ENS M1 General Relativity - Lecture 7 - Black-Holes and Stellar Structure

Nick Kaiser

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1 Static spherical space-times

- the line-element for a static, spherical space-time is usually taken to be, in t, r, θ, ϕ coordinates,
  - $\boxed{ds^2 = -e^{2\Phi(r)}dt^2 + e^{2\Lambda(r)}dr^2 + r^2 d\Omega^2}$
  - where $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$ is the intrinsic metric of the unit sphere
  - the metric is diagonal
  - lines of constant r, θ and ϕ are time-like and so represent possible world-lines for particles
    * though these will, in general, be accelerated
  - the time coordinate is assumed to range from $-\infty < t < \infty$
  - θ: it is assumed that $0 \leq \theta \leq \pi$ as for a normal sphere embedded in Euclidean space
  - ϕ; we identify $\phi = 0$ and $\phi = 2\pi$
  - it is not entirely unreasonable to question the requirement for the latter
    * for example, the 2-space with line element $dl^2 = d\theta^2 + \sin^2 d\phi^2$ and with $\phi = 0$ and $\phi = \pi$ identified has exactly the same *local* geometry
    * the rotation of a parallel transported vector is equal to $d\Omega = \int d\theta \sin\theta d\phi = \oint d\phi \cos\theta$
    * also, and interestingly, the metric of space in the vicinity of certain types of cosmic string is locally flat but does not have cylindrical azimuthal angle that ranges from 0 to 2π; the space is missing a thin wedge and is said to be 'conical' (by close analogy with a flat sheet of paper that has been cut and glued to make a cone)
    * so why do we assume we have to identify $\phi = 0$ and $\phi = 2\pi$?
    * the reason is that only with that choice is the surface $r =$ constant truly invariant under rotations; for any other choice the circumference around the equator would be different to the length of a loop passing through the poles, and the global geometry would then violate spherical symmetry
  - writing g_{tt} and g_{rr} as exponentials of $\Phi(r)$ and $\Lambda(r)$ is purely a matter of convenience
  - the fact that we have r^2 multiplying $d\Omega^2$ is also a matter of convention
    * we could have written $f(r)d\Omega^2$ here and made appropriate changes to the functions Φ and Λ to get a line element of otherwise identical form
    * by writing $r^2 d\Omega^2$ we are *defining* r to be such that the physical area of the surface of constant r is $4\pi r^2$
  - in Euclidean space r ranges from 0 to ∞, but here we make no such restrictions
    * we do not require that $r = 0$ be part of the space-time
    * and, in the closed FRW metric, for example, the range of r is finite

- an observer can, in principle, directly determine r from measurements of area or circumference, or, more locally, from summing angles of triangles (i.e. using parallel transport)

- the time coordinate can similarly be operationally defined:
  - in cases of actual interest, such as stars, $\Phi \to 0$ as $r \to \infty$ and one can then identify the coordinate t with the proper time measured by a constant-r observer at large r
  - one can also extend this to establish the time coordinate for all observers
    * without going into too many details, the idea is that our reference observer can exchange light signals with another observer at somewhat smaller radius and that observer can, after two such exchanges, set the zero point and the rate of his clock to be in synch with the reference observer
    * the rate needs to be adjusted because of the gravitational redshift factor $\lambda_{\rm obs}/\lambda_{\rm em} = e^{\Phi(r_{\rm obs}) - \Phi(r_{\rm em})}$
    * and in establishing this one uses that fact that the path of an ingoing signal is the mirror image – reflected in time – of an outgoing path
    * that observer can then exchange signals with another at still smaller radius and so on

- ∗ in this way one can realise a network of observers, each of whom has a clock measuring t.

- from this metric one can compute the Christoffel symbols and the curvature, Ricci and Einstein equations
- the field equations – with a spherically symmetric source term on the RHS – are then differential equations for the functions $\Phi(r)$ and $\Lambda(r)$ that involve the density ρ and pressure P
- we will return to that presently

2 The Schwarzschild metric

- The first non-trivial solution to the field equations was that of Karl Schwarzschild in 1916:
  - $\boxed{ds^2 = -(1-2M/r)dt^2 + (1-2M/r)^{-1}dr^2 + r^2 d\Omega^2}$
  - with $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$
  - or $g_{\alpha\beta} = \mathrm{diag}(g_{tt}, g_{rr}, g_{\theta\theta}, g_{\phi\phi}) = \mathrm{diag}(-(1-2M/r), (1-2M/r)^{-1}, r^2, r^2\sin^2\theta)$
  - with inverse $g^{\alpha\beta} = \mathrm{diag}(g^{tt}, g^{rr}, g^{\theta\theta}, g^{\phi\phi}) = \mathrm{diag}(-(1-2M/r)^{-1}, (1-2M/r), r^{-2}, (r^2\sin^2\theta)^{-1})$
  - making the calculations $g_{\alpha\beta} \to \Gamma^{\mu}{}_{\alpha\beta} \to R_{\alpha\beta\mu\nu} \to R_{\alpha\beta}, R \to G_{\alpha\beta}$ shows that $\mathbf{G} = 0$ so this is a vacuum solution
- Note that we are using geometrized units. Think of M as being GM'/c^2 where M' is the physical mass.

2.1 Relation to the weak field metric

- for $r \gg 2M$ – the *Schwarzschild radius*, or the *gravitational radius* – this is very similar to the weak field metric surrounding a point mass, for which $\phi = -M/r$
  - there we have $g_{tt} = -(1+2\phi)$, in agreement with Schwartzschild
  - the spatial part of the metric may seem slightly different in that there we had $dl^2 = (1-2\phi)(dx^2 + dy^2 + dz^2)$ which we could write as $(1-2\phi)(dR^2 + R^2 d\Omega^2)$
  - but this can be brought into concordance with Schwarzschild's metric (at linear order in ϕ) with the transformation $r = (1-\phi)R$
    - ∗ the angular part of the metric is then $(1-2\phi)R^2 d\Omega^2 = (1-2\phi)(1-\phi)^{-2} r^2 d\Omega^2 = r^2 d\Omega^2$ – with corrections that are only 2nd order in ϕ
    - ∗ while, in the radial part, $dr = (1-\phi)dR - R(d\phi/dR)dR \simeq dR$ (again with corrections only at 2nd order in ϕ)
- so the Schwartzschild metric is identical, at $r \gg M$, with the weak field metric for a point mass

2.2 Relation to the conventional spherically symmetric line element

- this Schwarzschild line element is actually something of a generalisation of the above form of the line element in that for $r < 2M = r_s$ the metric components g_{tt} and g_{rr} change sign (whereas $e^{2\Phi}$ and $e^{2\Lambda}$ cannot – for real Φ and Λ at least).

2.3 The light-cone structure

- as always, the first step to understanding the physical meaning of a metric is to study that form of the physically coordinate invariant light cones
- outside of $r = 2M$ the light-cones are an extension of what we found for the weak-field metric:
  - the deeper into the potential the narrower the light-cones become
  - so very close to $r = 2M$ any light-like or null particles are constrained to have world lines almost parallel to the t-axis

- but inside $r = 2M$ the situation is radically different:
  - the metric components g_{tt} and g_{rr} change sign – so time-like world-lines must have paths at angles less that $\pi/4$ from the r-axis rather than the t-axis – though the angular line element still involves r (not t)
  - the light cones are rotated by $\pi/2$ and transition from having very broad opening angle close to $r = 2M$ to being very narrow for $r \to 0$
  - it is often said that the timelikeness of the r-coordinate for $r < 2M$ implies that particles – including light – are constrained to move in the direction of decreasing r, and that any bundle of geodesics interior to $r = 2M$ form a 'closed trapped surface'
  - we will see that there's a bit more to it than that

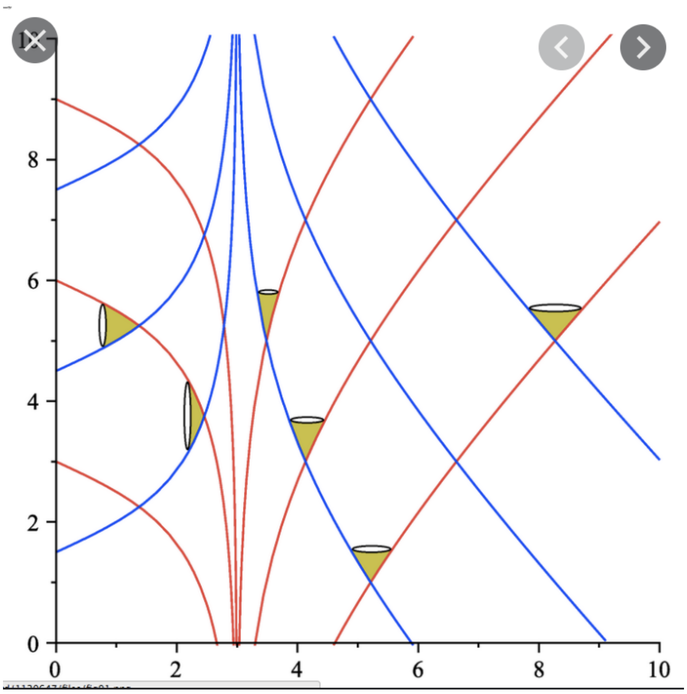


Figure 1: Light cones in Schwarzschild geometry. At large r these are like the light-cones in weak field gravity. But the narrowing becomes extreme as one approaches $r = 2M$ and, as is apparent, null rays from the exterior never reach $r = 2M$ (in finite co-ordinate time, at least). At $r < 2M$, however, they change radically and the radial coordinate becomes time-like.

2.4 Constant r observers

- for $r > 2M$ curves of constant r, θ and ϕ are time-like
  - so such curves are possible world-lines of (accelerated) observers
- for $r < 2M$ such curves – i.e. those with normalised tangent vector $\vec{U} \to (U^t, 0, 0, 0)$ are space-like
  - so no observers can have world-lines with constant r, θ and ϕ

2.5 Singularity of the metric at $r = 2M$

- approaching $r = 2M$ from outside $g_{tt} \to 0$ and $g_{rr} \to \infty$
- so, as one approaches $r = 2M$, possible world lines of particles, which must lie within or on the light-cone, are constrained to move, in the limit, only vertically in the $r - t$ plane
  - and looking at the light cones reinforces the idea that time-like or null world-lines can never reach $r = 2M$ in finite time
- for decades this was considered to be a physical barrier
- and it was thought that the 'singularity' in the behaviour of g_{rr} was a true physical singularity
- it was subsequently realised that this is only a 'coordinate singularity' and that, despite the fact that g_{rr} blows up,

- space-time itself is perfectly regular at $r = 2M$
- particles and photons can happily cross $r = 2M$

2.5.1 The tidal field at $r = 2M$

- the regular nature of the manifold is reflected in the fact that the *curvature* – as revealed by tidal stretching of the proper distance between freely falling particles – is finite at $r = 2M$
- consider two constant-r observers with the same θ and ϕ and at r and $r + \delta r$, both at $r > 2M$
- let them release two test particles at the same instant of coordinate time t
  - each particle has, initially, a 4-velocity $\vec{U} \to U^\alpha = (U^t, 0, 0, 0)$ with normalisation condition $g_{tt}(U^t)^2 = -1$, so $U^t = 1/\sqrt{-g_{tt}} = (1 - 2M/r)^{-1/2}$
  - their initial separation is $\vec{\xi} \to \xi^\alpha = (0, \delta r, 0, 0)$, which is orthogonal to the 4-velocities (by virtue of the diagonality of the metric)
  - i.e. the separation $\vec{\xi}$ lies in the *rest-frame* of the two test particles
  - what's more, the difference between the two 4 velocities agrees with the result of parallel transporting $\vec{U}$ along $\vec{\xi}$: i.e. $\delta U^\alpha = (\delta U^t, 0, 0, 0) = -\Gamma^\alpha{}_{\mu\beta} U^\mu \xi^\beta = -\Gamma^\alpha{}_{tr} U^t \delta r$
    - since $\Gamma^\alpha{}_{tr} = \frac{1}{2} g^{\alpha\gamma}(g_{t\gamma,r} + g_{r\gamma,t} - g_{rt,\gamma}) = \frac{1}{2}\delta^\alpha_t g^{tt} g_{tt,r}$
    - so $\delta U^r = 0$ and $\delta U^t = \frac{1}{2} g^{tt} g_{tt,r} U^t \delta r = -\frac{1}{2}(-g^{tt})^{3/2} g_{tt,r} \delta r = -(1 - 2M/r)^{-3/2}(M/r^2)\delta r$
    - which is the same as that given by $\delta U^t = \delta r dU^t/dr = \delta r d(1 - 2M/r)^{-1/2}/dr$
- so we have two test particles with initially parallel 4-velocities: $\delta\vec{U} = (\vec{\xi} \cdot \nabla)\vec{U} = 0$
  - that means that their physical separation is initially constant
    - something they can verify empirically by doing light-echo ranging
      - · in the process of which they will notice that the photons they exchange are not, initially at least, redshifted
      - · this is quite different to the constant-r observers who released them; they, in contrast, would see a 'gravitational' redshift, by virtue of the fact that they are being accelerated
    - or, more simply, with a ruler
- space-time curvature will cause the particles' 4-velocities to diverge
  - we would expect $\delta\vec{U}$ to grow linearly with proper time τ
  - and that the particles' physical separation will change by an amount that grows quadratically with τ and in linear proportion to their initial physical separation
    - just as for freely falling particles released in a Newtonian tidal field
  - we can calculate $\delta\vec{U}$ by taking the 4-velocity of the first particle after an interval $\Delta\tau$ and parallel transporting it over to the location of the other particle, whose 4-velocity at that time we subtract
  - the result, according to the definition of curvature (see figure 2), is given by $\Delta\delta\vec{U} = \mathbf{R}(\ ,\vec{U}, \vec{\xi}, \Delta\tau\vec{U}) = \Delta\tau\mathbf{R}(\ ,\vec{U}, \vec{\xi}, \vec{U})$
  - and dividing by $\Delta\tau$ and taking the limit gives $d\delta\vec{U}/d\tau = \mathbf{R}(\ ,\vec{U}, \vec{\xi}, \vec{U})$
- now $\vec{U} \equiv d\vec{x}/d\tau$, so $\delta\vec{U} = d\delta\vec{x}/d\tau = d\vec{\xi}/d\tau$ so we have
- $d^2\vec{\xi}/d\tau^2 = \nabla_{\vec{U}}\nabla_{\vec{U}}\vec{\xi} = \mathbf{R}(\ ,\vec{U}, \vec{\xi}, \vec{U})$
- here we are most interested in the rate of change of the component of $\vec{\xi}$ parallel to the separation; i.e. the radial component, and since $\vec{U}$ is parallel to the t-axis and $\vec{\xi}$ is (initially) parallel to the r axis the r^{th} component of the second rate of change of $\vec{\xi}$ is
- $(d^2\vec{\xi}/d\tau^2)^r = R^r{}_{trt} U^t \xi^r U^t$

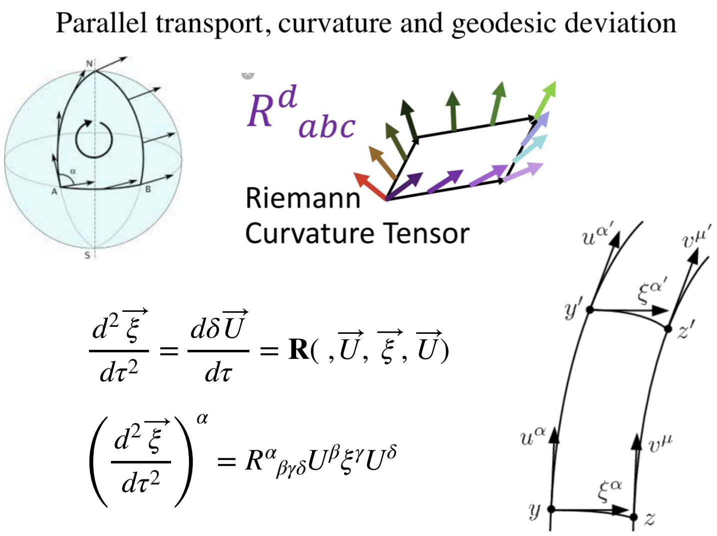


Figure 2: Geodesic deviation and its relation to curvature. Upper left illustrates how a vector – here on a 2-dimensional manifold – changes if parallel transported around a loop. To the right, the change (purple vector minus red vector) has components given by contracting the curvature tensor with the components of the initial (red) vector and those of the two vectors defining the parallelogram path. Bottom right shows how we are applying it here to calculate how two initially parallel trajectories diverge or converge; in this case the two vectors defining the path are the 4-velocity of one of the particles $\vec{U}$ and the initial separation $\vec{\xi}$.

