M-mode RIME explicit in beam, fringe and sky modes

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Abstract

This note adopts a somewhat cumbersome but well-defined approach to the expansion of the Radio Interferometer Measurement Equation (RIME). We describe the integrals (and also the decompositions) in RIME in terms of antenna coordinates, but stipulate that all spherical harmonic expansions are defined on the same static basis in the celestial system.

The result of this procedure is that: the expansion for a steady sky field remains unchanged; and the expansions for the beam and fringe vary with time; but for a drift scan, the only difference between the same mode at different times is a phase, $e^{\pm im\Delta t}$ (the sign depends on whether $Y_{\ell m}$ or $Y^*_{\ell m}$ is used for the expansion). Therefore, RIME at any time can be easily decomposed into the modes defined at a reference time. This is the essence of the m-mode. Since it is only required that the SH basis be sky-static, its choice (or rotation) can be arbitrary. In practical analysis we can choose a special basis that is completely coincident with the standard antenna SH basis at the reference time.

Key equations:

1. Eqn (13) shows the projection from the sky modes to the temporal data, in terms of the modes of the modulated beam.
2. Eqn (14) shows the projection from the sky modes to the m-mode data. The operator is just the SH coefficients of the modulated beam.
3. Eqn (22) shows the projection from the sky modes to the m-mode data, in terms of the modes of the primary beam and the fringe pattern. (The equation (20) defines the operator - I think it is tractable?)

1 Notations and conventions

- For a field point on the sphere:
  - $\hat{n}_a$ is its coordinates in an antenna-static (or ground-static) system.
  - $\hat{n}_s$ denotes the sky-static (or celestial) coordinates.
- An interferometric measurement (a.k.a. RIME):

$$V(\boldsymbol{b}_{ij}, \nu, t) = \int d^2\Omega \; \overbrace{A_i(\hat{n}_a, \nu)\, A_j^\dagger(\hat{n}_a, \nu)}^{\text{Beam } (\mathbf{B})} \; \overbrace{T(\hat{n}_s, \nu)}^{\text{Sky } (\mathbf{T})} \; \overbrace{e^{-2\pi i \nu \tau_{ij}(\hat{n}_a)}}^{\text{Fringe } (\mathbf{F})} \tag{1}$$

$$\text{Modulated Beam } (\mathbf{M})$$

- Each component of the equation is expressed in the system where it is static:
  * Both the beam and fringe parts vote for $\hat{n}_a$, while the stationary sky intensity field prefers $\hat{n}_s$.
  * 2 : 1 on the scoreboard, so we usually take the measurement as integral with respect to $\hat{n}_a$ for convenience.
    · $\int d^2\Omega \equiv \int d^2\hat{n}_a$
    · $\hat{n}_s$ is therefore understood as a function of $\hat{n}_a$ and t, the local sidereal time.
- The $\hat{n}_a \to \hat{n}_s$ mapping can be expressed by a rotation operator

$$\hat{n}_s = \hat{n}_s(\hat{n}_a, t) = \mathcal{R}(t)\hat{n}_a, \tag{2}$$

whose inverse mapping is given by

$$\hat{n}_a = \hat{n}_a(\hat{n}_s, t) = \mathcal{R}^{-1}(t)\hat{n}_s. \tag{3}$$

- Earth rotation changes the mapping between $\hat{n}_s$ and $\hat{n}_a$. For a drift-scan antenna,
  * The $\hat{n}_s(\hat{n}_a, t)$ coordinates of the same $\hat{n}_a$ at different times are related as follows

$$\begin{aligned} \hat{n}_s(\hat{n}_a, t_2) &= \mathcal{R}(t_2)\hat{n}_a \\ &= \mathcal{R}_z(t_2 - t_1)\mathcal{R}(t_1)\hat{n}_a = \mathcal{R}_z(t_2 - t_1)\, \hat{n}_s(\hat{n}_a, t_1) \end{aligned} \tag{4}$$

  where $\mathcal{R}_z$ represents the rotation of the azimuth (in the celestial system) and $(t_2 - t_1)$ is the rotation angle (in radians).
  * Correspondingly, the $\hat{n}_a(\hat{n}_s, t)$ coordinates of the same $\hat{n}_s$ at different times have the following relation

$$\hat{n}_a(\hat{n}_s, t_2) = \mathcal{R}_z^{-1}(\Delta t)\, \hat{n}_a(\hat{n}_s, t_1) \tag{5}$$

- Product of two or three fields
  - **'MT'** Interpretation: Measurement as the inner product of two fields, the modulated beam ('**M**') and the temperature sky ('**T**').
  - **'BFT'** Interpretation: Measurement as the product of three fields, the primary beam ('**B**'), the fringe ('**F**') and the sky ('**T**').

