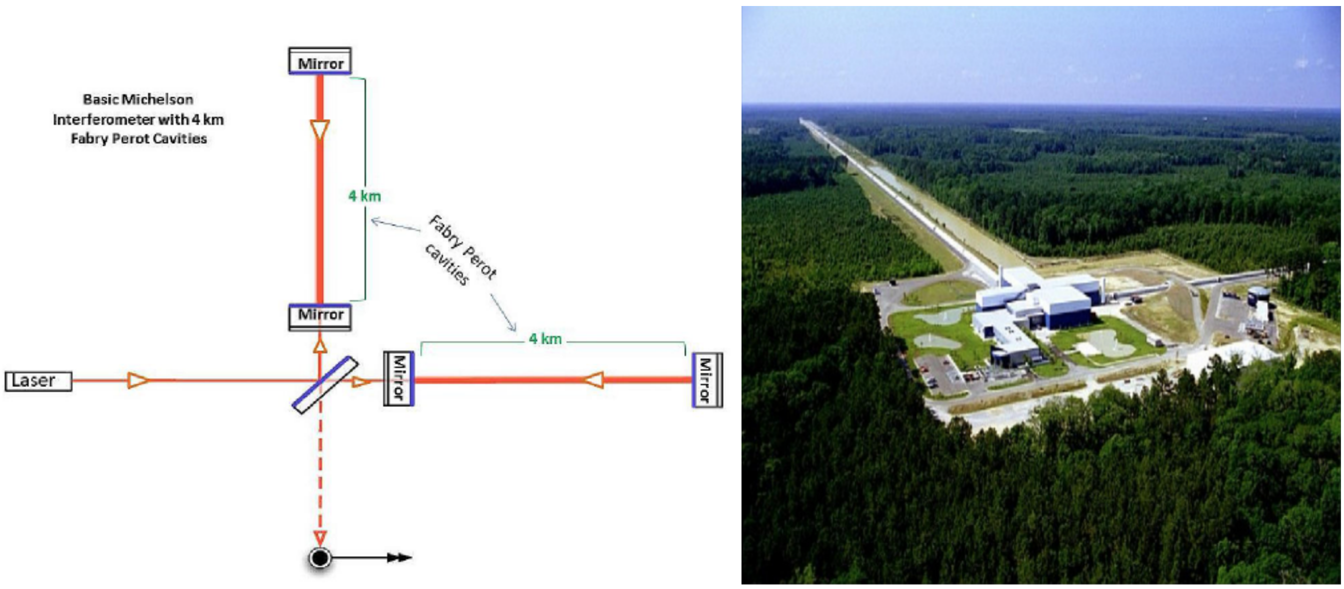

Figure 4: The Large Interferometric Graviatational-wave Observatory (LIGO) is, in essence, a Michelson interferometer; shown schematically at left. Photo at right is the detector at Livingston. The mirrors are suspended on fibres and so behave effectively like inertial test-masses. The Michelson interferometer is very sensitive to tiny changes in the lengths of the arms. If they are identical, no light comes out of the 'dark-port' as there is complete destructive interference. If they are changed slightly by the passage of a GW – which, for the appropriate polarisation – will stretch one arm and contract the other, some light will come out. The extra mirrors in the paths are to increase the effective distance the light travels in order to increase the sensitivity. With this set-up, the LIGO team are able to measure a strain of $h \sim 10^{-21}$. This corresponds to about 10^{-8} of the size of an atom!

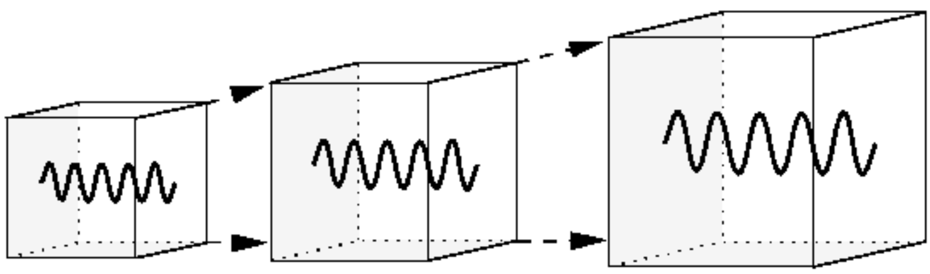

Figure 5: How can LIGO work?

6.3 Pulsar timing arrays

- Pulsars – and particularly the rapidly rotating 'milli-second pulsars' – are exceptionally stable clocks
  - individual pulses have 'arrival time residuals' – random errors in pulse arrival times – but these do not accumulate; there are systems where the pulse arrival times are stable to $\sim$ 100ns accuracy over a baseline of tens of years
  - this makes them as stable as the best terrestrial clocks
- Supermassive black holes live at the centres of most galaxies
- galaxies are thought to have grown 'hierarchically' by the merging of smaller galaxies so presumably the black holes have also merged, and so will have released gravitational wave energy – similar to that which LIGO has detected but of much lower frequency
- if this is broadly correct then from the abundance of supermassive BHs the expectation is for a 'background' of gravitational waves with root-mean-squared strain $\langle h \rangle^2 \sim 10^{-15}$
- this should be detectable by pulsar timing arrays in the foreseeable future

- such arrays are most sensitive, given the expected spectrum, which is very 'red', to waves of the lowest frequency measurable (i.e. periods of order a decade given the observable baseline)

- the effect that these arrays can measure is a time varying systematic drift in the pulse arrival time residuals; this effect – effectively the same as what is called the *Shapiro time delay* in the solar system – is on the order of the metric perturbation time the period of the waves and comes from the local effect of the random sea of waves from distant merging systems that are irradiating us

- one interesting feature of this which helps make this practical is that the waves, while coming from random directions and having random phases, have a distinctive signature in the form of correlation between the timing residuals from different pulsars that is a function of their celestial angular separation – this is called the *Helling-Downs curve*

- this gives the gravitational wave 'background' a 'signature' that makes it distinguishable from other sources of noise or 'nuisance effects'

- current limits – the best being from the Parkes survey – are right about at the level predicted