- it is perhaps, at this point, worthwhile to remind ourselves of the important distinction between $(d^2\vec{\xi}/d\tau^2)^r$ and $d^2\xi^r/d\tau^2$, these not, in general, being the same, and why we are more interested in the former rather than the latter. Skip to the next bullet if you are familiar with this.
  - the meaning of $d^2\vec{\xi}/d\tau^2$ (or $\nabla_{\vec{U}}\nabla_{\vec{U}}\vec{\xi}$) is that it is the second rate of change of $\vec{\xi}$ along the path *relative to what one would have obtained by parallel transporting $\vec{\xi}$ along the path*
    * the operator $\nabla_{\vec{U}}$, applied to any vector field, let's say $\vec{V}(\vec{x})$, is itself a vector, and has components $(\nabla_{\vec{U}}\vec{V})^\alpha = (\nabla_{\vec{U}}\vec{V})(\tilde{\omega}^\alpha) = U^\beta V^\alpha{}_{;\beta} = U^\beta(V^\alpha{}_{,\beta} + \Gamma^\alpha{}_{\mu\beta}V^\mu)$
    * these components are those four numbers that, when contracted with the four basis vectors $\vec{e}_\alpha$ *at the point in question where $\nabla_{\vec{U}}$ is being applied*, returns the vector $\nabla_{\vec{U}}\vec{V}$
    * applying $\nabla_{\vec{U}}$ to $\nabla_{\vec{U}}\vec{\xi}$, whose components are $(\nabla_{\vec{U}}\vec{\xi})^\alpha = U^\gamma\xi^\alpha{}_{;\gamma} = U^\gamma(\xi^\alpha{}_{,\gamma} + \Gamma^\alpha{}_{\nu\gamma}\xi^\nu)$, we get
      · $(d^2\vec{\xi}/d\tau^2)^\alpha = U^\beta([U^\gamma(\xi^\alpha{}_{,\gamma} + \Gamma^\alpha{}_{\nu\gamma}\xi^\nu)]_{,\beta} + \Gamma^\alpha{}_{\mu\beta}U^\gamma(\xi^\mu{}_{,\gamma} + \Gamma^\mu{}_{\nu\gamma}\xi^\nu))$
    * this is a bit of a mess, but if we work in locally inertial coordinates, so the Christoffel symbols (but not, in general, their derivatives) vanish, this says
      · $(d^2\vec{\xi}/d\tau^2)^\alpha = U^\beta U^\gamma(\xi^\alpha{}_{,\gamma\beta} + \xi^\nu\Gamma^\alpha{}_{\nu\gamma,\beta}) + U^\beta U^\gamma{}_{,\beta}\xi^\alpha{}_{,\gamma}$
    * but the particles are following geodesics, i.e. $\vec{U}$ is being parallel transported, so $U^\beta U^\gamma{}_{,\beta} = -\Gamma^\gamma{}_{\delta\beta}U^\delta U^\beta$, which vanishes (in a locally inertial frame), and, with $d^2\xi^\alpha/d\tau^2 = U^\beta U^\gamma\xi^\alpha{}_{,\gamma\beta}$, we have
      · $(d^2\vec{\xi}/d\tau^2)^\alpha = d^2\xi^\alpha/d\tau^2 + U^\beta U^\gamma\xi^\nu\Gamma^\alpha{}_{\nu\gamma,\beta}$
    * which is sufficient to show the essential, and non-trivial, difference between the components of the second derivative of a vector and the second derivatives of its components.
  - regarding the change in $\vec{\xi}$ over the interval $\tau = \tau_0$ to $\tau = \tau_0 + \delta\tau$, the vector $\vec{\xi} + \frac{1}{2}(\delta\tau)^2 d^2\vec{\xi}/d\tau^2$ is a vector that, in some sense, 'lives' at $\vec{x}(\tau = \tau_0)$
  - its $\alpha^{\rm th}$ component $(\vec{\xi})^\alpha + \frac{1}{2}(\delta\tau)^2(d^2\vec{\xi}/d\tau^2)^\alpha$, is the result of letting $\vec{\xi} + \frac{1}{2}(\delta\tau)^2 d^2\vec{\xi}/d\tau^2$ act on the $\alpha^{\rm th}$ basis 1-form $\tilde{\omega}^\alpha$ *at the point on the particle trajectory* $\vec{x}(\tau = \tau_0)$
  - in contrast $\xi^\alpha + \frac{1}{2}(\delta\tau)^2 d^2\xi^r/d\tau^2$ is the $\alpha^{\rm th}$ component of $\vec{\xi}(\tau_0 + \delta\tau)$ which is the result of letting $\vec{\xi}(\tau_0 + \delta\tau)$ act on the $\alpha^{\rm th}$ basis 1-form at the position $\vec{x}(\tau_0 + \delta\tau)$
  - as such, $d^2\xi^\alpha/d\tau^2$, by and of itself, is an incomplete and potentially misleading measure of how $\vec{\xi}$ is changing along the path. It could be that $d^2\xi^\alpha/d\tau^2 \neq 0$, for instance, simply because of the way the basis 1-forms and basis vectors are changing, while $(d^2\vec{\xi}/d\tau^2)^\alpha = 0$ indicating that $\vec{\xi}$ is not in fact changing (with respect to a parallel transported copy of itself)

- to evaluate $R^r{}_{trt} = \Gamma^r{}_{tt,r} - \Gamma^r{}_{tr,t} + \Gamma^r{}_{\gamma r}\Gamma^\gamma{}_{tt} - \Gamma^r{}_{\gamma t}\Gamma^\gamma{}_{tr}$ we need some more Christoffel symbols:
  - in addition to $\Gamma^\alpha{}_{tr} = \frac{1}{2}\delta^\alpha_t g^{tt} g_{tt,r}$,
  - we readily obtain $\Gamma^\alpha{}_{tt} = -\frac{1}{2}\delta^\alpha_r g^{rr} g_{tt,r}$ and $\Gamma^\alpha{}_{rr} = \frac{1}{2}\delta^\alpha_r g^{rr} g_{rr,r}$, so

* $\Gamma^r{}_{tt} = -\frac{1}{2}g^{rr}g_{tt,r} = (1 - 2M/r)M/r^2$
* $\Gamma^r{}_{rr} = \frac{1}{2}g^{rr}g_{rr,r} = -(1 - 2M/r)^{-1}M/r^2$
* $\Gamma^t{}_{tr} = \Gamma^t{}_{rt} = \frac{1}{2}g^{tt}g_{tt,r} = (1 - 2M/r)^{-1}M/r^2$
* with all other symbols with lower indices rr, tt or rt vanishing

- using these, and the fact that $\Gamma^r{}_{tr,t} = 0$ (as nothing is dependent on t here) gives
  - $\boxed{R^r{}_{trt} = -(1 - 2M/r) \times 2M/r^3}$
- This is not divergent at $r = 2M$. In fact it is zero. But this does not mean that there is no tidal stretching there. The quantity appearing in the GDE is $R^r{}_{trt}U^tU^t$ in which the divergence of $(U^t)^2 = -1/g_{tt} = (1-2M/r)^{-1}$ cancels the factor $(1-2M/r)$ in $R^r{}_{trt}$ and the GDE is, finally, simply
  - $\boxed{(d^2\vec{\xi}/d\tau^2)^r = (2M/r^3)\xi^r}$
- to get the change in physical separation as a function of the physical separation itself, we still need to multiply both sides by a basis vector. But this is a constant basis vector - as it is the basis vector at a specific point - so it enters on both sides of this equation. So we can justifiably think of the components here as measuring physical quantities
- thus there is no divergence (nor indeed vanishing) of the tide at $r = 2M$. The result is identical to the Newtonian one. The radial tidal stretching – which, being $\ddot{\xi}/\xi = 2M/r^3$, has units of inverse time squared – is on the order of (G times) the density one obtains by taking the mass divided by the volume. I.e. the same as the (inverse squared) dynamical or orbital time for a system of mass M and size r.
- for the massive black-holes that are found at the centres of massive galaxies, the tidal field that we would feel if located at, or close to, $r = 2M$ is no more than that near the surface of the earth
  - or, for that matter, the tidal field produced by one's own body if one were floating in empty space – i.e. not much!
  - though the tide scales as M/r^3, or as $1/M^2$ at the gravitational radius, so less massive BHs are potentially more dangerous
- Q: A star falling into a black-hole may get tidally torn apart - with interesting observational effects provided the tidal disruption occurs before the star crossed the event horizon. For a star like the sun, what are the conditions on the black-hole mass such that we would be able to observe this?

3 Radial orbits in Schwarzschild geometry

3.1 The cycloidal solution for bound orbits

- the metric coefficients are independent of t so $p_t =$ constant.
  - let us denote it by $p_t = -E$ with E a positive constant
  - the minus sign is obligatory if we demand that, exterior to $r = 2M$, the contravariant time component $p^t = mdt/d\tau > 0$
  - i.e. if we require that proper time τ increases with increasing t
- for radial orbits, $p^\theta = 0$ and $p^\phi = 0$
- so $p^2 = -m^2 = g_{\alpha\beta}p^\alpha p^\beta = g^{tt}(p_t)^2 + g_{rr}(p^r)^2 = (1 - 2M/r)^{-1}((p^r)^2 - (p_t)^2)$
- or, with $p^r = mdr/d\tau$ with τ the proper time
  - $(dr/d\tau)^2 = E^2/m^2 - 1 + 2M/r$ or
  - $\boxed{(dr/d\tau)^2 = 2M/r - \mathcal{E}}$
  - with $\boxed{\mathcal{E} \equiv 1 - E^2/m^2}$

* we will consider here the case that $\mathcal{E}$ is positive
    * this corresponds to a bound orbit – i.e. $E < m$ that will reach some 'apogee' and then return
    * negative $\mathcal{E}$ is allowed. This corresponds to an unbound – or 'hyperbolic' – orbit (with total energy $E > m$) that will escape to $r = \infty$
  - these equations look exactly the same as their Newtonian counterparts
    * if we replace r and proper time τ by Newton's absolute space and time
    * and with $\mathcal{E}$ being minus half the total energy (kinetic plus potential)
  - but there is one important difference
    * from the definition $\mathcal{E} \equiv 1 - E^2/m^2$ we have $E^2/m^2 = 1 - \mathcal{E}$, but E^2 is obviously non-negative, which implies that $\mathcal{E} \leq 1$
    * that means that the turning point – where $dr/d\tau = 0$ – must be at $r > 2M$
    * the relativistic energy-momentum relation does not allow an orbit with apogee at $r < 2M$
    * this seems sensible, given that, interior to $r = 2M$, the r-coordinate plays the role of time (since g_{rr} is negative)
    * so particles do not reverse their direction of travel in time
  - the energy equation $\dot{r}^2 = 2M/r - \mathcal{E}$ has a parametric solution (a cycloid)
  - $$\boxed{\begin{array}{c} r = (M/\mathcal{E})(1 - \cos\eta) \\ \tau = (M/\mathcal{E}^{3/2})(\eta - \sin\eta) \end{array}}$$

- so this is an orbit which starting from $r_{\rm max} = 2M/\mathcal{E} > 2M$ (at $\eta = \pi$) falls to $r = 0$ after a finite proper time $\tau = \pi M/\mathcal{E}^{3/2}$
- evidently there is no physical barrier at $r = 2M$

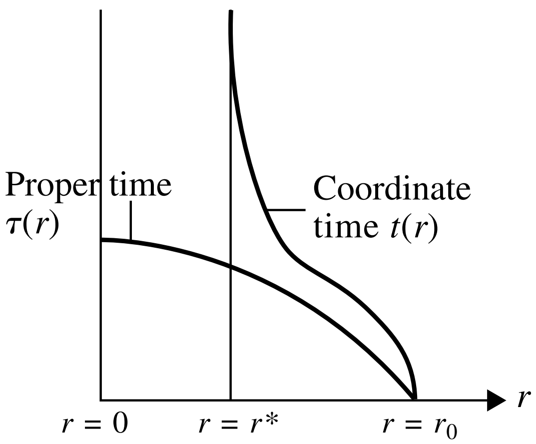


Fig. 6.8 The contrasting behavior of proper time $\tau(r)$ vs. coordinate time $t(r)$ at the Schwarzschild surface.

Figure 3: Infalling radial orbit in Schwarzschild geometry. If we plot radius vs proper time $r = r(\tau)$ – the lower curve – the particle is seen to fall to the singularity at $r =$ in finite proper time. However, if we plot $t = t(r)$ (upper curve) we see that the particle heads off to $t = +\infty$. But this simply reflects a pathology of the t-coordinate. The full trajectory in $r - t$ coordinates is shown below.

