Fast simulation of the polarized galactic synchrotron

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1 The Polarized Sky

The total intensity of the synchrotron emission coming from a volume element $\mathrm{d}V = s^2\,\mathrm{d}s\delta\Omega$ in a frequency interval $\delta\nu$ is given by the emission coefficient $j_I(s, \hat{\mathbf{n}}, \nu)$, which can be written as

$$j_I(s, \hat{\mathbf{n}}, \nu) = C_I \left(\frac{2\pi m_e c}{3e}\nu\right)^{\frac{1-p}{2}} n_{CR} B_\perp^{\frac{p+1}{2}}, \tag{1}$$

where n_{CR} is the cosmic ray electron density, $B_\perp$ is the transverse galactic magnetic field and we are assuming a power law energy distribution for the CR electrons $N(E) \propto E^{-p}$. The coefficient C_I is given by

$$C_I = \frac{\sqrt{3}e^3}{4\pi m_e c^2(p+1)}\Gamma\left(\frac{3p-1}{12}\right)\Gamma\left(\frac{3p+19}{12}\right). \tag{2}$$

As the synchrotron photons undergo Faraday rotations, the observed polarization angle and the initial polarized angle are related by $\phi = \phi_0 + \psi(s, \hat{\mathbf{n}})(c/\nu)^2$, where ψ is the Faraday rotation measure, given as

$$\psi(s, \hat{\mathbf{n}}) = \frac{e^3}{2\pi(m_e c^2)^2}\int_0^s n_e(s', \hat{\mathbf{n}}) B_\parallel(s', \hat{\mathbf{n}})\,\mathrm{d}s'. \tag{3}$$

Thus, the polarized synchrotron intensity can be written as

$$I_P(\nu, \hat{\mathbf{n}}) = \Pi_0 \int_0^\infty j_I(s, \hat{\mathbf{n}}, \nu) e^{2i\phi_0(s,\hat{\mathbf{n}})} e^{i\psi(s,\hat{\mathbf{n}})x_\nu}\,\mathrm{d}s, \tag{4}$$

where $x_\nu = 2(c/\nu)^2$.

If we use the Faraday depth $\psi(s, \hat{\mathbf{n}})$ as LOS coordinate, instead of s, then we can rewrite the polarized intensity as

$$I_P(\nu, \hat{\mathbf{n}}) = \int k(\psi, \hat{\mathbf{n}}, \nu) e^{i\psi x_\nu}\,\mathrm{d}\psi, \tag{5}$$

where $k(\psi_0) = \int \delta(\psi(s) - \psi_0) j_I(s) e^{2i\phi_0(s)}\,\mathrm{d}s$ is the collective emission from regions with Faraday depth ψ.

2 Model assumptions

The galactic synchrotron model used in Alonso 2014 bases on the following assumptions:

1. The spectral dependence of the emission is basically the same at all depths and can be factorized:

$$k(\psi, \hat{\mathbf{n}}, \nu) = b(\hat{\mathbf{n}}, \nu) k_0(\psi, \hat{\mathbf{n}}), \tag{6}$$

   where $b(\hat{\mathbf{n}}, \nu) = (\nu/\nu_{ref})^{\alpha(\hat{\mathbf{n}})}$.[1]

2. For each direction, ψ is normally scattered, in a mean-zero way with variance $\sigma^2(\hat{\mathbf{n}})$, i.e.[2],

$$\text{the number of regions with } \psi \propto \exp\left[-\frac{1}{2}\left(\frac{\psi}{\sigma^2(\hat{\mathbf{n}})}\right)^2\right]. \tag{7}$$

3. The collective emission at some ψ is proportional to the number of regions with that Faraday depth so that k_0 is modeled as

$$k_0(\psi, \hat{\mathbf{n}}) = B \exp\left[-\frac{1}{2}\left(\frac{\psi}{\sigma^2(\hat{\mathbf{n}})}\right)^2\right] \mu(\psi, \hat{\mathbf{n}}). \tag{8}$$

4. The field $\mu(\psi, \hat{\mathbf{n}})$ has the same angular structure as the unpolarized emission and it's correlated in Faraday space on scales smaller than some correlation length ξ_ψ

$$\langle \mu_{lm}(\psi) \mu^*_{l'm'}(\psi') \rangle \propto \delta_{ll'} \delta_{mm'} \left(\frac{l_{ref}}{l}\right)^\beta e^{-\frac{1}{2}\left[\frac{\psi-\psi'}{\xi_\psi}\right]^2}, \tag{9}$$

5. $\tilde{\mu}(x)$, the Fourier transform of $\mu(\psi)$, are uncorrelated.

