Directional average of the radiative transfer

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Abstract

The transfer equations for physical quantities are often derived from first principles, and thus these equations are inherently local due to the local nature of the interactions. However, in cosmological theories and observations, coarse-grained treatments are sometimes unavoidable. It is often possible to discuss the dynamics of coarse-grained physical quantities in terms of these local first-principles equations, but this is not always correct. Take, for example, the radiative transfer equation, which accurately describes the intensity of radiation along a single propagation path, but the same differential equation does not necessarily describe the evolution of the average over multiple light paths. For this reason, I am puzzled by the many 21cm papers that discuss the global signal by binning, using averaged parameters, assuming constant absorption coefficients, and so on. In this note I will show that we need to know not only the mean radiative parameters but also their fluctuations in order to describe the evolution of the average (or global) signal.

Eqn (11) shows a modified radiative transfer equation (though in the form of an integral), where the first term is the usually considered radiative damping (dissipation) term, while the second and third terms are the usually overlooked leading order corrections, which can be understood as the contribution to the intensity from the ensemble variance (fluctuation). This formalism could be used as a refined treatment of the dynamics of the global 21cm signal, etc.

1 Evolution of the LOS signal

The radiative transfer equation:

$$\frac{\mathrm{d}I}{\mathrm{d}s} = -\alpha I + j, \tag{1}$$

where $\alpha = \alpha(s)$ and $j = j(s)$ are general radiative coefficient functions that depend on various environmental parameters that depend on the affine parameter s.

Integral expression

For convenience we can define

$$\mu(s) \equiv e^{\int_{s_i}^{s} \alpha(s')\,\mathrm{d}s'} \tag{2}$$

then the radiative transfer equation can be rewritten as

$$\mu(s)I(s) = \int_{s_i}^{s} \mu(s')j(s')\,\mathrm{d}s'. \tag{3}$$

The general solution is

$$I(s) = \frac{1}{\mu(s)} \int_{s_i}^{s} \mu(s')j(s')\,\mathrm{d}s', \tag{4}$$

or more explicitly,

$$
\begin{aligned}
I(s) &= e^{-\int_{s_i}^{s} \alpha \, \mathrm{d}s'} \int_{s_i}^{s} e^{\int_{s_i}^{s''} \alpha \, \mathrm{d}s'''} j(s'') \, \mathrm{d}s'' \\
&= \int_{s_i}^{s} \mathrm{d}s' \, j(s') \, e^{-\int_{s'}^{s} \alpha(s'') \, \mathrm{d}s''},
\end{aligned} \tag{5}
$$

where the reference point s_i has been chosen such that $I(s_i) = 0$. In other words, $j(s_i)$ is the injection of the first lights.

Taylor expansion

We now expand the above expression

- Pivot values:

$$
j(s, \hat{\Omega}) = \bar{j}(s) + \delta j(s, \hat{\Omega}) \tag{6}
$$

$$
\alpha(s, \hat{\Omega}) = \bar{\alpha}(s) + \delta\alpha(s, \hat{\Omega}) \tag{7}
$$

  where the overbar indicates averaging over directions, for example,

$$
\bar{j}(s) = \langle j(s) \rangle_{\Omega} \equiv \int j(s, \hat{\Omega}) \, \mathrm{d}\Omega.
$$

  Consequently, we have $\int \delta j(s, \hat{\Omega}) \, \mathrm{d}\Omega = 0$.

- We also define

$$
\begin{aligned}
\tau(s', s, \hat{\Omega}) \equiv \int_{s'}^{s} \alpha(s'', \hat{\Omega}) \, \mathrm{d}s'' &= \int_{s'}^{s} \bar{\alpha}(s'') \, \mathrm{d}s'' + \int_{s'}^{s} \delta\alpha(s'', \hat{\Omega}) \, \mathrm{d}s'' \\
&\equiv \bar{\tau}(s', s) + \delta\tau(s', s, \hat{\Omega})
\end{aligned} \tag{8}
$$

- The exponential (damping) term in Eqn (5) can be expanded as

$$
\begin{aligned}
e^{-\int_{s'}^{s} \alpha(s'', \hat{\Omega}) \, \mathrm{d}s''} &= \exp\left[-\bar{\tau}(s', s) \left(1 + \frac{\delta\tau(s', s, \hat{\Omega})}{\bar{\tau}(s', s)} \right) \right] \\
&= e^{-\bar{\tau}(s', s)} \sum_{n=0}^{\infty} \frac{\left[-\delta\tau(s', s, \hat{\Omega}) \right]^n}{n!}
\end{aligned} \tag{9}
$$

  Now Eqn (5) can be rewritten as

$$
I(s, \hat{\Omega}) = \sum_{n=0}^{\infty} \frac{1}{n!} \int_{s_i}^{s} \mathrm{d}s' \left(\bar{j}(s') + \delta j(s', \hat{\Omega}) \right) e^{-\bar{\tau}(s', s)} \left[-\delta\tau(s', s, \hat{\Omega}) \right]^n \tag{10}
$$

2 Evolution of the directional averaged signal

After averaging Eqn (10) over a wide field for all directions, we get

$$
\begin{aligned}
\bar{I}(s) = \int_{s_i}^{s} \mathrm{d}s' \, \bar{j}(s') \, e^{-\bar{\tau}(s', s)} &+ \frac{1}{2} \int_{s_i}^{s} \mathrm{d}s' \, \bar{j}(s') \, e^{-\bar{\tau}(s', s)} \langle \left[\delta\tau(s', s, \hat{\Omega}) \right]^2 \rangle_{\Omega} \\
&- \int_{s_i}^{s} \mathrm{d}s' \, \bar{j}(s') \, e^{-\bar{\tau}(s', s)} \langle \left[\delta j(s', \hat{\Omega}) \delta\tau(s', s, \hat{\Omega}) \right] \rangle_{\Omega}
\end{aligned} \tag{11}
$$

where we have omitted the third and higher order terms. The last term in the above equation is kept because it is usually the case that δj and $\delta\tau$ have some parameters in common, i.e. some of their terms might be correlated.