3.1.1 Trajectories in $r - t$ space

- a peculiarity of the r, t coordinates is clearly seen if one considers the trajectory of the particles on radial orbits in r, t space
  - we have seen that $p_t = -E = -m\sqrt{1 - \mathcal{E}}$ implies $dt/d\tau = p^t/m = \sqrt{1 - \mathcal{E}}/(1 - 2M/r)$
  - while, for an infalling particle, $dr/d\tau = -\sqrt{2M/r - \mathcal{E}}$
  - so $dr/dt = (dr/d\tau)/(dt/d\tau) = -\sqrt{(2M/r - \mathcal{E})/(1 - \mathcal{E})}(1 - 2M/r)$
  - close to $r = 2M$, and letting $r = 2M(1 + \epsilon)$, so $dr = 2M d\epsilon$, we have $dr/dt = 2M d\epsilon/dt \simeq (1 - 2M/r) \simeq -\epsilon$ so $t = \text{constant} - 2M \log \epsilon$

- starting from $r_{\rm max}$ (at $t = 0$ say) it heads off to $t = +\infty$ on its way to $r = 2M$
- thereafter it returns from $t = +\infty$ heading in the *negative* t direction before reaching $r = 0$ (a true singularity)

- perhaps most interestingly, if we start at $\eta = 0$, the full solution is that of a particle that starts from $r = 0$; flies out to $r_{\rm max} > 2M$ and then returns
  - the outgoing path is the reflection of the infalling path about the r-axis
    - so the particle initially moves towards negative t and heads off to $t = -\infty$ on its way out to $r = 2M$ and thereafter returns from $t = -\infty$ moving in the positive t direction
  - so Schwarzschild space-time allows orbits that carry particles outward from inside $r = 2M$
  - this is not some mathematical curiosity, but a very real phenomenon
    - if one considers any large spherical region of our universe then, not so long ago, it would have had $r = 2M$ and was, at earlier times, inside $r = 2M$
    - it turns out that the behaviour of such a spherical sub-section of a homogeneous universe is unaffected by the exterior universe
    - and particles on the boundary behave just like the cycloidal solutions above
    - this is sometimes called a 'white-hole'

3.2 The Oppenheimer-Snyder model for BH formation

- the foregoing leads to the Oppenheimer-Snyder model for the formation of a black-hole
- their starting point is the 'closed' Friedmann-Robertson-Walker (FRW) metric for a homogeneous matter-dominated cosmology
  - for which the line-element can be written either as the 'hyper-sphere'
    - $ds^2 = -d\tau^2 + a(\tau)^2(d\chi^2 + \sin^2\chi(d\theta^2 + \sin^2\theta d\phi^2))$
    - where τ is the proper time measured by the 'fundamental observers'
    - these FOs being observers who maintain constant χ, θ and ϕ.
    - you can check, if you wish, that the Christoffel symbols for this metric give a geodesic equation consistent with the FOs world-lines being geodesics (i.e. being unaccelerated)
  - or, if you prefer, as
    - $ds^2 = -d\tau^2 + a(\tau)^2(dr^2/(1 - r^2) + r^2(d\theta^2 + \sin^2\theta d\phi^2))$
    - since the transformation $r = \sin\chi \rightarrow dr = \cos\chi d\chi \rightarrow d\chi^2 = dr^2/\cos\chi^2 = dr^2/(1 - \sin^2\chi) = dr^2/(1 - r^2)$
  - and for which the scale factor $a(\tau)$ obeys
    - $\boxed{(da/d\tau)^2 = (8\pi/3)G\rho a^2 - 1}$
    - which follows from the field equations
    - but can also be 'derived' by realising that the evolution of a small spherical region of such a universe should be independent of what is happening outside and so should obey the Newtonian equations of motion for an expanding sphere of 'dust' (zero pressure fluid)
- but since $\rho \propto 1/a^3$, this equation for $a(\tau)$ is identical in form to the relativistic equation for a radial test particle
  - if we set $r(\tau) = \sqrt{\mathcal{E}}a(\tau)$ then this says (in units such that Newton's constant $G = 1$)
  - $\dot{r}^2 = 2M/r - \mathcal{E}$ with $M = (4\pi/3)\rho r^3$ a constant
  - which is identical to the equations of motion of a particle on a radial trajectory in Schwartzschild space-time
- and so $a(\tau)$ is a cycloid with $a = (M/\mathcal{E}^{3/2})(1 - \cos\eta)$ and $\tau = (M/\mathcal{E}^{3/2})(\eta - \sin\eta)$ as before
- Oppenheimer and Snyder's insight was to realise that a *finite* portion of the entire FRW solution

  - the matter within some radius $\chi = \chi_{\rm max}$
  - and with vacuum outside

- is also a solution of the field equations (these being local)
  - with the FRW interior geometry being smoothly 'stitched on' to the external Schwarzschild space-time
    * this stitching evidently requires that the physical area of the sphere
    * $A = 4\pi a^2(\tau) \sin^2 \chi_{\rm max}$ according to the FRW metric
    * $A = 4\pi r^2$ according to Schwarzschild
    * be equal, which implies
    * $\boxed{\sin \chi_{\rm max} = \sqrt{\mathcal{E}}}$
  - the picture this engenders – in the light of the discussion of the curvature of space in the Newtonian limit – is that the space is positively curved in the constant density region like a parabolic bowl, and that this is connected – smoothly – to an exterior trumpet-horn like geometry
  - though one must firmly denounce any suggestion that the motion of massive particles such as planets in the solar system are responding to the curvature of *space* caused by the sun – as we have seen, it is only the warping of time that tells planets how to move
  - it is interesting to note that there are two possible solutions of $\sin \chi_{\rm max} = \sqrt{\mathcal{E}}$
    * one with $\chi_{\rm max} < \pi$ – i.e. containing the region from the 'South pole' to a latitude below the 'equator' – and another with $\chi_{\rm max} > \pi$ which contains the entire 'southern hemisphere' and part of the north as well
    * in the latter case, on approaching the edge of the dust sphere from inside, the radius r – and therefore also the surface area $4\pi r^2$ is decreasing
    * so if this is going to match smoothly on to some exterior geometry, this must resemble something more like the neck of a flask than a trumpet horn – how can that be possible?
    * these two possibilities have the same 'active gravitational mass' as sensed by an external observer, but very different total proper mass

- a fundamental observer at the boundary $\chi = \chi_{\rm max}$ can be considered both as 'fundamental observer' in the FRW space-time and as a test particle on a radial orbit in the Schwarzschild space-time

- their model for a star which has for some reason run out of fuel and lost pressure support is that it will behave like this partial closed FRW model – starting at the point of maximum expansion $\eta = \pi$ – and collapse to a singularity at $r = 0$ leaving Schwarzschild geometry outside in its wake, as illustrated in figure 4

- but if we consider the full cycloidal solution we have the 'white-hole' phenomenon
  - one can imagine a rocket-borne accelerated observer at rest (maintaining constant r, that is) outside of $r = 2M$ who would suddenly be engulfed in particles emanating from inside of $r = 2M$
  - and, as we shall see, distant observers can see radiation that was emitted from the surface when it was smaller than $r = 2M$
  - and even photons emitted from arbitrarily close to the 'naked' singularity at $r = 0$ that existed before (in the sense of being at more negative coordinate time) the sphere started expanding

- this may seem surprising
  - the conventional wisdom is that the matter falling into the black-hole – in the later stage of this model – is irreversibly trapped and nothing can escape
  - but if outdoing radial orbits are allowed, what is to stop an infalling observer firing some matter or radiation onto an outgoing orbit after it crosses $r = 2M$ but before it reaches $r = 0$?

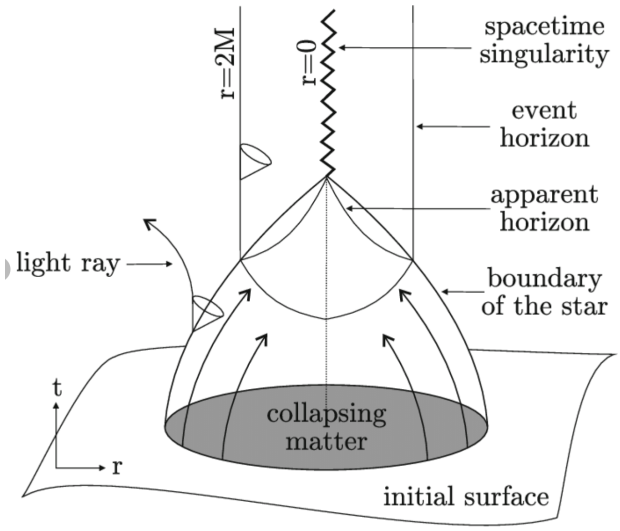


Figure 4: Oppenheimer-Snyder model for collapse of a star to form a black-hole.

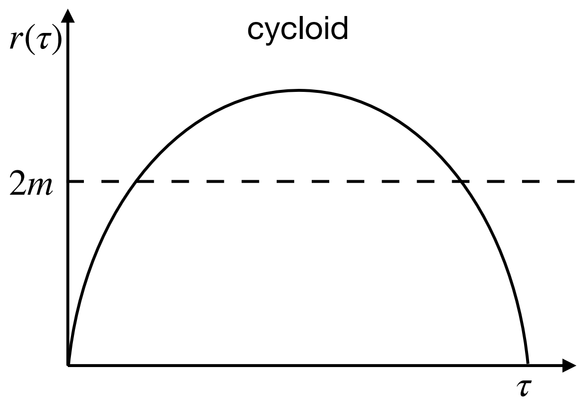


Figure 5: The full cycloidal soution.

3.3 Radial orbits and particle dynamics interior to $r = 2M$

3.3.1 Energy of outgoing particles as seen by infalling observers

- let's look a bit more carefully at the radial orbit solutions, and particularly their behaviour interior to $r = 2M$
- invariance of the metric with respect to t provides $p_t = -E = -m\sqrt{1-\mathcal{E}}$, where the sign of p_t is determined by the requirement that, exterior to $r = 2M$, the proper time increases in the same direction as so
  - $p^t = mU^t = mdt/d\tau = g^{tt}p_t = E/(1-2M/r) = m\sqrt{1-\mathcal{E}}/(1-2M/r)$
  - which, for $r < 2M$, is negative for both infalling and outgoing orbits
- the energy momentum relation $p^2 = -m^2$ $(U^r)^2 = 2M/r - \mathcal{E}$ gives the 4-velocity component
  - $U^r = dr/d\tau = \pm\sqrt{2M/r - \mathcal{E}}$
  - which is positive (negative) for outgoing (infalling) particles
  - the same being true of the radial component of the 4-momentum $p^r = mU^r$
- it is interesting to ask, what would be the *energy* of an outgoing particle P as measured by an infalling observer O at the same location?
  - or, equivalently, the energy of a particle emitted from O on an outgoing orbit – assuming for the moment that this is possible

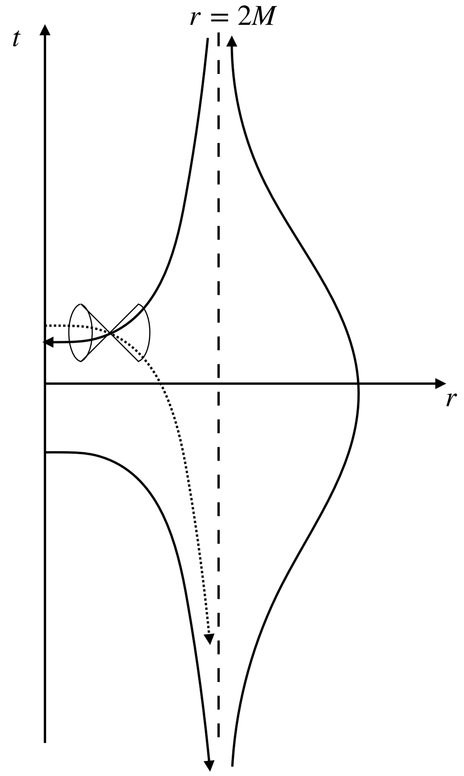


Figure 6: Radial orbits in Schwarzschild coordinates. This shows how an infalling observer would seem to be able to meet an outgoing observer. And he could pass that observer a message; and thus communicate with the outside world from inside the event horizon!

- recall that in a locally inertial frame that is instantaneously comoving with O
  - i.e. that frame in which $\vec{U}_O \rightarrow (1, 0, 0, 0)$
- the energy of P is the time component of P's 4-velocity times its proper mass:
  - $E = mU_P^t = -m\vec{U}_O \cdot \vec{U}_P = -m\mathbf{g}(\vec{U}_O, \vec{U}_P) = -mg_{\alpha\beta}U_O^\alpha U_P^\beta$
  - the last being a tensor expression and therefore valid in all frames
- let us assume, for simplicity, that they have the same $\mathcal{E}$ (i.e. the same apogee)
- then, since the particles have identical U^t but opposite U^r, we have
  - $E/m = -g_{tt}U_O^t U_P^t - g_{rr}U_O^r U_P^r = -g^{tt}(U_t)^2 + g_{rr}(U^r)^2$
- or equivalently
  - $E/m = [(1 - \mathcal{E}) + (2M/r) - \mathcal{E})]/(1 - 2M/r)$
- but both terms in parentheses are positive, so for $r > 2M$ the observed energy is positive,
- but inside $r = 2M$ the energy of the outgoing particle as measured by the infalling particle is *negative!*
- this seems bizarre
  - taken seriously, it would suggest that our infalling observer can emit outgoing particles at *negative* cost in energy – which might seem both a recipe for instability and a nice way to generate energy
- but it is, in fact, not as crazy as it looks. To see why we need to consider interactions between such particles.

3.3.2 'Emission' of an outgoing particle

- Imagine that, at some point on infalling O's world-line, when O is at $r_0 < 2M$, he emits a particle P onto an outgoing radial orbit
- as P has negative energy as observed by O that means that after the emission event O will have *more* energy than before – which seems bizarre
- but recall that this is interior to $r = 2M$ where it is r that plays the role of time and the axes of the light-curves are horizontal in r, t coordinates
- so the events along the world-line of P are at $r > r_0$ and so are actually in the *past* of the 'emission' event from O's perspective
- so, from O's point of view, this is not the emission of a particle at all. Rather it is an *absorption* event. And it is the absorption of a particle that, from O's perspective, is not outgoing at all; it is another infalling particle
- and since O's energy is increased in the process, O would say that this was the absorption of a *positive* energy particle

3.3.3 'Absorption' of an outgoing particle

- we can similarly imagine a particle P, who considers himself to be outgoing from $r = 0$, being 'absorbed' by the observer O at the point on O's world-line at radius r_0
- and taking the view that P has negative energy in O's frame energy conservation would require that O must have lower energy after the absorption event than before
- but the events on P's world-line – lying at $r < r_0$ are all in the *future* of the 'absorption' event from O's perspective
- so O considers this to be the *emission* of a particle that, like himself, heads off towards the singularity
- and, doing the energy book-keeping, O would conclude that he had emitted a particle of positive energy
- it is interesting that, in both of these examples, the infalling observer considers the other particles with whom he can interact – and who definitely consider themselves to have a r which is increasing with their proper time – to be infalling too, and to have positive, not negative, energy

3.3.4 Relation between the energy and the 'arrow of proper-time'

- in deriving the standard formula $E = -m\vec{U}_O \cdot \vec{U}_P = -mg_{\alpha\beta}(dx^\alpha_O/d\tau)(dx^\beta_P/d\tau)$ there was an implicit assumption that, in the rest-frame of O, where $E = mU^t_P = mdt_P/d\tau$, that P's 'affine parameter' τ is increasing with t_P, the time coordinate of P as seen by O
- but in the examples above, that was not the case
  - the affine parameter of the infalling particle increases in the direction of decreasing r while that of the 'outgoing' particle increases with increasing r
  - and it is this difference that gives rise to the negative energy
- so one way to avoid the seemingly unphysical negative energy would be to 'correct' the standard formula - and, say, replace $dt_P/d\tau$ by its modulus
- it certainly seems that the energy of particles that observers can measure by absorbing, emitting or interacting with them more generally should be positive and independent of the direction in which the proper-time of the observer is increasing
- we can see this if we think about a matter-field wave packet occupying some 'world-tube'

- from some observer O's perspective – that is to say expressed in terms of his locally inertial coordinates $x^\alpha = (t, \mathbf{x})$ – the wave might be $\phi = \cos(-\omega t)$ times some large, smoothly varying, 'envelope'
- this would correspond to a zero 3-momentum packet
- the integrated 4-momentum of the packet is readily shown (for e.g. a massive scalar field with stress-energy $T^{\mu\nu} = \phi^{,\mu}\phi^{,\nu} - \eta^{\mu\nu}(\phi^{,\alpha}\phi_{,\alpha} + m^2\phi^2)/2$) to be
- $\begin{bmatrix} p^0 \\ \mathbf{p} \end{bmatrix} = \int d^3x\omega \begin{bmatrix} T^{00} \\ T^{0i} \end{bmatrix} = \int d^3x\omega \begin{bmatrix} \omega \\ \mathbf{k} \end{bmatrix} \phi^2(\mathbf{x})$
- and so the energy p^0 in O's frame is positive
- but if there is another observer O' for whom proper time is increasing in the opposite direction doing the measuring, for them their locally inertial coordinates will have a flip the sign of the time coordinate
- so they will say the wave is $\phi = \cos(+\omega t)$
- but they will conclude that the energy is positive also