With the above assumptions one can rewrite the polarized intensity as

$$I_P(\nu, \hat{\mathbf{n}}) = \int B(\nu/\nu_{ref})^{\alpha(\hat{\mathbf{n}})} \exp\left[-\frac{1}{2}\left(\frac{\psi}{\sigma^2(\hat{\mathbf{n}})}\right)^2\right] \mu(\psi, \hat{\mathbf{n}}) e^{i\psi x_\nu} \, \mathrm{d}\psi, \tag{10}$$

where $\mu(\psi, \hat{\mathbf{n}})$ is realized using assumptions 4 and 5. Thus, the goal of modeling the sky breaks down to how to generate $\mu(\psi)$ with $\tilde{\mu}(x)$. I used both Alonso's and my own way to do this. Both are described as below.

[1] I use a full sky spectral index map as Alonso did.

[2] The variance is estimated from the Oppermann 2012 maps of ψ_∞ by smoothing ψ^2_∞ on a large angular scale (Alonso 2014 has used 5 deg).

3 Alonso's simulation: $\tilde{\mu}_{lm}(x) \xrightarrow{\textbf{SHT}} \tilde{\mu}(x)$ maps $\longrightarrow I_P(\nu, \hat{\mathbf{n}})$

In Alonso's simulation, $\tilde{\mu}(x)$ is written as

$$\tilde{\mu}(x) \equiv \int \frac{d\psi}{\sqrt{2\pi}} \mu(\psi) e^{i\psi x}, \tag{11}$$

which is uncorrelated for different values of x with variance $l^{-\beta} e^{-(\xi_\psi x)^2/2}$. Thus, we can rewrite the polarized intensity as

$$\begin{aligned}
I_P(\nu, \hat{\mathbf{n}}) &= \int k(\psi, \hat{\mathbf{n}}, \nu) e^{i\psi x_\nu} \, \mathrm{d}\psi \\
&= \int b(\nu, \hat{\mathbf{n}}) k_0(\psi, \hat{\mathbf{n}}) e^{i\psi x_\nu} \, \mathrm{d}\psi \\
&= \int b(\nu, \hat{\mathbf{n}}) B \exp\left[-\frac{1}{2}\left(\frac{\psi}{\sigma(\hat{\mathbf{n}})}\right)^2\right] \mu(\psi, \hat{\mathbf{n}}) e^{i\psi x_\nu} \, \mathrm{d}\psi \\
&= \int b(\nu, \hat{\mathbf{n}}) B \exp\left[-\frac{1}{2}\left(\frac{\psi}{\sigma(\hat{\mathbf{n}})}\right)^2\right] \left(\int \frac{1}{\sqrt{2\pi}} \tilde{\mu}(x) e^{-i\psi x} \, \mathrm{d}x\right) e^{i\psi x_\nu} \, \mathrm{d}\psi \\
&= \iint b(\nu, \hat{\mathbf{n}}) \frac{B}{\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{\psi}{\sigma(\hat{\mathbf{n}})}\right)^2\right] \tilde{\mu}(x) e^{-i\psi x} e^{i\psi x_\nu} \, \mathrm{d}\psi \, \mathrm{d}x \\
&= \int b(\nu, \hat{\mathbf{n}}) \frac{B}{\sqrt{2\pi}} \tilde{\mu}(x) \left[\int e^{-\frac{1}{2}\left(\frac{\psi}{\sigma(\hat{\mathbf{n}})}\right)^2} e^{i\psi(x_\nu - x)} \, \mathrm{d}\psi\right] \mathrm{d}x \\
&= \int b(\nu, \hat{\mathbf{n}}) \frac{B}{\sqrt{2\pi}} \tilde{\mu}(x) \left[\sqrt{2\pi\sigma^2(\hat{\mathbf{n}})} e^{-2\pi^2\sigma^2(\hat{\mathbf{n}})\left(\frac{x_\nu - x}{2\pi}\right)^2}\right] \mathrm{d}x \\
&= b(\nu, \hat{\mathbf{n}}) B' \sigma(\hat{\mathbf{n}}) \int \tilde{\mu}(x) e^{-\frac{(x_\nu - x)^2}{2\sigma^{-2}(\hat{\mathbf{n}})}} \, \mathrm{d}x
\end{aligned} \tag{12}$$

