Wave function in topological sense

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Smooth and normalized field:

$$m^a = \frac{M^a}{\|M\|}$$

Skyrmion number, or winding number:

$$Q^{skyrmion} = \frac{1}{4\pi} \int \vec{m} \cdot (\partial_x \vec{m} \times \partial_y \vec{m}) \, \mathrm{d}x \, \mathrm{d}y$$

Topological charge density:

$$q = \frac{1}{8\pi} \epsilon^{0\mu\nu} \epsilon_{abc} m^a \partial_\mu m^b \partial_\nu m^c$$

Topological phase transition:

$$\partial_t q = D \left(\frac{M}{z} \right) \delta(\vec{M})$$

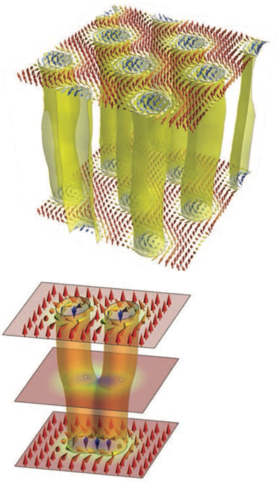


Figure: Topological analysis in normalised smooth vector field.

Wave function as a vector field?

- Non-relativistic wave function, continuous parameter space, e.g., position and momentum space:

$$\Psi = \phi_1 + i\phi_2$$

- Unitarity:

$$\partial_t \rho + \nabla \cdot \vec{j} = 0 \text{ and } \rho = \|\Psi\|^2 = \phi_1^2 + \phi_2^2$$

where $\vec{j}$ is derived from the Schrodinger equation (free particle):

$$j_\mu = -\frac{i\hbar}{2m}\left(\Psi^* \partial_\mu \Psi - \Psi \partial_\mu \Psi^*\right) = \frac{\hbar}{m}\left(\phi_1 \partial_\mu \phi_2 - \phi_2 \partial_\mu \phi_1\right)$$

- We can thus define the **velocity vector field**, $\vec{v} = \frac{\vec{j}}{\rho}$, of the probability density:

$$v_\mu = \frac{\hbar}{m} \frac{\phi_1 \partial_\mu \phi_2 - \phi_2 \partial_\mu \phi_1}{\phi_1^2 + \phi_2^2}$$

- The velocity field:

$$v_\mu = \frac{\hbar}{m}\frac{\phi_1\partial_\mu\phi_2 - \phi_2\partial_\mu\phi_1}{\phi_1^2+\phi_2^2} = \frac{\hbar}{m}\partial_\mu\left[\arctan(\frac{\phi_2}{\phi_1})\right] = \frac{\hbar}{m}\partial_\mu\theta$$

where $\theta \equiv \arctan(\frac{\phi_2}{\phi_1})$.

- Path integral of the velocity field = adiabatic evolution in the parameter space
- Closed path integral = adiabatic periodic evolution:

$$\oint v_\mu \, \mathrm{d}x^\mu = \frac{\hbar}{m}\oint \partial_\mu\theta \, \mathrm{d}x^\mu = \frac{\hbar}{m}\oint \mathrm{d}\theta = \frac{\hbar}{m}\oint \frac{d\theta}{d\Psi}\,\mathrm{d}\Psi$$

- The last term is complex integral, for which Cauchy integral theorem applies:

$$W \equiv \oint \frac{d\theta}{d\Psi}\,\mathrm{d}\Psi = \frac{m}{\hbar}\oint v_\mu \, \mathrm{d}x^\mu \tag{1}$$

where W is the “winding number”, which is understood as the count of (counter-clockwise) circles around the singularity of Ψ.

With EM fields

- The probability current density reads

$$\vec{j} = \vec{j}_{free} - \frac{q}{mc}\mathbf{A}\|\Psi\|^2$$

and the velocity field is thus:

$$v_\mu = \frac{\hbar}{m}\partial_\mu\theta - \frac{q}{mc}A_\mu$$

- The adiabatic periodic evolution:

$$\oint v_\mu \, \mathbf{d}x^\mu = \frac{\hbar}{m}\cdot W - \frac{q}{mc}\oint A_\mu \, \mathbf{d}x^\mu$$

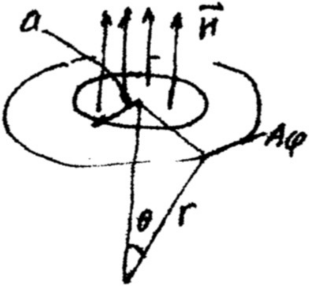
where the last term is the flux of magnetic field.

Magnetic Flux and Angular Momentum Quantization

- Angular momentum operator along $\hat{z}$:

$$\hat{L}_z = x\hat{P}_y - y\hat{P}_x = \frac{\hbar}{i}\frac{\partial}{\partial \phi}$$

- The eigenfunctions are $\psi_n = e^{in\phi}$.



$$\begin{aligned}\hat{L}_z &= x(\hat{P}_y - \frac{q}{c}A_y) - y(\hat{P}_x - \frac{q}{c}A_x) \\ &= x\hat{P}_y - y\hat{P}_x - \frac{q}{c}(xA_y - yA_x) \\ &= \frac{\hbar}{i}\frac{\partial}{\partial \phi} - \frac{q}{c}(xA_y - yA_x)\end{aligned}$$

Change basis: $xA_y - yA_x = r\sin\theta A_\phi$
Closed integral outside the solenoid:

$$\oint \mathbf{A}\cdot \mathbf{dl} = \oint A_\phi r \sin\theta \, \mathrm{d}\phi = A_\phi r \sin\theta \cdot 2\pi$$

$$LHS = \iint \mathbf{H}\cdot \mathbf{dS} = \Phi$$

where Φ denotes the flux of the magnetic field.

- $\hat{L}_z = \frac{\hbar}{i}\frac{\partial}{\partial\phi} - \frac{q}{c}(xA_y - yA_x)$
- $xA_y - yA_x = r\sin\theta A_\phi$

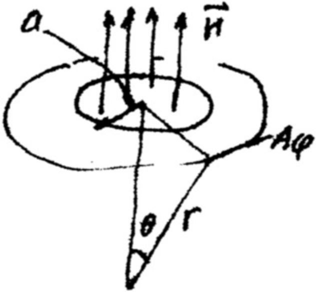


Figure: Circular Loop (solenoid)

Closed integral outside the solenoid:

- $\oint \mathbf{A}\cdot d\mathbf{l} = A_\phi r\sin\theta \cdot 2\pi$
- $\oint \mathbf{A}\cdot d\mathbf{l} = \iint \mathbf{H}\cdot d\mathbf{S} = \Phi$

It obvious that

$$A_\phi = \frac{\Phi}{r\sin\theta 2\pi}$$

i.e., $xA_y - yA_x = \frac{\Phi}{2\pi}$. Thus, outside the solenoid, we have

$$\hat{L}_z = \frac{\hbar}{i}\frac{\partial}{\partial\phi} - \frac{q\Phi}{2\pi c}$$

$$\hat{L}_z\psi_n = \left(n\hbar - \frac{q\Phi}{2\pi c}\right)\psi_n = n'\hbar\psi_n$$

$$\Phi = \frac{2\pi c}{q}(n - n')\hbar$$