3.3.5 The orientability of the space-time manifold.

- saying that O and P have affine parameters that increase in opposite sense with respect to the time-like r-coordinate sounds rather anodyne
  - but the affine parameter is proper time
  - so if P were carrying a clock then O would see it running backwards and vice versa
  - more dramatically, O would see P getting younger!
- is this a problem?
  - it is certainly bizarre and counter to what we normally observe
  - and also counter to what would be seen if O and P were to meet at $r > 2M$
  - but it is not clear that this actually violates any laws of physics
- the question here, to my mind, is whether Nature ever allows there to be regions of space-time where there are different observers who would observe each other, if in close proximity, to have proper time running in opposite directions.
- I suspect that the answer is no. If so, that would mean that there is a way to label one direction of the light-cones as the 'future' that is consistent over the whole of the manifold.
- as we shall see, this is suggested by the 'maximal extension' of Schwartzschild space-time found by Kruskal and Szekeres in which, in effect, one can have a 'white-hole' singularity at $r = 0$, with particles coming out of it, and these particles can fall into a 'black-hole' singularity, but these singularities live in distinct regions of the manifold – this avoids an infalling observer ever meeting up with an outgoing observer at $r < 2M$

3.3.6 The fate of matter falling through the event horizon

- the discussion above covers some wild and crazy ideas
- it is not clear that they apply to our universe
  - most text-books treat white-holes as far-flung science-fiction
  - but the basic observational facts of cosmology suggest that they should be taken seriously
  - however, it could be that they have no particular relevance to black-holes
  - as we shall see, in the case of the Oppenheimer and Snyder model, the 'white-hole' and parallel universe regions in the maximally extended Schwartzschild geometry do not exist

- nor is there any observational evidence for matter emerging from black-holes – even though, as we have seen, radial orbits emerging from $r = 2M$ do not seem to be forbidden

- regarding the existence of 'closed trapped surfaces' and 'cosmic censorship', we should emphasis that if an infalling observer O *does* attempt to emit a real particle – something that he could potentially attach a message to – then the events on the messenger's world-line must lie in the *future* of the emission event
  - and these all lie at smaller r
  - and so the emitted particle and any message it might carry must, inevitably, end up at the singularity
  - and the message never gets out

4 Rindler space-time

- Rindler geometry is empty flat space-time viewed from the perspective of a family of accelerated observers
  - it uses a 'spatial' coordinate that labels the observers
  - and a 'temporal' coordinate that increases along the world lines (it is a function of the proper time)
- it turns out it has a horizon and a 'coordinate singularity' that are similar to those in Schwarzschild geometry
  - which also has coordinates tied to accelerated observers (who maintain constant r – outside of $r = 2M$ at least)
- Rindler space-time provides a helpful insight into what's wrong with Schwarzschild r, t coordinates
  - and points the way towards a better coordinate system for Schwarzschild geometry
- consider then a particle subject to a steady acceleration $\mathbf{a}$
  - in the frame of reference of the particle at some proper time τ_0 its 3-velocity will change to $\mathbf{v} = \mathbf{a}\Delta\tau$ after a small interval of proper time $\Delta\tau$, so its 4-velocity in this frame is $\vec{U} = (1 + \mathcal{O}(a^2\Delta\tau^2), \mathbf{a}\Delta\tau)$
  - let's assume for simplicity that the acceleration is parallel to the x-axis
  - if this frame is moving at velocity v relative to the lab-frame at $\tau = \tau_0$, then after the short interval, the 4-velocity in the lab-frame will be
  - $\vec{U}' = \begin{bmatrix} \gamma' \\ \gamma' v' \end{bmatrix} = \begin{bmatrix} \gamma & \gamma v \\ \gamma v & \gamma \end{bmatrix} \begin{bmatrix} 1 \\ a\Delta\tau \end{bmatrix} = \begin{bmatrix} \gamma(1 + va\Delta\tau) \\ \gamma(v + a\Delta\tau) \end{bmatrix}$
  - and hence
  - $v' = (v + a\Delta\tau)/(1 + va\Delta\tau) = v(1 + (a/v - va)\Delta\tau + \ldots)$
  - so the change in velocity in the lab-frame $\Delta v = a(1 - v^2)\Delta\tau$
  - with that change occurring over an interval of time in the lab-frame $\Delta t = \gamma\Delta\tau = \Delta\tau/\sqrt{1 - v^2}$
- so $\Delta v/\Delta t = a/\gamma^3$ or, with $\dot{x} = v = dx/dt$ and $\ddot{x} = dv/dt$ the equation for the trajectory is
  - $\ddot{x} = a(1 - \dot{x}^2)^{3/2}$
- a solution to this equation is the hyperboloid
  - $x^2 = \chi^2 + t^2$
  - with $\chi = 1/a$
  - the physical significance of this distance ($\chi = c^2/a$) is that if you travel this distance you will reach a velocity $v \sim c$

– the general solution is obtained by replacing $x \to x - x_0$ and $t \to t - t_0$ for an arbitrary constant displacement (t_0, x_0) – but we won't use that here
– we will consider a family of particles, all with $(t_0, x_0) = (0, 0)$ but with different accelerations (and therefore different minimum x-coordinates) as illustrated in figure 7

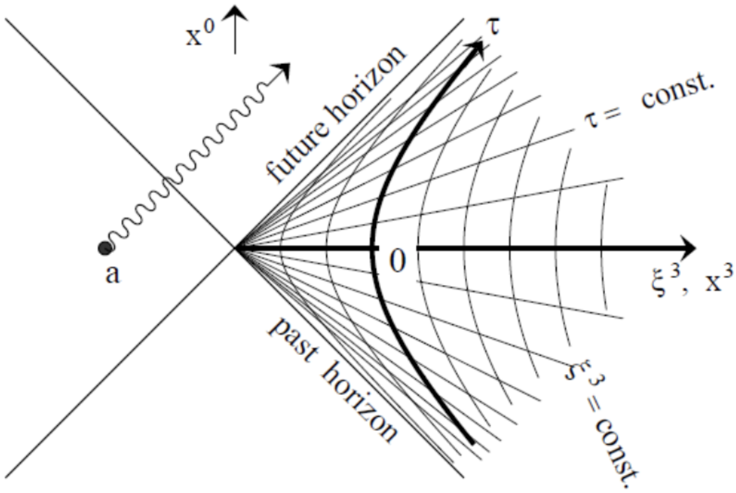


Figure 7: Rindler space-time - we use χ and ν rather than ξ and τ used here. This shows, in flat Minkowskian space-time, the trajectories of a set of particles each undergoing constant acceleration. Their world-lines are a set of hyperbolae $x^2 = t^2 + \chi^2$ where $\chi = 1/a$ labels the particles. So they have the same asymptotic trajectories $\pm x \to \pm t$ as $t \to \pm\infty$. These world-lines foliate Minkowski space, but only part of it. There are horizons as indicated. A photon emitted from an event such as a will never reach any of the accelerated observers.

- this can also be expressed parametrically as
  – $$\begin{aligned} t &= a^{-1}\sinh(a\tau) = \chi\sinh(\tau/\chi) \\ x &= a^{-1}\cosh(a\tau) = \chi\cosh(\tau/\chi) \end{aligned}$$
  – with τ the proper time
    * since $dt^2 - dx^2 = \cosh^2(\tau/\chi)d\tau^2 - \sinh^2(\tau/\chi)d\tau^2 = d\tau^2$
- Now consider a *family* of accelerated observers
  – with different accelerations, but each with $\chi = 1/a$
- this family of trajectories has some interesting characteristics
  – these trajectories foliate the wedge-like region of space-time $x > 0$ and $|t| < x$
  – all the trajectories have the same asymptote at $\tau \to \infty$ which is $x = t$
  – this means that $x = t$ is a *horizon* in the sense that any light ray emitted in the future of the origin $(x, t) = (0, 0)$ will never catch up with any of the particles
  – these trajectories are paths of constant proper distance $\sqrt{x^2 - t^2} = \chi$ from the origin
  – the straight lines $t = \nu x$ radiating from the origin are (relativistically) orthogonal to the trajectories – lines of constant χ – at their intersection points
    * this looks plausible graphically and is easily verified:
    * for constant χ, $xdx = tdt$ or $(dt, dx) = (1, t/x)dt = (1, \nu)dt$
    * while for constant ν, $dt' = \nu dx'$ so $(dt', dx') = (\nu, 1)dx'$
    * so the scalar product $-dtdt' + dxdx' = 0$
  – the interval (dt', dx') lies in the rest-frame of the particles whose trajectories it connects
  – and its length is just $d\chi$
  – as an aside, another interesting feature is that neighbouring trajectories maintain a constant proper separation from one another
    * despite the fact that they have *different* accelerations
    * the trailing particle has to have a greater acceleration if it is to keep up with its leading neighbour

    * which has the corollary that two particles with the *same* acceleration will have a proper separation that increases
    * this leads to Bell's famous space-ship paradox: if two identically accelerating rockets are connected by a thread, will it break?

- the orthogonality of elements of constant χ and constant ν mean that we can write the line element ds^2 as the sum of their squared lengths
  - we have seen that an elemental separation at constant ν is $ds^2 = d\chi^2$
  - while at constant χ the fact that $xdx = tdt$ implies $ds^2 = -dt^2 + dx^2 = -dt^2(1-\nu^2)$
  - but $\nu^2 = t^2/x^2 = t^2/(\chi^2 + t^2)$ implies $t^2(1-\nu^2) = \chi^2\nu^2$ or $t = \chi\nu/\sqrt{1-\nu^2}$
  - from which $dt = \chi d\nu(1+\nu^2)/(1-\nu^2)^{3/2}$ and so $ds^2 = -dt^2(1-\nu^2) = -\chi^2 d\nu^2(1+\nu^2)^2/(1-\nu^2)^2$
- combining these gives
  - $ds^2 = -\chi^2 d\nu^2(1+\nu^2)^2/(1-\nu^2)^2 + d\chi^2$
- or, finally, if we change variable to $\rho \equiv \log\chi$
  - $ds^2 = e^{2\rho}(-d\nu^2(1+\nu^2)^2/(1-\nu^2)^2 + d\rho^2)$
- what we have done here is a coordinate transformation from (t, x) to (ν, ρ) with spatial coordinate $\rho = \log\chi = \log\left(\sqrt{x^2 - t^2}\right)$ which labels the trajectories and a temporal coordinate $\nu = t/x$ which increases along each trajectory
  - the temporal coordinate is not in fact proper time τ – which might seem a more natural choice – rather it is $\nu = t/x = \tanh(\tau/\chi)$
- the particular choice of spatial variable has has resulted in the common 'conformal factor' $e^{2\rho} = \chi^2$
  - this means that we can infer the light-cone structure in the (ν, ρ) coordinate system simply by inspecting the terms in the parentheses
  - evidently the light-cones are everywhere vertical
  - on the plane $t = 0$ they light cones have opening angle $\pi/4$ as in Minkowski space
  - but moving away from the plane they open up, and the null rays become parallel to the ρ axis as $|\nu| \to 1$
  - thus it might seem that light-rays can never reach $\nu = 1$ (the horizon $t = x$)
- this is reminiscent of the behaviour of light-rays in Schwarzschild geometry in (t, r) coordinates
- but here we know that this is simply an artefact of the choice of coordinate system
- if one had discovered the metric expressed in terms of (ν, ρ) coordinates as a vacuum solution of the field equations it might not be obvious that if one were to make the inverse transformation
  - $t = e^\rho/\sqrt{\nu^{-2} - 1}$
    $x = e^\rho/\sqrt{1-\nu^2}$
- then one would recover the much simpler metric
  - $ds^2 = -dt^2 + dx^2$
- and one would realise
  - that the line $t = x$ (or $\nu = 1$) is not in any way a physical barrier and that space-time is regular there
  - that the (ν, ρ) coordinate system only covers part of the entire space-time
  - and – as a bonus – the causal connectedness of different regions in the entire space-time would be self-evident

- as we will now show, there is a closely analogous transformation from the bad t, r Schwarzschild coordinates – the analogue of ν, ρ coordinates here – to 'well-behaved' v, u coordinates – the analogue of t, x here – found by Kruskal and Szekeres in 1960
- this coordinate system further elucidates the nature of the surface $r = 2M$ in Schwarzschild space-time and makes the causal structure of the space-time clear
- and it also seems to suggest that what we have been considering as the range of the space-time is incomplete and only covers half of the total manifold
  - rather similar to the way that Rindler's wedge only covers part of Minkowski space

5 Kruskal-Szekeres coordinates

- the apparent 'singularity' in g_{rr} at $r = 2M$ stems from a badly behaved coordinates
- in the 60's Kruskal and Szekeres found a coordinate transformation $r, t \to u, v$
  - $\begin{array}{l} u = |r/2M - 1|^{1/2} e^{r/4M} \cosh(t/4M) \\ v = |r/2M - 1|^{1/2} e^{r/4M} \sinh(t/4M) \end{array}$ for $r > 2M$
  - and the same but with $\cosh \leftrightarrow \sinh$ for $r < 2M$
- in terms of which the line element is
  - $ds^2 = (32M^3/r) r^{-r/2M} (-dv^2 + du^2) + r^2(d\theta^2 + \sin^2\theta d\phi^2)$
- where $r(u, v)$ is to be considered the solution of $(r/2M - 1)e^{r/2M} = u^2 - v^2$
- and the other part of the inverse transformation is $t = 4M \tanh^{-1}(v/u)$ (for $r > 2M$) and $t = 4M \coth^{-1}(v/u)$ otherwise

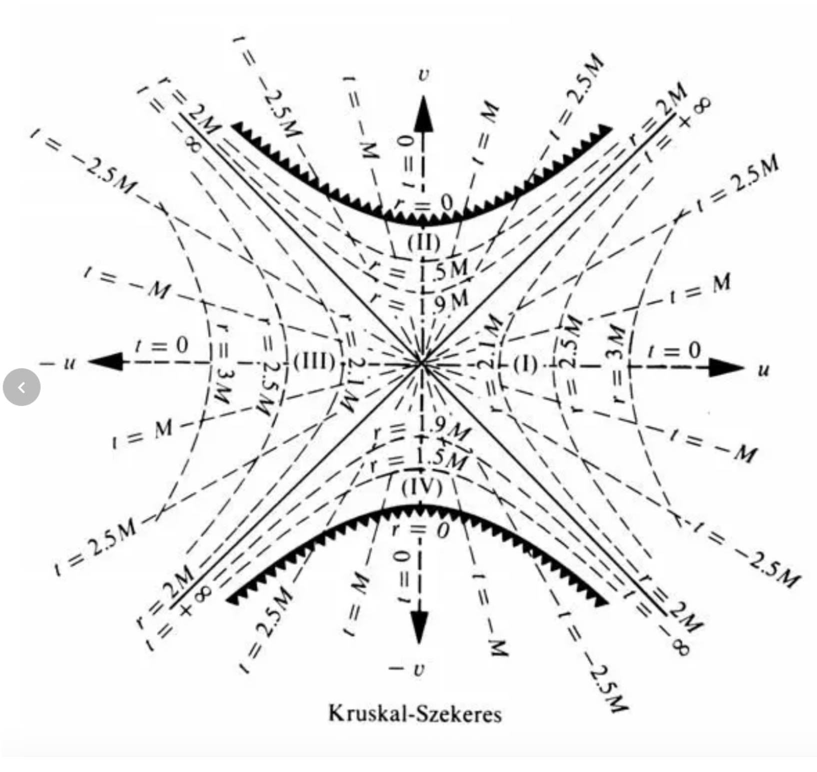


Figure 8: The Kruskal-Szekeres $u - v$ coordinate system. Each point represents a sphere in space. Light cones (for radial rays) are at 45 degrees from the vertical axis. Time-like trajectories at less than 45 degrees from the vertical. The horizon is along the diagonals. The hyperbolae are surfaces of constant r. The original Schwarzschild $r - t$ coordinates cover the regions labelled I and II but the full $u - v$ plane includes another exterior universe III and another region with $r < 2M$ (IV). It is not clear whether these are really distinct regions. But if they are, a trajectory like the cycloid solution would leave the lower (white hole) singularity in region IV, emerge from $r = 2M$ into region I (or III), turn around, and then fall back into the black-hole singularity in region II.