So what Alonso really did is, for each frequency,

1. Generating $\tilde{\mu}(x, \hat{\mathbf{n}})$ maps using Gaussian random realizations with power spectrum $C_l \propto l^{-\beta} e^{-x^2 \xi_\psi{}^2/2}$;
2. Applying mask $e^{-\frac{(x_\nu - x)^2}{2\sigma^{-2}(\hat{\mathbf{n}})}}$;
3. Numerical integration: integrating all x slices;
4. Multiplying trivial terms, $b(\nu, \hat{\mathbf{n}})$ and $\sigma(\hat{\mathbf{n}})$, and doing normalization[3].

Thus, we can get $I_P(\nu, \hat{\mathbf{n}})$ maps for each frequency frame using Alonso's code.

[3] Assume a "reasonable" polarized fraction at high latitude regions and then normalize to the Haslam map.

4 Zhang's simulation:

$$\tilde{\mu}_{lm}(x) \xrightarrow{\textbf{FFT}} \mu_{lm}(\psi) \xrightarrow{\textbf{spinSHT}} \mu(\psi), k_0(\psi) \longrightarrow I_P(\nu)$$

In Zhang's simulation, $\mu_{lm}(\psi)$ is written as

$$\mu_{lm}(\psi) \equiv \int \tilde{\mu}_{lm}(x) e^{-2\pi i \psi x}\, \mathrm{d}x. \tag{13}$$

Thus, the left hand side (LHS) of eq.(9) can be rewritten as

$$\langle \mu_{lm}(\psi)\mu^*_{l'm'}(\psi')\rangle = \langle \left[\int \tilde{\mu}_{lm}(x)e^{-2\pi i\psi x}\,\mathrm{d}x\right]\left[\int \tilde{\mu}^*_{lm}(x)e^{2\pi i\psi' x}\,\mathrm{d}x\right]\rangle \tag{14a}$$

$$\equiv \int_{-\infty}^{+\infty} \langle \tilde{\mu}_{lm}(x)\tilde{\mu}^*_{lm}(x)\rangle\, e^{-2\pi i(\psi-\psi')x}\,\mathrm{d}x \tag{14b}$$

The identity of eq.(14a) and eq.(14b) is implied by the assumption 5[4] of Alonso's model. Using eq.(14), we can thus take the inverse Fourier transform with respect to $(\psi - \psi')$ at both sides of eq.(9):

$$\langle \tilde{\mu}_{lm}(x)\tilde{\mu}^*_{lm}(x)\rangle \propto l^{-\beta}\sqrt{2\pi}\xi e^{-2\pi^2\xi^2x^2}, \tag{15}$$

with constant terms dropped, it reads

$$\langle \tilde{\mu}_{lm}(x)\tilde{\mu}^*_{lm}(x)\rangle \propto l^{-\beta} e^{-2\pi^2\xi^2x^2}. \tag{16}$$

Now we can get the explicit form of the Gaussian realization of $\tilde{\mu}_{lm}(x)$ as

$$\tilde{\mu}_{lm}(x,p) \propto C_p\left(\mathbf{X} + i\mathbf{Y}\right) l^{-\beta/2} e^{-\pi^2\xi^2x^2}, \tag{17}$$

where $\mathbf{X}, \mathbf{Y} \sim \mathbf{N}(\mathbf{0}, \mathbf{1})$ are Gaussian random number generators, and $p = B, E$ denotes the B mode and E mode. Here, I simply set $C_B = C_E = 1$.

The next is to get $\mu_{lm}(\psi)$ out of $\tilde{\mu}_{lm}(x)$ using the discrete form of eq.(13). Then step further to get $\mu(\psi, \hat{\mathbf{n}})$ maps from $\mu_{lm}(\psi)$'s. In my simulation, I produced $\mu_{lm}(\psi)$'s for B mode and E mode separately, and then performed the spin-2 spherical harmonic transform.

Appendices

A Input maps

B Documentation of Z. Zhang's simulation

[4] Assumption 5 implies that, in order to realize Alonso's model, one should do random Gaussian realization for not only each lm-mode, but also each x-mode.