Sky static spherical harmonics in antenna coordinates

- To study the statistics of sky fields, we consider the spherical harmonic (SH) basis fixed on the celestial sphere.
  - The basis as functions of sky coordinates are independent of time, whereas in antenna coordinates they are in rotation from time to time:

$$Y_{\ell m} = Y_{\ell m}(\hat{n}_s) = Y_{\ell m}(\mathcal{R}(t)\hat{n}_a).$$

  - In drfit-scan mode, the spherical harmonic in the antenna coordinates at any time can be expressed in terms of the harmonics at a **reference time** $t_{\rm ref}$ as follows

$$\begin{aligned} Y_{\ell m}(\mathcal{R}(t)\hat{n}_a) &= Y_{\ell m}(\mathcal{R}_z(t - t_{\rm ref})\mathcal{R}(t_{\rm ref})\hat{n}_a) \\ &= \sum_{m'} D^{\ell}_{m'm}(t - t_{\rm ref}, 0, 0) Y_{\ell m'}(\mathcal{R}(t_{\rm ref})\hat{n}_a) \\ &= Y_{\ell m}(\mathcal{R}(t_{\rm ref})\hat{n}_a) e^{im(t - t_{\rm ref})} \end{aligned} \tag{6}$$

  where the second equality has applied the Wigner D matrices to account for $\mathcal{R}_z(t - t_{\rm ref})$.

Antenna field decomposition with sky static spherical harmonics

Next we decompose a field in the antenna system, for example the beam intensity $B(\hat{n}_a)$, on an SH basis $Y_{\ell m}$ (or $Y^*_{\ell m}$) fixed to the sky:

$$\mathbf{B}(\hat{n}_a) = \sum_{\ell m} B_{\ell m}(t) Y_{\ell m}(\hat{n}_s), \tag{7}$$

where $\hat{n}_s = \mathcal{R}(t)\hat{n}_a$ and

$$B_{\ell m}(t) \equiv \int d^2\Omega\, \mathbf{B}(\hat{n}_a) Y^*_{\ell m}(\hat{n}_s). \tag{8}$$

As a result of eqn (6) we have

$$B_{\ell m}(t) e^{im(t-t_{\mathrm{ref}})} = B_{\ell m}(t_{\mathrm{ref}}), \tag{9}$$

or equivalently,

$$\mathbf{B}(\hat{n}_a) = \sum_{\ell m} B_{\ell m}(t_{\mathrm{ref}}) Y_{\ell m}(\hat{n}_s) e^{-im(t-t_{\mathrm{ref}})}. \tag{10}$$

2 M-mode formalism (MT interpretation)

Now we expand the modulated beam and temperature sky with $Y^*_{\ell m}$ and $Y_{\ell m}$ respectively.

$$\mathbf{M}(\hat{n}_a) = \sum_{\ell m} M_{\ell m}(t_{\mathrm{ref}}) Y^*_{\ell m}\, e^{im(t-t_{\mathrm{ref}})}, \tag{11}$$

$$\mathbf{T}(\hat{n}_s) = \sum_{\ell m} T_{\ell m}\, Y_{\ell m}. \tag{12}$$

This method of expansion was chosen for ease of later notations. Substituting the above equations into RIME, we get

$$\begin{aligned} V(\boldsymbol{b}_{ij}, \nu, t) &= \int d^2\Omega\, \mathbf{M}(\hat{n}_a)\, \mathbf{T}(\hat{n}_s) \\ &= \sum_{\ell m} \sum_{\ell' m'} T_{\ell m}\, M_{\ell' m'}(t_{\mathrm{ref}})\, e^{im'(t-t_{\mathrm{ref}})} \int d^2\Omega\, Y_{\ell m} Y^*_{\ell' m'} \\ &= \sum_{m} \left(\sum_{\ell} T_{\ell m}\, M_{\ell m}(t_{\mathrm{ref}}) \right) e^{im(t-t_{\mathrm{ref}})} \end{aligned} \tag{13}$$

Since $V(\boldsymbol{b}_{ij}, \nu, t+2\pi) = V(\boldsymbol{b}_{ij}, \nu, t)$ is periodic, the last equality of eqn (13) is the Fourier series expansion of $V(\boldsymbol{b}_{ij}, \nu, t)$ as a function of time. In other words,

$$\tilde{V}_m(\boldsymbol{b}_{ij}, \nu) = \sum_{\ell \geq m} T_{\ell m}\, M_{\ell m}(t_{\mathrm{ref}}) \tag{14}$$

where $\tilde{V}_m(\boldsymbol{b}_{ij}, \nu)$ represents the Fourier coefficients of the time series data,

$$\tilde{V}_m(\boldsymbol{b}_{ij}, \nu) \equiv \frac{1}{2\pi} \int_{t_{\mathrm{ref}}}^{t_{\mathrm{ref}}+2\pi} V(\boldsymbol{b}_{ij}, \nu, t) e^{-imt} \mathrm{d}t. \tag{15}$$

The formalism discussed in this section is also known as the m-mode analysis.