- there are many interesting features of the Kruskal-Szekeres diagram
  - lines of constant r are hyperbolae in (v, u) coordinates, just like the constant χ (or ρ) world-lines of Rindler's accelerated observers in (t, x) coordinates

- lines of constant t radiate out from the origin in (v, u) space just like lines of constant ν in (t, x) coordinates
- the hyperbolic sines and cosines are reminiscent of the expressions for t and x in Rindler space-time
- radial null paths are lines of 45 degrees everywhere – this makes it very easy to visualise the paths of particles of any kind
- though it must be kept in mind that each point in u, v space corresponds to a sphere spanned by θ and ϕ
- however, there appear to be four distinct regions:
  * the Eastern sector (conventionally sector I) is the 'exterior' region $r > 2M$
  * the Northern sector (sector II) is the 'interior black-hole' region $r < 2M$ and contains the singularity $r = 0$
  * the Southern sector (sector IV) is the 'interior white-hole' region $r < 2M$, which also contains $r = 0$
  * then, off to the West, is sector III, the 'parallel exterior region'
- the full Kruskal space-time is known as the 'maximal extension' or 'continuation' of Schwarzschild space-time

• one can infer many useful things from this diagram
  - if a particle falls into a black-hole, emitting pulses of light as it does so, then for a distant observer at $r \gg 2M$, where coordinate time faithfully reflects proper time, it will seem to take an infinite time for the particle to reach the horizon $r = 2M$
    * that is because the wavelength of the emitted radiation, and hence also the time between pulses, gets redshifted – infinitely so as $r \to 2M$ – and while the particle emits only a finite number of pulses these take an infinite amount of time to reach the distant observer
    * note that for a freely falling particle this is different to the redshift we discussed for light emitted by an observer at constant r – such an observer would see the infalling observer redshifted too, so the net effect is much larger

• this diagram allows one to depict and clarify many properties of radial trajectories and the Oppenheimer-Snyder model
  - if one had a star of constant radius $r > 2M$ which, at some point, lost its source of pressure support and went into free-fall collapse as in the Oppenheimer & Snyder model then only the sectors I and II would be relevant
  - only the region exterior to the surface of the star is Schwarzschild space-time
  - if it has constant r, for $t < 0$ say, then this would be one of the hyperbolae in region I
  - after $t = 0$ the surface of the freely collapsing star would follow a roughly vertical trajectory

• as mentioned, the world-line of a particle – or the surface of a dust sphere – following the entire cycloidal trajectory, on the other hand, would be a roughly vertical line (actually somewhat 'bowed out' away from the origin in u, v space) starting in region IV, emerging into region I (or perhaps III), and then continuing into region II to eventually meet the black-hole singularity
  - radiation from the outgoing particle on a cycloidal orbit escapes to infinity even from inside $r = 2M$ (in sector IV)
    * the external observer can also see photons emitted from the naked singularity 'before' – i.e. at earlier coordinate time t – than the event of the emergence of the particle from the singularity
  - all trajectories of physical particles from any event inside region II will inevitably end on the singularity
  - all events in the exterior region I have at least some of their future that remains in sector I

• are the black- and white-hole regions (and regions I and III for that matter) really distinct?

  - all points in t, r space get mapped to *two* points in u, v coordinates
  - those points being diametrically opposite: $u', v' = -u, -v$
  - if one takes the view that u', v' is just the image of the same event u, v then one is forced to conclude that in any region of the space-time there can be particles whose direction of proper time proceeds in the opposite direction to that of neighbours
  - but infalling particles in region II still cannot communicate to the outside world
    * the only way they can 'meet' outgoing particles is for such particles to collide with them in their (the infalling particle's) past
    * any particles that consider themselves to be outgoing are seen by the infalling observer to be infalling also

- the more conventional view (that events reflected through the origin but having the same t, r are truly distinct) has considerable appeal, for two reasons
  - it avoids the temporal counter-flow problem (if indeed it is a problem)
  - the extra spatial region to the left allows one to 'stitch' Schwarzschild geometry onto the exterior of a partial closed universe with $\chi > \pi/2$
    * as we remarked, the boundary here – where area is *decreasing* with increasing radius – seems to require that one connect this to a vacuum exterior that is something like the neck of a flask that narrows and then opens out
    * this is exactly what adding section III seems to provide
    * in this picture, the exterior of the FRW partial sphere would be a geodesic on the left side of the figure, say, with the region to the left side of that being outside the Schwarzschild region (i.e. inside the FRW hypersphere)
    * considering the horizontal $t = 0$ plane, this would be a neck that starts at $r > 2M$, narrows down to $r = 2M$ and then opens up again into region I.

- Zero and negative energy solutions:
  - radial orbits with $E = 0$ are perfectly acceptable
    * these have $\mathcal{E} = 1$ and so the apogee is at $r = 2M$
    * and (since $E = -p_t = g_{tt}(dt/d\tau)$) they are lines of constant t
    * so they pass through the origin
  - orbits with $E < 0$ are also possible – indeed they are essential if we take seriously the idea that the extra regions are truly distinct parts of the manifold and we want to avoid temporal counter-flows
    * in region III time is decreasing in the vertical direction – so if particles in that region are observed by constant r observers to have positive energy they must be moving downwards!
    * and in the left hand side of region IV time is increasing as one moves up and away from the white-hole singularity
    * but we had concluded earlier that for $r < 2M$ particles have $dt/d\tau < 0$ – which would require particles on the LHS to be moving downwards too
    * that, however, was assuming that $E > 0$
    * the resolution is that any geodesic particles that have paths that take them through region III must have $E < 0$
    * then all geodesics have proper time that is generally increasing as they proceed in an upward direction

- Are regions III and IV physically relevant?
  - many reputable books on the subject (e.g. Hobson et al.) question whether white-holes really exist
  - Wald dismisses the idea that regions III and IV are real
  - there is the long standing conjecture of 'cosmic censorship' that holds that nature never allows a singularity to be seen from the outside
  - my feeling is that the big-bang is a prima facie counter example to this

6 Non-radial orbits and the precession of the perihelion of Mercury

6.1 Newtonian orbits

- The equation of conservation of energy for a Newtonian unit mass test particle in orbit around a mass M is
  - $\dot{r}^2/2 + (r\dot{\phi})^2/2 = E + M/r$
- and with conservation of angular momentum $L = r^2\dot{\phi}$ this is
  - $\dot{r}^2/2 = E - V(r)$
  - with effective potential energy
  - $V(r) \equiv L^2/2r^2 - M/r$
- the potential has a single minimum at $r_{\min} = L^2/M$ and with $V(r_{\min}) = -M^2/2L^2$
- if $E > V(r_{\min})$ then r will oscillate between two turning points
- so $r_{\min}$ is the radius of a circular orbit
  - for which the angular frequency is
  - $\omega_\phi = \dot{\phi} = L/r^2 = M^2/L^3$
- differentiating the energy equation with respect to time and dividing by $\dot{r}$ gives
  - $\ddot{r} = -M/r^2 + L^2/r^3$
- and substituting $r = L^2/M + u$ gives, to first order in u/r
  - $\ddot{u} = -(M^4/L^6)u$
- so the radial displacement undergoes simple harmonic oscillatory motion with frequency $\omega_r = M^2/L^3$, exactly the same as the angular frequency
- so this proves that for nearly circular orbits, the radial motion has the same frequency as the angular motion, so the orbits are closed
- of course Newton proved that this is exactly true even if the orbits are highly non-circular, but the simple proof here is just a 'warm-up' for the analogous relativistic problem
- Footnote:
  - Newton was apparently impressed that there were only two types of potential that allowed closed orbits – and thought that these were the ones that would be chosen by Nature.
  - These are $\phi \propto -1/r$ and $\phi \propto r^2$.
  - It is very interesting that the latter is what you get (aside from the sign) for 'dark energy' – so Newton may be said to have invented dark energy as well as Newtonian gravity!

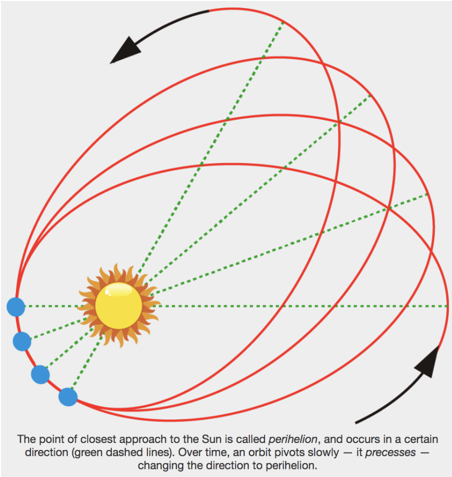
6.2 Nearly circular relativistic orbits

- the metric components are independent of t
  - so $p_t = -E$ is constant
- and are also independent of ϕ
  - so $p_\phi = L$ is constant
- considering a particle orbiting in the equatorial plane:
  - i.e. $\theta = \pi/2$ so $g_{\phi\phi} = r^2$

- the energy-momentum relation (for a unit mass test-particle) is
  - $-1 = g_{\alpha\beta}p^\alpha p^\beta = -(1-2M/r)(p^t)^2 + (1-2M/r)^{-1}(p^r)^2 + r^2(p^\phi)^2$
- or
  - $-1 = -(1-2M/r)^{-1}(E^2 - \dot{r}^2) + L^2/r^2$
- or
  - $\dot{r}^2 = E^2 - V(r)$
- qualitatively similar to the Newtonian formula, but now with effective potential
  - $V(r) = (1 + L^2/r^2)(1 - 2M/r)$
- differentiating $V(r)$ gives a quadratic equation for radii for which $\dot{r} = 0$ – i.e. the allowed circular orbits – with solutions
  - $r = (L^2/M \pm \sqrt{L^4/M^2 - 12L^2})/2$
- so circular orbits are only possible for angular momentum $L > \sqrt{12}M$
  - quite different from the Newtonian result
  - an incoming particle with impact parameter such that $L < \sqrt{12}M$ will fall directly to $r = 0$
- the negative square-root solution corresponds to a maximum of the potential
  - for which the orbit is unstable
- so we need to take the positive square root:
  - $r = (L^2/M + \sqrt{L^4/M^2 - 12L^2})/2$
- in the limit $M^2/L^2 \ll 1$ this gives $r = L^2/M$, in accord with the Newtonian result
  - which is reasonable since $M/L = M/(r\dot{r}) = (M/r)/\dot{r} = \dot{r}$
    - since $\dot{r}^2 = M/r$ by virtue of the virial theorem
  - so $M^2 \ll L^2 \to \dot{r}^2 \ll 1$
    - which is the Newtonian limit
- keeping the lowest order 'post-Newtonian' modifications, the radii of circular orbits are
  - $r = L^2/M(1 - 3M^2/L^2 + \ldots)$
  - where $\ldots$ indicates terms $\mathcal{O}(M^4/L^4)$ or smaller
- and the angular frequency is
  - $\dot{\phi} = p^\phi = g^{\phi\phi}p_\phi = L/r^2 = (M^2/L^3)(1 + 6M^2/L^2 + \ldots)$

6.3 Precession of orbits

- the fact that the frequency ω_r for radial displacements is identical to the angular speed $\dot{\phi}$ in Newtonian theory derives from the particular form for the effective potential $V(r) = L^2/2r^2 - M/r$
- for relativistic orbits the potential is different, and the orbits are not, in general, closed, and the point of closest approach to the sun – the perihelion – will not remain fixed but will rotate or precess
- the effect is a change in the angle of perihelion that is on the order of v^2/c^2 – so largest for Mercury, whose orbit will appear to precess as compared to those of the other planets (for which the effect is much smaller)



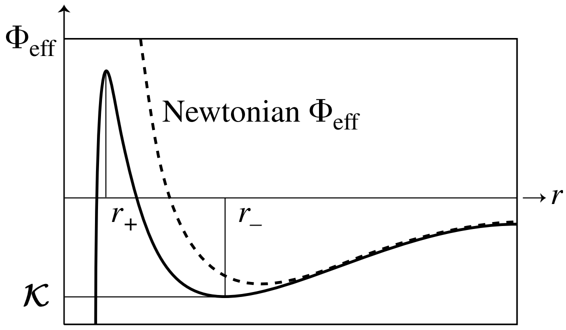


Fig. 6.12 Schwarzschild vs. Newtonian effective potential.

Figure 9: Precession of the perihelion.

- the total precession of Mercury is about 500"/century – mostly caused by the tidal effect of the other planets – but it was known since the late 19th century that about 40"/century could not be explained by tides
- to calculate this it is usual, at this point, to change variable from proper time to angle, but we can find the effect simply by looking at the equation of motion for small displacements relative to the circular orbit
- writing r as the sum of the approximate circular orbit radius $r = (L^2/M)(1 - 3M^2/L^2)$ plus a small perturbation u:
  - $r = (L^2/M)(1 - 3M^2/L^2) + u$
- or
  - $r = (L^2/M)(1 - 3M^2/L^2 + uM/L^2)$
- and inserting this in the equation of motion
  - $\ddot{r} = L^2/r^3 - M/r^2 - 3ML^2/r^4$
  - which is obtained by differentiating the energy-momentum relation with respect to proper time and dividing by $2\dot{r}$
  - and in which the last term is smaller than the first two by on the order of $\sim L^2/r^3 \sim v^2/c^2$
- and Taylor expanding we find, keeping the leading order terms which are linear in u, plus the leading order u-independent term,
  - $\ddot{u} = -(M^4/L^6)(1 + 6M^2/L^2 + \ldots)u - 6M^7/L^8 + \ldots$
- note that you need to go to second order when expanding $r = (L^2/M)(1 + \epsilon)$ in the first two terms in the equation of motion since $\epsilon^2 = (-3M^2/L^2 + uM/L^2)^2$ contains a component that is linear in u
- the last term above gives a slight correction to the stable orbit radius – remember, we used as a baseline the approximate formula for this. The stable circular orbit must have $\ddot{u} = 0$ from which we find $u \simeq 06M^4/L^2$ so $r_{\rm circ} = L^2/M * 1 - 2M^2/L^2 06M^4/L^4 + \ldots$.
- otherwise this offset has no significant impact and we find that the frequency of radial oscillations ω_r for displacements about $r_{\rm circ}$ is
  - $\omega_r = \sqrt{-\ddot{u}/u} = (M^2/L^3)(1 + 3M^2/L^2 + \ldots)$
- while the orbital angular speed is, as we found above,
  - $\omega_\phi = \dot{\phi} = (M^2/L^3)(1 + 6M^2/L^2 + \ldots)$

- so these are indeed different. And it is this difference that accounts for the anomalous precession of the perihelion of Mercury

7 The equations of stellar structure

In this section, we will use 'geometrized' units such that, numerically, $c = 1$ and $G/c^2 = 1$ and we will be sloppy/lazy and not write c or G/c^2 and leave it up to the reader to figure out where they go. The symbol m represents mass, but appears here to have the same units as r which is a length. You should think of m as being equal to the physical mass times G/c^2. Similarly ρ represents mass density, but appears here to have the same units as pressure P, which has units of energy- rather than mass-density. So you should think of ρ as representing physical mass density times c^2. We will refer to the time coordinate as $x^0 = ct$ and thus avoid explicit reference to t. The potentials Φ and Λ here are dimensionless.

7.1 The field equations

The Ricci tensor for the static, spherically symmetric metric with line element

$$ds^2 = -e^{2\Phi(r)}(dx^0)^2 + e^{2\Lambda(r)}dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2 \tag{1}$$

is shown below (§A) to have non-vanishing components

$$\begin{aligned} R_{00} &= (\Phi'' + \Phi'^2 - \Phi'\Lambda' + 2\Phi'/r)e^{2(\Phi-\Lambda)} \\ R_{rr} &= -(\Phi'' + \Phi'^2 - \Phi'\Lambda' - 2\Lambda'/r) \\ R_{\theta\theta} &= 1 - (r\Phi' - r\Lambda' + 1)e^{-2\Lambda} \\ R_{\phi\phi} &= \sin^2\theta R_{\theta\theta} \end{aligned} \tag{2}$$

Its trace (the Ricci scalar) is

$$R = g^{\mu\nu}R_{\mu\nu} = 2/r^2 - 2e^{-2\Lambda}(\Phi'' + \Phi'^2 - \Phi'\Lambda' + 2(\Phi' - \Lambda')/r + 1/r^2) \tag{3}$$

from which we find the non-vanishing components of the Einstein tensor $G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R$ are

$$\begin{aligned} G_{00} &= \frac{e^{2\Phi}}{r^2}\frac{d}{dr}(r(1 - e^{-2\Lambda})) \\ G_{rr} &= -\frac{e^{2\Lambda}}{r^2}(1 - e^{-2\Lambda}) + (2/r)\Phi' \\ G_{\theta\theta} &= r^2 e^{-2\Lambda}(\Phi'' + (\Phi')^2 + \phi'/r - \Phi'\Lambda' - \Lambda'/r) \\ G_{\phi\phi} &= \sin^2\theta G_{\theta\theta} \end{aligned} \tag{4}$$

Einstein's equations relate these to the corresponding components of the stress-energy tensor. Assuming a perfect fluid with density ρ and pressure P this is

$$T_{\mu\nu} = (\rho + P)U_\mu U_\nu + g_{\mu\nu}P \tag{5}$$

where $\vec{U}$ is the 4-velocity of an observer who sees there to be neither energy flux density nor momentum density. The validity of this may be established as in a locally inertial frame that is instantaneously comoving with the fluid this gives $T_{\mu\nu} = \text{diag}\{\rho c^2, P, P, P\}$.

The 4-velocity of an observer at constant r, θ and ϕ is $\vec{U} \to (U_0, 0, 0, 0)$ and the normalisation condition $g^{\mu\nu}U_\mu U_\nu = -c^2$ gives $U_\mu = c\delta^0_\mu e^\Phi$ which, working in geometrized units as we are doing here, gives

$$\begin{aligned} T_{00} &= e^{2\Phi}\rho \\ T_{rr} &= e^{2\Lambda}P \\ T_{\theta\theta} &= r^2 P \\ T_{\phi\phi} &= \sin^2\theta T_{\theta\theta} \end{aligned} \tag{6}$$

and the field equations are obtained by equating corresponding components in (4) and (6).

7.2 The equation of hydrostatic equilibrium

7.2.1 Hydrostatic equilibrium in static spherically symmetric space-times

The $\alpha = r$ component of $T^{\alpha\beta}{}_{;\beta} = 0$ provides one of the equations of stellar structure. From $T^{\alpha\beta}{}_{;\gamma} = T^{\alpha\beta}{}_{,\gamma} + \Gamma^{\alpha}{}_{\mu\gamma} T^{\mu\beta} + \Gamma^{\beta}{}_{\mu\gamma} T^{\alpha\mu}$ this is

$$T^{r\beta}{}_{;\beta} = T^{r\beta}{}_{,\beta} + \Gamma^{r}{}_{\mu\beta} T^{\mu\beta} + \Gamma^{\beta}{}_{\mu\beta} T^{r\mu} = 0 \tag{7}$$

or, since the stress tensor is diagonal,

$$T^{rr}{}_{,r} = -\Gamma^{r}{}_{\mu\beta} T^{\mu\beta} - \Gamma^{\beta}{}_{r\beta} T^{rr} \tag{8}$$

The contravariant components of the stress tensor are readily found to be $T^{00} = (g^{00})^2 T_{00} = e^{-2\Phi}\rho$ etc. so the left hand side is $T^{rr}{}_{,r} = (Pe^{-2\Lambda})'$ while the sum in the last term $\Gamma^{\beta}{}_{r\beta} = \Phi' + \Lambda' + 2/r$ and using the Christoffel symbols from §A to calculate the first we find

$$\boxed{(\rho + P) d\Phi/dr = -dP/dr} \tag{9}$$

which is the relativistic version of the equation of hydrostatic equilibrium (which looks the same as the Newtonian equation but with the enthalpy $\rho + P$ in place of ρ on the left hand side). Since pressure is the momentum flux density, and the right hand side is (minus) its divergence, so equal to the rate at which the momentum density would be increasing in the absence of gravity, this equation expresses conservation of momentum.

7.2.2 Hydrostatic equilibrium from the equivalence principle

A simpler way to obtain (9) is to invoke the principle of equivalence. Consider an observer in flat space-time who is being accelerated along the z-direction with acceleration a and who is holding a container containing a perfect fluid with $T^{\mu\nu} = \mathrm{diag}(\rho, P, P, P)$. The metric of space-time, in the vicinity of the observer, and with spatial coordinates such that the observer is at the origin, is $g_{\mu\nu} = \mathrm{diag}(-(1 + az), 1, 1, 1)$ where we see the familiar 'warping' of time – or 'gravitational time dilation' – as perceived in an accelerated frame. The non-vanishing Christoffel symbols are $\Gamma^{z}{}_{00} = \Gamma^{0}{}_{0z} = a$, and so we find that the equation of continuity of z-momentum is

$$T^{z\beta}{}_{;\beta} = 0 = T^{z\beta}{}_{,\beta} + \Gamma^{z}{}_{\alpha\beta} T^{\alpha\beta} + \Gamma^{\beta}{}_{\gamma\beta} T^{z\gamma} \tag{10}$$

or, since $T^{z\beta}{}_{,\beta} = T^{zz}{}_{,z} = dP/dz$ and $\Gamma^{z}{}_{\alpha\beta} T^{\alpha\beta} + \Gamma^{\beta}{}_{\gamma\beta} T^{z\gamma} = \Gamma^{z}{}_{00} T^{00} + \Gamma^{0}{}_{0z} T^{zz} = (\rho + P)a$

$$dP/dz = -(\rho + P)a \tag{11}$$

in, as in (9), we see that pressure gradient needed to accelerate the fluid is minus the enthalpy times the acceleration.

This means that the appearance of $\rho + P$ in place of ρ in the equation of hydrostatic equilibrium is a purely special (rather than general) relativistic effect.

The physical reason for this relativistic correction is a little subtle but interesting. Consider an element of gas with kinetic pressure from the thermal motion of the atoms. If an atom's thermal velocity in the frame of the gas is $\mathbf{u}$ then the z-momentum of its momentum is $p_z = m\gamma_{\mathbf{u}} u_z$. Boosting into the (primed) 'lab-frame', with respect to which the gas is moving at small z-velocity $\overline{v}$ (because the gas has been accelerated for a small time) the momentum is $p'_z = m\gamma_{\overline{v}}\gamma_{\mathbf{u}}(\overline{v} + u_z)$. Averaging this over $\mathbf{u}$ would suggest that the mean momentum would be $\langle p'_z \rangle = m\gamma_{\overline{v}} \langle \gamma_{\mathbf{u}}(\overline{v} + u_z) \rangle$. But $\langle \gamma_{\mathbf{u}} u_z \rangle = 0$, so $\langle p'_z \rangle = m\gamma_{\overline{v}}\overline{v}\langle \gamma_{\mathbf{u}} \rangle = m\langle \gamma_{\mathbf{u}} \rangle \overline{v} \times (1 + \mathcal{O}(\overline{v}^2))$. With $\overline{v} = a\Delta t$, the change in the mean momentum per particle is, at linear order in Δt, just $d\langle p'_z \rangle/dt = m\langle \gamma_{\mathbf{u}} \rangle a\Delta t$. But $m\langle \gamma_{\mathbf{u}} \rangle$ (times c^2) is the mean particle energy and n times this is ρ so the rate of change of z-component of the momentum density is $d\pi_z/dt = \rho a$ – with no extra contribution from the pressure – whereas the left hand side of (11) is the divergence of P, and P is the momentum flux density. So this might seem to say that conservation of momentum would require simply $dP/dz = -\rho a$.

The flaw in the above argument is that we are taking the average $\langle \gamma_{\mathbf{u}}(\overline{v} + u_z) \rangle$ in the frame of the gas whereas we should be doing it in the lab-frame. Consider a gas consisting of two streams of particles; one moving in the z direction with speed $u = +u_0$ and the other moving in the $-z$ direction with the same speed, i.e. $u = -u_0$. In the gas frame they have the same space density[1] $n_+ = n_- = n/2$. *But in the lab-frame they have different densities.* If the gas is moving at small positive velocity $\overline{v} \ll u_0$ then both of these streams will be length contracted in the lab-frame, but the positively moving stream will have its density enhanced by a factor $\gamma_+ = \gamma(u_0 + \overline{v})$ while that of the negatively moving stream will be enhanced by a factor $\gamma_1 = \gamma(u_0 - \overline{v})$ so $\gamma_+ > \gamma_-$ and there therefore is an excess of positively moving particles.

Using the definition $\gamma = 1/\sqrt{1-v^2}$ we have $\gamma_\pm \simeq \gamma(u_0)(1 \pm \gamma(u_0)^2 \overline{v} u_0)$. Let's consider a gas of non-relativistic atoms for simplicity, so the densities are $n_\pm = \frac{1}{2} n(1 \pm \overline{v} u_0)$. The momentum density for the streams is $\pi_{z\pm} = mn_\pm \gamma_{u_0}(\overline{v} \pm u_0)$ so their sum is

$$\pi_z = \tfrac{1}{2} mn\gamma_{u_0}[(1 + \overline{v}u_0)(\overline{v} + u_0) + (1 - \overline{v}u_0)(\overline{v} - u_0)) \tag{12}$$

or, to first order in $\overline{v}$,

$$\pi_z = mn\gamma_{u_0}\overline{v}(1 + u_0^2). \tag{13}$$

Now $mn\gamma_{u_0}$ is ρ the energy density and $mn\gamma_{u_0}u_0^2$ is the $z - z$ component of the pressure tensor (let's call it P), so the momentum density changes with time as

$$d\pi_z/dt = (\rho + P)d\overline{v}/dt. \tag{14}$$

In more detail, one would average over a distribution of velocities and generalise to the case where u_0 may be comparable to c (where one has to be more careful about how velocities add). But hopefully the above argument is sufficient to convince you that this is the correct physical explanation of the presence of the enthalpy rather than just the energy density in the equation of hydrostatic equilibrium.

7.2.3 The acceleration of constant-r observers

Equation (11) might seem to suggest that $d\Phi/dr$ can be identified with the acceleration of a constant-r observer in the space-time described by (1). This is, in essence, correct, but we need to be careful. The derivatives in (9) are with respect to coordinate position r while in (11) we have the derivative of P with respect to physical distance. To obtain the physical acceleration a for a constant-r observer we can use the geodesic equation that describes the motion of a test particle that such an observer would release. At the moment the test particle is released, when it has $U^\mu = (e^{-\Phi}, 0, 0, 0)$, this is

$$d^2r/d\tau^2 = -\Gamma^r{}_{00}U^0U^0 = -\Phi' e^{-2\Lambda} \tag{15}$$

but the physical distance travelled is related to coordinate distance by $dr_{\rm phys} = \sqrt{g_{rr}}dr = e^\Lambda dr$ so this equation says

$$d^2r_{\rm phys}/d\tau^2 = -d\Phi/dr_{\rm phys} \tag{16}$$

so the acceleration that the constant-r observer perceives, which is of course opposite to that of the test particle he releases, is

$$a = d\Phi/dr_{\rm phys} \tag{17}$$

so Φ here plays the same role as the Newtonian potential (rendered dimensionless by dividing by c^2), but $d\Phi/dr \neq a$, rather $d\Phi/dr = e^{-\Lambda}d\Phi/dr$.

7.3 The other equations of stellar structure

7.3.1 The $G_{rr} = 8\pi T_{rr}$ and $G_{00} = 8\pi T_{00}$ equations

Two more stellar structure equations can be obtained from the field equations. It proves convenient at this point to replace $\Lambda(r)$ with

$$m(r) \equiv r(1 - e^{-2\Lambda(r)})/2 \tag{18}$$

in terms of which

$$g_{rr} = e^{2\Lambda} = (1 - 2m(r)/r)^{-1} \tag{19}$$

so this is like g_{rr} in the Schwarzschild solution, but now with a r-dependent mass.

[1]These densities are not the same as the space-density in the rest frame of the respective streams, as both are length contracted, and their gas-frame densities are higher than the rest-frame by a factor $\gamma(u_0)$.