3 M-mode formlism (BFT interpretation)

In some scenarios we want to separate the roles of beam and fringe when projecting sky modes into visibility space. Of course that can be done by directly decompose each of the three fields and then there product is the sum a number of Gaunt's integrals,

$$\int d^2\Omega\, Y_{\ell_1 m_1} Y_{\ell_2 m_2} Y_{\ell_3 m_3} = \left(\frac{(2\ell_1+1)(2\ell_2+1)(2\ell_3+1)}{4}\right)^{\frac{1}{2}} \begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \equiv \mathcal{G}(\ell_1, \ell_2, \ell_3; m_1, m_2, m_3). \tag{16}$$

However, since we have already expressed the measurement in terms of $T_{\ell m}$ and $M_{\ell m}$ (see eqn (13)). We can simply first rewrite $M_{\ell m}$ in terms of beam and fringe modes, and then substitute it into the m-mode RIME. Working this out helps to understand the role played by beam and fringe in m-mode analysis.

By definition, we have

$$M_{\ell m}(t_{\mathrm{ref}}) = \int d^2\Omega\, \mathbf{M}(\hat{n}_a)\, Y_{\ell m}(\hat{n}_s) = \int d^2\Omega\, \mathbf{B}(\hat{n}_a)\, \mathbf{F}(\hat{n}_a)\, Y_{\ell m}(\hat{n}_s). \tag{17}$$

We define the SH modes of $\mathbf{B}$ and $\mathbf{F}$ with $Y^*_{\ell m}$:

$$B_{\ell m}(t_{\mathrm{ref}}) \equiv \int d^2\Omega\, \mathbf{B}(\hat{n}_a) Y^*_{\ell m}(\hat{n}_s), \qquad F_{\ell m}(t_{\mathrm{ref}}) \equiv \int d^2\Omega\, \mathbf{F}(\hat{n}_a) Y^*_{\ell m}(\hat{n}_s). \tag{18}$$

Then eqn (17) can be rewritten as

$$\begin{aligned} M_{\ell m}(t_{\mathrm{ref}}) &= \sum_{\ell_1 m_1} \sum_{\ell_2 m_2} B_{\ell_1 m_1}(t_{\mathrm{ref}}) F_{\ell_2 m_2}(t_{\mathrm{ref}}) \int d^2\Omega\, Y_{\ell_1 m_1} Y_{\ell_2 m_2} Y_{\ell m} \\ &= \sum_{\ell_1 m_1} \sum_{\ell_2 m_2} B_{\ell_1 m_1}(t_{\mathrm{ref}}) F_{\ell_2 m_2}(t_{\mathrm{ref}}) \mathcal{G}(\ell_1, \ell_2, \ell; m_1, m_2, m) \end{aligned} \tag{19}$$

Since the selection rule

$$m_1 + m_2 + m = 0$$

must be fulfilled, eqn (19) actually only has three nested loops

$$M_{\ell m}(t_{\mathrm{ref}}) = \sum_{l_1 \ge |m_1|} \sum_{l_2 \ge |m_1+m|} \sum_{m_1} B_{\ell_1 m_1}(t_{\mathrm{ref}}) F_{\ell_2, -m_1-m}(t_{\mathrm{ref}})\, \mathcal{G}(\ell_1, \ell_2, \ell; m_1, -m_1-m, m). \tag{20}$$

We can further write $F_{\ell_2 m_2}(t_{\mathrm{ref}})$ in the plane wave expansion, which gives

$$F_{\ell m}(t_{\mathrm{ref}}) = 4\pi (i)^{\ell} j_{\ell}(2\pi u) Y_{\ell m}\left(\mathcal{R}(t_{\mathrm{ref}})\hat{u}_a\right). \tag{21}$$

where $u = |\boldsymbol{b}_{ij}/\lambda|$, and $\hat{u}_a$ is the baseline directional coordinates in the antenna system. $\mathcal{R}$ transforms them to the sky coordinates.

Substituting eqn (20) into eqn (14), we have

$$\tilde{V}_m(\boldsymbol{b}_{ij}, \nu) = \sum_{l_1 \ge |m_1|} \sum_{l_2 \ge |m_1+m|} \sum_{m_1} \sum_{\ell \ge |m|} T_{\ell m} B_{\ell_1 m_1} F_{\ell_2 m_2}\, \mathcal{G}(\ell_1, \ell_2, \ell; m_1, -m_1-m, m), \tag{22}$$

where it may be worth emphasising again that the $F_{\ell m}$ and $B_{\ell m}$ are evaluated in the Antenna system using the Sky-static SH basis at t_{ref}, and the m-mode data is defined over $(t_{\mathrm{ref}}, t_{\mathrm{ref}} + 2\pi)$!