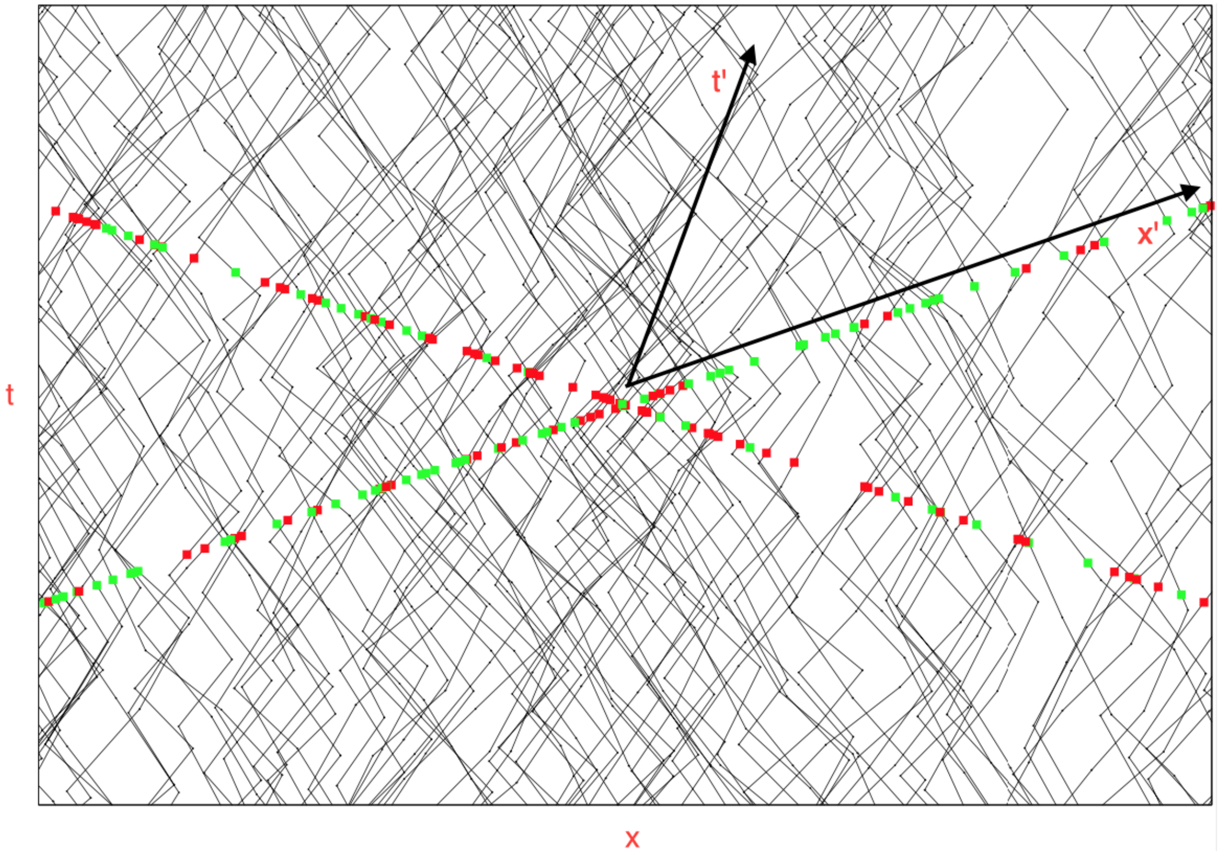


Figure 10: Why enthalpy $\rho + P$ (rather than just energy density ρ) appears in the equation of hydrostatic equilibrium. Space-time diagram showing trajectories of particles undergoing random deflections. Coloured symbols show intersection with hypersurfaces $t = 0$ for an observer moving with respect to the frame of rest of the particles. Right (left) moving intersections are plotted as red (green) symbols. Right (left) moving observer intercepts more left (right) moving trajectories. Conversely, if the gas is moving towards $+x$ in the lab-frame then the lab-frame observer will see a higher density of relatively faster moving particles than relatively slower moving particles.

The time-time component of the field equations $G_{00} = 8\pi T_{00}$ then implies

$$\boxed{dm/dr = 4\pi r^2 \rho} \tag{20}$$

which relates our new parameterisation of the radial metric coefficient $g_{rr} = (1 - 2m(r)/r)^{-1}$ to the (energy) density ρ, while the $r - r$ component $G_{rr} = 8\pi T_{rr}$ gives

$$\boxed{d\Phi/dr = (m + 4\pi r^3 P)/(r(r - 2m))} \tag{21}$$

which is evidently the relativistic equivalent to the Newtonian $d\Phi/dr = (G_{\rm N}/c^2)m/r^2$ – being equal to this in the limits that $r \gg m$ and $P \ll m/r^3 \sim \rho$ – which gives the gravitational potential gradient (here being the gradient of the logarithm of $\sqrt{g_{00}} = e^{\Phi}$).

The appearance of the pressure in the first factor on the right hand side along side m is often described as saying *"pressure gravitates in GR"*. We will discuss this further below. But here we should emphasise that this is *not* simply reflecting the fact that pressure – in the form of radiation pressure or kinetic pressure – makes a contribution to the energy density. That – the extra kinetic energy density of moving atoms or molecules or radiation (or whatever) over an above their rest-mass energy density – is already included in ρ appearing in the field equations.

7.3.2 Stellar structure of stars undergoing nuclear fusion

The boxed equations above give 3 equations for the 4 unknown functions $\rho(r)$, $P(r)$, $\Phi(r)$ and $m(r)$. This is unfortunate, but to be expected. What is needed to 'close the loop' is some rule to give the pressure, for example. For a star undergoing nuclear fusion this requires 2 extra components: First there is nuclear physics, which gives the rate of energy generation $\dot{\mathcal{E}}$, which is a function of density and temperature as well as chemical composition. Second there is radiative transfer, which relates the radiative energy flux density $\mathcal{F}$ to the gradient of the temperature (which is a function of density and pressure: $T = T(\rho, P)$ being the 'equation of state'). This also involves the opacity κ, which again depends on density and temperature as well as chemical composition. In steady state, $\dot{\mathcal{E}}$ and the gradient of $\mathcal{F}$ are related by the equation of energy conservation.

So with the equations of a) nuclear physics, b) radiative transfer, c) the equation of state and d) energy conservation together with the equations above for e) hydrostatic equilibrium (or momentum conservation), f) the $m(r)$-density relation and g) the $G_{rr} = 8\pi T_{rr}$ equation giving the response of Φ to this component of the matter stress, we have 7 equations for 7 unknowns[2] $\rho(r)$, $P(r)$, $\Phi(r)$, $m(r)$, $\dot{\mathcal{E}}$, $\mathcal{F}$ and T.

7.3.3 Stellar structure of white dwarfs

Another, much simpler but important, situation is for white dwarfs and other stellar remnants which have exhausted their supply of nuclear energy and are supported, in the case of WDs, by electron degeneracy pressure. There the pressure P, being the flux density of momentum, is the product of the (electron) density n, the typical momentum, which is on the order of the Fermi-momentum p_F (which, being the momentum of an electron whose quantum mechanical wavelength is on the order of the mean separation $\sim n^{-1/3}$, varies as the 1/3 power of the density) and the velocity of the electrons, which is proportional to p_F for densities such that the $p_F \ll m_e c$, i.e. non-relativistic electrons, giving an equation of state $P \propto \rho^{5/3}$, but which saturates at $v = c$ for highly relativistic electrons, giving, in that limit, the 'softer' equation of state $P \propto \rho^{4/3}$. This gives, in the non-relativistic limit, a 1-parameter family of solutions, in which the radius R varies as $M^{-1/3}$, but, as shown by Chandrasehkar, with inclusion of the relativistic effects described above the radius falls to zero at the *Chandrasehkar mass* – about $1.4 \times M_\odot$ for realistic chemical composition – which gives an upper limit to the allowed mass of WDs.

7.3.4 The exterior solution

Outside the star we have $\rho = P = 0$ and so $dm/dr = 0$ and $d\Phi/dr = m/(r(r - 2m))$, the solution of which gives the Schwarzschild metric.

7.4 The Tolman-Oppenheimer-Volkov equation

The equation of hydrostatic equilibrum implies $d\Phi/dr = -(\rho+P)^{-1}dP/dr$ while the equation of momentum conservation (the G_{rr} equation), gives another independent expression for $d\Phi/dr$. Eliminating $d\Phi/dr$ gives the TOV equation

$$dP/dr = -(\rho + P)(m + 4\pi r^3 P)/(r(r - 2m)). \tag{22}$$

In the non-relativistic limit we have $P \ll \rho$ (and therefore $4\pi r^3 P \ll m$) and $r \gg m$ so this becomes the usual Newtonian expression for hydrostatic equilibrium $dP/dr = -\rho d\phi/dr = -\rho Gm/r^2$. We see above that including relativistic effects acts to increase all three factors on the right hand side.

Together with the equation for dm/dr and the equation of state (for degenerate electrons say) this can be solved to give $P(r)$, $\rho(r)$ and $m(r)$ where the the boundary conditions are $m(0) = 0$ and the value of the central density $P(0)$ (or the central density $\rho(0)$). Given these conditions at $r = 0$ the TOV equations can be integrated (numerically) out to the point $r = R$ where $P(r) = 0$ which is the surface of the star, giving a 1-parameter family specified by the central density.

7.5 The meaning of $m(r)$

The equation for $m(r)$:

$$dm/dr = 4\pi r^2 \rho \tag{23}$$

makes it look like its integral

$$M = m(R) = \int dr 4\pi r^2 \rho \tag{24}$$

[2]Or 8 equations for 8 unknowns if we include the 'constituency relation' for the opacity κ and add that to the list of variables. And it's actually more complicated still as stars may have regions that are convectively unstable – if the specific entropy decreases with height – and one has to then allow for the fact that convection will render such regions isentropic.

is just the volume integral of the energy density.

But that is incorrect, as $4\pi r^2 dr$ is not the *proper* volume element, which is $e^{\Lambda} 4\pi r^2 dr$. So the integrated energy density is

$$\tilde{M} = \int dr 4\pi r^2 e^{\Lambda} \rho = 4\pi \int dr r^2 \rho(r)(1 - 2m/r)^{-1/2} \tag{25}$$

which is greater than the active gravitational mass M appearing in the exterior metric. The origin of the difference is the gravitational binding energy. A given amount of proper mass density in a potential well has lower energy and reduces, for instance, its influence on velocity of a planet orbiting the star.

7.6 Does pressure really gravitate in GR?

It is interesting that the active mass M does *not* include the gravitational attraction of pressure. This raises the interesting question: does pressure *really* gravitate in GR? If it does, why does it not appear in M?

We see above that pressure appears along with the energy density in the equation for $\Phi' = d\Phi/dr$. And it appears also in the equation of momentum conservation for that matter, which also involves Φ'. But how do we know this is a real physical change of the gravitational field – with real locally observable consequences – rather than some kind of coordinate artefact?

One way to see that there is a real observable effect is to consider two solutions to the structure equations that have the identical $m(r)$, and therefore the same energy density ρ and the same spatial metric potential $\Lambda(r)$, but with slightly different pressure. There are two equations, (9) and (21), involving Φ'. The first gives, for changes in the pressure, its gradient, and the potential gradient (holding ρ fixed):

$$(\rho + P)\delta\Phi' + \Phi'\delta P = -\delta P' \tag{26}$$

while the second gives

$$\delta\Phi' = \frac{4\pi r^3}{r(r - 2m)} \delta P. \tag{27}$$

Using the second to eliminate $\delta\Phi'$ from the first gives

$$\delta P' = -\left(\Phi' + \frac{4\pi r^3(\rho + P)}{r(r - 2m)}\right) \delta P \tag{28}$$

or

$$\frac{d}{dr} \log \delta P = -\left(\Phi' + \frac{4\pi r^3(\rho + P)}{r(r - 2m)}\right) \tag{29}$$

with solution

$$\delta P = A \exp\left\{-\left(\Phi' + \frac{4\pi r^3(\rho + P)}{r(r - 2m)}\right) r\right\} \tag{30}$$

where A is a constant. Now for some stellar profiles this would not make much sense; for any profile with an edge, this would imply a non-zero δP in the exterior. But for edge-less solutions this is physically reasonable. For example, for something like an isothermal sphere where the pressure is proportional to the density and where, in the Newtonial limit $\rho \propto 1/r^2$ and $\Phi' \propto 1/r$, so the argument of the exponential is constant, and the solution would be the rather reasonable constant offset to the pressure. The point here is not to treat solutions for realistic stars; it is simply to find a 'test-case' and to explore what would be the physical consequences of making a change to the pressure, along with the corresponding change in the potential gradient given by (27).

So, given some suitable solution the the structure equations we can consider making a change to the pressure P and potential Φ according to (30) and (27) while maintaining hydrostatic equilibrium and conserving energy and maintaining the spatial metric unchanged. If we did this, the temporal potential $\Phi(r)$ would change instantaneously. The question is then: does this have any observable effect?

There are various ways to see that the answer is yes: For one thing, as we have seen, $\Phi' e^{-\Lambda}$ is the acceleration felt by a constant-r observer. Another observable consequence of a change in the potential Φ is the gravitational redshift.

And perhaps the most direct way to see that there is a physical change to the gravitational field – the curvature, that is – is to calculate the tidal deviation of a pair of test particles – which are assumed not to feel the pressure – that are released from rest with radial separation ξ. The geodesic deviation equation is

$$d^2\xi/d\tau^2 = R^r{}_{0r0} U^0 U^0 \xi \tag{31}$$

in which the tide is

$$R^r{}_{0r0}U^0U^0 = e^{-2\Lambda}(-\Phi'' - \Phi'^2 + \Phi'\Lambda') \tag{32}$$

so, even though these solutions have the same $\Lambda(r)$, as they have different Φ' and, in general, different Φ'', the tidal deviation of the test particles is observably different.

Making a change to the pressure profile – while maintaining equilibrium and keeping the energy density profile fixed – therefore changes the gravitational field inside the star. So pressure really does gravitate. But, in the example we have explored here, the pressure can be changed without changing the spatial geometry; it only affects the warping of time through its effect on the potential Φ (the spatial potential Λ was unchanged). It is the energy density alone that determines the spatial curvature.

Q1: Consider the situation illustrated in figure 11 where we have a uniform density sphere with negligible pressure (just enough to stop it from contracting). At a certain time the matter spontaneously combusts, creating a very strong pressure. But it is enclosed in a membrane that stops it from expanding – which must, of course, develop a tension in order to do so. What do you think happens to the external gravitational mass, as sensed by e.g. the orbit of a satellite? For fun, draw some arrows on the right hand figure showing the flux density of x-momentum in the fluid and the membrane.

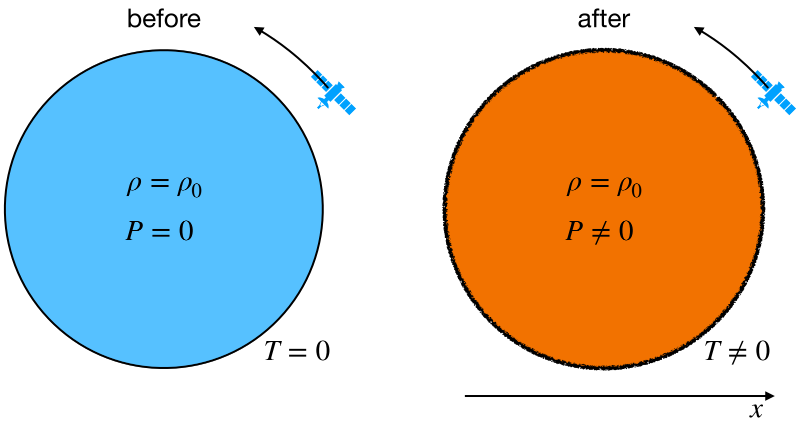


Figure 11: Bombs in a balloon. Imagine we have a sphere of uniform density with vanishing, or negligible, pressure, and a satellite in orbit around it. The sphere is actually composed of bombs, which, at a certain instant of time, explode, creating a huge pressure (but conserving energy). And the sphere is actually enclosed in a very strong membrane, which was slack before the explosion, but afterwards develops whatever tension is necessary to keep the sphere from expanding. What happens to the orbit of the satellite?

Q2: Along the same lines as Q1, imagine one has a star in hydrostatic equilibrium that converts some of its rest-mass to radiation, thus changing the pressure, which then adjusts to a new equilibrium configuration. In the process, the fluid composing the star may do PdV work, so the integrated energy density will decrease. Assuming no material is blown off in the process, what do you think happens to the exterior mass? Does it change?

7.7 Limits to the masses of stars

The details of stellar structure require detailed understanding of the equation of state for realistic matter, which is beyond our scope.

A simplistic, but still interesting, model is to assume that the density is independent of radius. This assumption replaces the equation of state and integration of the TOV equation then gives the pressure. It turns out that the central pressure would have to be greater than infinity for

$$M > M_{\rm max} = 4R/9. \tag{33}$$

It is impossible to put more mass than this inside a sphere of radius R. If stellar evolution were to lead to this – which will happen for sufficiently large M – the result would be the formation of a black-hole.

7.8 Gravity in the core of a star

Let's assume we have a star, in the centre of which the pressure and density are close to uniform. The G_{00} equation ($m' = 4\pi r^2\rho$) says that $m \simeq (4\pi/3)\rho_0 r^3$ where ρ_0 is the central energy density. This means $m/r \to 0$ as $r \to 0$, so we can put $r - 2m \to r$ in the G_{rr} which then says

$$\Phi' \simeq (m + 4\pi r^2 P)/r^2 \simeq \tfrac{4\pi}{3}(\rho_0 + 3P_0)r \tag{34}$$

which admits a solution

$$\Phi \simeq \text{constant} + \tfrac{2\pi}{3}(\rho_0 + 3P_0)r^2. \tag{35}$$

In the equation of hydrostatic equilibrium Φ' from (34) says that the pressure, unsurprisingly, cannot be exactly constant since it must have gradient $dP/dr = -(\rho + P)d\Phi/dr$ which will be proportional to r and so $P \simeq P_0 - \frac{2\pi}{3}(\rho + P)(\rho_0 + 3P_0)r^2$. This shows that a constant central density and pressure is a consistent equilibrium solution.

So the potential Φ is quadratic in the core. And in the formula (32) for the tidal field $R^r{}_{0r0}U^0U^0 \simeq \Phi''$. Thus the equation of geodesic deviation for test particles is

$$\ddot{\xi}/\xi = \tfrac{4\pi}{3}(\rho_0 + 3P_0). \tag{36}$$

We will use this in cosmology where this appears as one of the Friedmann equations.

8 The gravitational action principle

The route we have taken in developing GR followed that charted by Einstein. We assert that gravitational phenomena are the influence of curvature of the space-time manifold. Requiring agreement with Newtonian theory we are led – aside from possible ambiguities associate with the cosmological constant – to a unique rank-two contraction **G** of the curvature tensor that is 'sourced' by the matter stress-tensor **T**.

An alternative is to show that Einstein's equations can be obtained by requiring that the so-called 'Einstein-Hilbert' action

$$\boxed{S = \int d^4x\sqrt{-g}\left(\frac{R}{16\pi\kappa} + \mathcal{L}_\mathrm{m}\right)} \tag{37}$$

where R is the Ricci scalar and where $\mathcal{L}_\mathrm{m}$ is the Lorentz scalar Lagrangian density for the matter fields, be stationary with respect to variation of the metric $\delta g_{\mu\nu}$.

This is useful for two reasons. One is that it is a good starting point for thinking about possible modifications to Einstein's gravity (beyond just adding a cosmological term $\Lambda\mathbf{g}$ to **G**). The other is that the stress-energy tensor $T^{\mu\nu} = \delta\mathcal{L}_\mathrm{m}/\delta g_{\mu\nu}$ for the matter is guaranteed to be symmetric, whereas, as we saw earlier, in electromagnetism for instance, this was not the case.

8.1 The gravitational action

The Ricci scalar is $R = g^{\mu\nu}R_{\mu\nu}$ where $R_{\mu\nu} = R^\alpha{}_{\mu\alpha\nu}$ with $R^\alpha{}_{\mu\beta\nu} = \Gamma^\alpha{}_{\mu\nu,\beta} - \Gamma^\alpha{}_{\mu\beta,\nu} + \Gamma^\alpha{}_{\gamma\beta}\Gamma^\gamma{}_{\mu\nu} - \Gamma^\alpha{}_{\gamma\nu}\Gamma^\gamma{}_{\mu\beta}$ and where $\Gamma^\alpha{}_{\mu\nu} = \frac{1}{2}g^{\alpha\beta}(g_{\beta\mu,\nu} + g_{\beta\nu,\mu} - g_{\mu\nu,\beta})$.

In mechanics, the Lagrangian – the integrand in the action integral $S = \int dtL$ – would contain functions of positions of particles and, quadratically, their velocities. For fields, the Lagrangian density – the integrand in $S = \int dt \int d^3x\mathcal{L}$ – contains the fields and, again quadratically, their first derivatives with respect to space-time coordinates. Here, the Ricci scalar contains, quadratically, derivatives of the metric components in the terms coming from products of Christoffel symbols. But it also contains second derivatives via the first two terms in the curvature tensor. But these only appear at first order, and the Ricci scalar can be written as a combination of terms that are quadratic in the metric first derivatives plus total derivatives that do not contribute to the variation of the action when we vary this keeping the metric and its derivatives fixed on the boundary.

Following Dirac, we split gravitational action into two parts:

$$S = \frac{1}{16\pi\kappa}\int d^4x\sqrt{-g}R = \frac{1}{2\kappa}\int d^4x\sqrt{-g}(\underbrace{g^{\mu\nu}(\Gamma^\alpha{}_{\mu\nu,\alpha} - \Gamma^\alpha{}_{\mu\alpha,\nu})}_{R^\star} - \underbrace{g^{\mu\nu}(\Gamma^\alpha{}_{\gamma\nu}\Gamma^\gamma{}_{\mu\alpha} - \Gamma^\alpha{}_{\gamma\alpha}\Gamma^\gamma{}_{\mu\nu})}_{L}). \tag{38}$$

The integrand involving $R^\star$ – which contains the unwanted second derivatives of the metric – can be written as

$$\sqrt{-g}R^\star = \cancel{(\sqrt{-g}g^{\mu\nu}\Gamma^\alpha{}_{\mu\nu})_{,\alpha}} - \cancel{(\sqrt{-g}g^{\mu\nu}\Gamma^\alpha{}_{\mu\alpha})_{,\nu}} - \Gamma^\alpha{}_{\mu\nu}(\sqrt{-g}g^{\mu\nu})_{,\alpha} + \Gamma^\alpha{}_{\mu\alpha}(\sqrt{-g}g^{\mu\nu})_{,\nu} \tag{39}$$

where the slashed terms are the total derivates that we will discard. We now want to express the remaining two terms as products of Christoffel symbols. To this end, we will use the fact that the metric is covariantly constant so $g^{\mu\nu}{}_{;\alpha} = 0 \Rightarrow g^{\mu\nu}{}_{,\alpha} = -\Gamma^\mu{}_{\gamma\alpha}g^{\gamma\nu} - \Gamma^\nu{}_{\gamma\alpha}g^{\mu\gamma}$ and the fact that $(\sqrt{-g})_{,\alpha} = \frac{1}{2}\sqrt{-g}g^{\gamma\delta}g_{\gamma\delta,\alpha} = \sqrt{-g}\Gamma^\nu{}_{\alpha\nu}$, which you showed in the homework problem. Expanding the derivatives of the products in parentheses in (39) gives 6 terms, two of which cancel, leaving 4 which are two duplicate pairs, that turn out to be $2\sqrt{-g}L$. The result is that

$$\delta S = \frac{1}{16\pi\kappa}\delta\int d^4x\sqrt{-g}R = \frac{1}{2\kappa}\delta\int d^4x\underbrace{\sqrt{-g}L}_{\mathcal{L}(g_{\mu\nu},g_{\mu\nu,\gamma})} \tag{40}$$

where, as indicated, the integrand – the Lagrangian density $\mathcal{L}$ – now contains only the metric and its first derivatives.

It is now straightforward but tedious to carry out the variation $\delta\mathcal{L}$ and show, again by discarding some total derivatives that do not contribute to δS, that $\delta\mathcal{L} = R_{\mu\nu}\delta(g^{\mu\nu}\sqrt{-g})$ so

$$\delta S = \frac{1}{16\pi\kappa}\delta\int d^4x\mathcal{L} = \frac{1}{2\kappa}\int d^4x R_{\mu\nu}\delta(g^{\mu\nu}\sqrt{-g}) \tag{41}$$

which is sufficient – by requiring that it vanish for arbitrary $\delta(g^{\mu\nu}\sqrt{-g})$ – to give the vacuum form of Einstein's field equations $R_{\mu\nu} = 0$.

In order to obtain a more useful expression (one involving $\delta g_{\alpha\beta}$) we proceed as follows: First, the fact that $g^{\mu\gamma}g_{\gamma\nu} = \delta^\mu_\nu$ means that $g^{\mu\gamma}\delta g_{\gamma\nu} = -g_{\gamma\nu}\delta g^{\mu\gamma}$ which implies $\delta g^{\mu\nu} = -g^{\mu\alpha}g^{\nu\beta}\delta g_{\alpha\beta}$. Second, just as $(\sqrt{-g})_{,\gamma} = \frac{1}{2}\sqrt{-g}g^{\mu\nu}g_{\mu\nu,\gamma}$, the variation $\delta\sqrt{-g} = \frac{1}{2}\sqrt{-g}g^{\mu\nu}\delta g_{\mu\nu}$.

Combining these gives, for the variations in (41) above,

$$\delta(g^{\mu\nu}\sqrt{-g}) = -\sqrt{-g}\left(g^{\mu\alpha}g^{\nu\beta} - \tfrac{1}{2}g^{\mu\nu}g^{\alpha\beta}\right)\delta g_{\alpha\beta} \tag{42}$$

which in (41) yield

$$\delta S = -\frac{1}{16\pi\kappa}\int d^4x\sqrt{-g}\left(R^{\alpha\beta} - \tfrac{1}{2}g^{\alpha\beta}R\right)\delta g_{\alpha\beta} \tag{43}$$

so requiring that δS vanish, now for arbitrary $\delta g_{\alpha\beta}$, gives Einstein's field equations in the form

$$G^{\alpha\beta} = R^{\alpha\beta} - \tfrac{1}{2}g^{\alpha\beta}R = 0 \tag{44}$$

and, when combined with the variation of the matter Lagrangian density, this furnishes Einstein's equations in the presence of matter:

$$\boxed{G^{\alpha\beta} = 16\pi\kappa\,\delta\mathcal{L}_{\rm m}/\delta g_{\alpha\beta}.} \tag{45}$$

As an example, consider the classical scalar field, whose Lagrangian density in flat space, we recall, is

$$\mathcal{L}(\phi,\phi_{,\mu}) = -\tfrac{1}{2}\phi_{,\mu}\phi^{,\mu} - \tfrac{1}{2}m^2\phi^2 \tag{46}$$

and whose invariance under time and space translations implied $\vec{\nabla}\cdot\mathbf{T} = 0$ (or, if you prefer $T_{\mu\nu}{}^{,\mu} = 0$) where

$$T_{\mu\nu} = -\phi_{,\mu}\frac{\partial\mathcal{L}}{\partial\phi^{,\nu}} + \eta_{\mu\nu}\mathcal{L} = \phi_{,\mu}\phi_{,\nu} + \eta_{\mu\nu}\left(-\tfrac{1}{2}\phi_{,\alpha}\phi^{,\alpha} - \tfrac{1}{2}m^2\phi^2\right). \tag{47}$$

The action principle, if correct, tells us we should be able to obtain this by writing the Lagrangian density in a covariant manner

$$\mathcal{L}(\phi,\phi_{,\mu},g_{\mu\nu}) = 2\sqrt{-g}\left(-g^{\mu\nu}\tfrac{1}{2}\phi_{,\mu}\phi_{,\mu} - \tfrac{1}{2}m^2\phi^2\right) \tag{48}$$

and then performing the variation of this with respect to the metric coefficients $g_{\mu\nu}$. Doing this, working in locally inertial coordinates, for which $g_{\mu\nu} = \eta_{\mu\nu}$ and $\sqrt{-g} = \sqrt{-g_{00}g_{xx}g_{yy}g_{zz}} = 1$ gives

$$T_{\mu\nu} = -\frac{\delta\mathcal{L}}{\delta g^{\mu\nu}} = \phi_{,\mu}\phi_{,\nu} + \eta_{\mu\nu}\mathcal{L} \tag{49}$$

so this does indeed work.

A The Schwarzschild metric

We seek a spherically symmetric metric

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = -e^{2\Phi(r)}(dx^0)^2 + e^{2\Lambda(r)}dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2 \tag{50}$$

that describes the gravitational field outside a point mass. I.e. a solution of Einstein's equations in the absence of matter: $G_{\mu\nu} = 8\pi\kappa T_{\mu\nu} = 0$. This implies $\overline{G}_{\mu\nu} = 0$ also, where $\overline{G}_{\mu\nu} \equiv G_{\mu\nu} - \frac{1}{2}g_{\mu\nu}G$. But the definition $G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R$ implies $G = -R$ and therefore $\overline{G}_{\mu\nu} = R_{\mu\nu}$, so Einstein's equations, in vacuum, are equivalent to

$$R_{\mu\nu} = 0. \tag{51}$$

The non-vanishing Christoffel symbols are found, from (50), to be

$$\begin{array}{ccc} \Gamma^0{}_{0r} = \Gamma^0{}_{r0} = \Phi' & \Gamma^r{}_{rr} = \Lambda' & \Gamma^r{}_{00} = \Phi' e^{2(\Phi-\Lambda)} \\ \Gamma^r{}_{\theta\theta} = -re^{-2\Lambda} & \Gamma^r{}_{\phi\phi} = -r\sin^2\theta e^{-2\Lambda} & \Gamma^\theta{}_{\phi\phi} = -\sin\theta\cos\theta \\ \Gamma^\theta{}_{r\theta} = \Gamma^\theta{}_{\theta r} = 1/r & \Gamma^\phi{}_{r\phi} = \Gamma^\phi{}_{\phi r} = 1/r & \Gamma^\phi{}_{\theta\phi} = \Gamma^\phi{}_{\phi\theta} = \cot\theta \end{array} \tag{52}$$

where primes denote derivatives with respect to r and which in

$$R_{\mu\nu} = R^\alpha{}_{\mu\alpha\nu} = \Gamma^\alpha{}_{\mu\nu,\alpha} - \Gamma^\alpha{}_{\mu\alpha,\nu} + \Gamma^\alpha{}_{\gamma\alpha}\Gamma^\gamma{}_{\mu\nu} - \Gamma^\alpha{}_{\gamma\nu}\Gamma^\gamma{}_{\mu\alpha} \tag{53}$$

yield

$$\begin{aligned} R_{00} &= (\Phi'' + \Phi'^2 - \Phi'\Lambda' + 2\Phi'/r)e^{2(\Phi-\Lambda)} \\ R_{rr} &= -(\Phi'' + \Phi'^2 - \Phi'\Lambda' - 2\Lambda'/r) \\ R_{\theta\theta} &= 1 - (r\Phi' - r\Lambda' + 1)e^{-2\Lambda} \\ R_{\phi\phi} &= \sin^2\theta R_{\theta\theta} \end{aligned} \tag{54}$$

with all the other components of the Ricci tensor vanishing.

According to (51) all of the above vanish. Vanishing of R_{00} and R_{rr} require that

$$\Phi' = -\Lambda' \tag{55}$$

and if we demand that both Φ and Λ vanish at infinity that requires

$$\Phi = -\Lambda \tag{56}$$

which, in (50), implies $g_{rr} = 1/g_{00}$.

Vanishing of $R_{\theta\theta}$ implies

$$1 = (r\Phi' - r\Lambda' + 1)e^{-2\Lambda} = (2r\Phi' + 1)e^{2\Phi} = (re^{2\Phi})' \tag{57}$$

with solution

$$re^{2\Phi} = r - 2M \tag{58}$$

where the length $M = G_\mathrm{N} M_\mathrm{phys}/c^2$ is a constant of integration and so we find

$$g_{00} = e^{2\Phi} = 1 - 2M/r \tag{59}$$

and so, together with $g_{rr} = 1/g_{00}$, we have Schwarzschild's famous metric

$$\boxed{ds^2 = (1 - 2M/r)c^2 dt^2 + (1 - 2M/r)^{-1}dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2.} \tag{60}$$